Upper bound involving parameter σ_2 for the rainbow connection number*

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Abstract

The minimum degree-sum $\sigma_2(G)$, or simply denoted by σ_2, of a graph G is defined as $min\{d(u) + d(v)|u, v \in V(G), uv \notin E(G)\}$. *The rainbow connection number* $rc(G)$ of a graph G was introduced by Chartrand et al. Chandran et al. proved that if G is a connected graph of order n with minimum degree δ, then $rc(G) \leq 3n/(\delta + 1) + 3$, and they gave an example to show that the bound is tight up to additive factors. In this paper, we prove that if G is a connected graph of order n, then $rc(G) \leq \frac{6n}{\sigma_2+2} + 8$. Moreover, we give two examples to show that our bound is tight up to additive factors. We also give another example G to show that from our bond one gets that $rc(G) < 14$ which is a constant, whereas from their bound one can only get that $rc(G) \leq \frac{3n}{2} + 3$ which is very large, linear in n.

Keywords: rainbow coloring, rainbow connection, connected two-step dominating set, parameter $\sigma_2(G)$

AMS subject classification 2010: 05C15, 05C40

1 Introduction

All graphs in the paper are finite, undirected and simple. Let $G = (V(G), E(G))$ be a graph and define $\sigma_2(G) = min\{d(u) + d(v)|u, v \in V(G), uv \notin E(G)\}$ as the minimum degree-sum of G, or simply denoted by σ_2. Let $Y \subseteq V(G)$. The subgraph $G[Y]$ of G induced by Y is the graph with vertex set Y and edge set consisting of the edges

*Supported by NSFC No.11071130 and "the Fundamental Research Funds for the Central Universities".

of G with both ends in Y. The distance between two vertices u and v in G, denoted by $d(u,v)$, is the length of a shortest path between them in G. The eccentricity of a vertex v is defined as $ecc(v) := \max_{x\in V(G)} d(v,x)$. The diameter of G is then defined as $diam(G) := \max_{x\in V(G)} ecc(x)$. Undefined terminology and notation can be found in [1]

The following notations were introduced in [5]. A path in an edge-colored graph with no two edges sharing the same color is called *a rainbow path*. An edge-colored graph is said to be *rainbow connected* if every pair of distinct vertices are connected by at least one rainbow path. Such a coloring is called a *rainbow coloring* of the graph. The minimum number of colors required to rainbow color a connected graph G is called its *rainbow connection number*, denoted by $rc(G)$. Note that disconnected graphs cannot be rainbow colored and hence the rainbow connection number for them is left undefined. An easy observation is that $rc(G) \geq diam(G)$.

It was shown by Chakraborty et al. [3] that computing the rainbow connection number of an arbitrary graph is NP-Hard. Chandran et al. [4] proved that if D is a connected two-way two-step dominating set in a graph G, then $rc(G) \leq rc(G[D]) + 6$, and they showed that if G is a connected graph of order n with minimum degree δ, then $rc(G) \leq 3n/(\delta+1) + 3$ and the bound is tight up to additive factors. The result nearly settles the investigation for an upper bound of rainbow connection number in terms of minimum degree, which was initiated by Caro et al. in [2]. However, if a graph has a small minimum degree δ and a large number n of vertices, then the upper bound could be very large, even linear in n. How to find a parameter to replace the parameter minimum degree δ and get a better upper bound is a natural question to ask for. Since σ_2 is a generalization of δ, a natural idea is to investigate the the bound in terms of the parameter σ_2. In the following we will see that in some graphs, their δ is very small, but their σ_2 could be very large, and this gives a better upper bound for the rainbow connection number.

Theorem 1. If G is a connected graph of order n, then $rc(G) \leq \frac{6n}{\sigma_2+2} + 8$.

We give two examples, Examples 1 and 2, to show that our bound is tight up to additive factors. Furthermore, we give another example, Example 3, to show that from our bound one gets that $rc(G) < 14$ which is a constant, whereas from their bound one can only get that $rc(G) \leq \frac{3n}{2} + 3$ which is very large, linear in n.

The following notions are needed in the sequel. Given a graph G, a set $D \subseteq V(G)$ is called a *k-step dominating set* of G if every vertex in G is at a distance at most k from D. Furthermore, if D induces a connected subgraph of G, it is called a *connected k-step dominating set* of G. The *k-step open neighborhood* of a set $D \subseteq V(G)$ is defined as $N^k(D) := \{x \in V(G) | d(x,D) = k\}$ for $k = \{0,1,2,\cdots\}$. A dominating set D in a graph G is called a *two-way dominating set* if every pendant vertex of G is included in

D. In addition, if $G[D]$ is connected, we call D a *connected two-way dominating set.* A connected two-step dominating set D of vertices in a graph G is called a *connected two-way two-step dominating set* if (i) every pendant vertex of G is included in D and (ii) every vertex in $N^2(D)$ has at least two neighbors in $N^1(D)$.

2 Examples

First of all, we denote by $K^*_{a,b}$ the graph obtained from the complete bipartite graph $K_{a,b}$ by joining every pair of vertices in the b-part by a new edge.

Example 1: When σ_2 is even, we denote $K^*_{2,\sigma_2/2-1}$ by H, and denote $K^*_{2,\sigma_2/2}$ by H'. Take t copies of H, denoted by $H_1, \cdots, H_t$ and label the two non-neighbor vertices of H_i by $x_{i,1}, x_{i,2}$ for $1 \leq i \leq t$. Take two copies of H', denoted by H_0, H_{t+1} and label the two non-neighbor vertices of H_0 by $x_{0,1}, x_{0,2}$, and label the two non-neighbor vertices of H_{t+1} by $x_{t+1,1}, x_{t+1,2}$. Now, join $x_{i,2}$ and $x_{i+1,1}$ for $i = 0, \cdots, t$ by an edge. The obtained graph G has $n = (t+2)(\sigma_2/2+1)+2$ vertices. We can get that $d(x_{0,2}) = \sigma_2/2+1$, $d(x_{t+1,2}) = \sigma_2/2+1$, and for any $x \in \{x_{i,1}, x_{i,2}, x_{0,1}, x_{t+1,1}\}$, $1 \leq i \leq t$, $d(x) = \sigma_2/2$. Hence, $min\{d(x)+d(y)|xy \notin E(G)\} = \sigma_2$. It is straightforward to verify that a shortest path from $x_{0,1}$ to $x_{t+1,2}$ has length $2(t+2)+t+1 = 3t+5 = diam(G) = \frac{6n}{\sigma_2+2} - (\frac{12}{\sigma_2+2}+1)$. Since $rc(G) \geq diam(G)$ and $rc(G) \leq \frac{6n}{\sigma_2+2} + 8$, our bound is tight up to addition factors.

Example 2: When σ_2 is odd, we denote $K^*_{2,(\sigma_2-1)/2}$ by H, and denote $K^*_{2,(\sigma_2+1)/2}$ by H'. Take $t+1$ copies of H, denoted by $H_0, \cdots, H_t$ and label the two non-neighbor vertices of H_i by $x_{i,1}, x_{i,2}$ for $0 \leq i \leq t$. Take a copy of H', denoted by H_{t+1} and label the two non-neighbor vertices of H_{t+1} by $x_{t+1,1}, x_{t+1,2}$. Now, join $x_{i,2}$ and $x_{i+1,1}$ for $i = 0, \cdots, t$ by an edge. The obtained graph G has $n = (t+2)(\sigma_2-1)/2+1$ vertices. We can get that $d(x_{0,1}) = (\sigma_2-1)/2$, $d(x_{t+1,2}) = (\sigma_2+1)/2+1$, and for any $x \in \{x_{i,1}, x_{i,2}, x_{0,2}, x_{t+1,1}\}$, $1 \leq i \leq t$, $d(x) = (\sigma_2+1)/2$. Hence, $min\{d(x)+d(y)|xy \notin E(G)\} = \sigma_2$. It is straightforward to verify that a shortest path from $x_{0,1}$ to $x_{t+1,2}$ has length $2(t+2)+t+1 = 3t+5 = diam(G)$. Since $rc(G) \geq diam(G)$ and $rc(G) \leq \frac{6n}{\sigma_2+2} + 8$, similar to the discussion of Example 1, our bound is tight up to addition factors.

Example 3: Let $V_1 = \{v_1\}$, and let K_2 be a complete graph of order 2 and K_{n-3} be a complete graph of order $n-3$. Denote $V(K_2) = \{v_2, v_3\}$. We join v_1 to every vertex of K_2 by an edge, join v_2 to every vertex of K_{n-3} by an edge, and join v_3 to every vertex of K_{n-3} by an edge. Thus we have constructed a graph G. In G, $\delta = d(v_1) = 2$, and for any $v \in K_{n-3}$, $vv_1 \notin E(G)$, $d(v) = n-2$. So we can get that $\sigma_2 = n$. Thus, in this example, our bound gives that $\frac{6n}{\sigma_2+2} + 8 = \frac{6n}{n+2} + 8 < 14$, which is a constant. However, from the bound above in terms of minimum degree δ we get that $rc(G) \leq \frac{3n}{\delta} + 3 = \frac{3n}{2} + 3$, which is very large, linear in n.

3 Proof of Theorem 1

If G is a complete graph, then $rc(G) = 1$, the theorem holds. In the following we assume that G is not complete. When $\sigma_2 \leq 4$, $\frac{6n}{\sigma_2+2} + 8 \geq n + 8$. However, $rc(G) \leq n - 1$, and hence when $\sigma_2 \leq 4$, the theorem is obvious true.

In the following we assume $\sigma_2 \geq 4$. So we can get that G has at most a pendant vertex. First, we prove the following two claims:

Claim 1. If G is a connected graph of order n with $\sigma_2 \geq 4$, then G has a connected two-step dominating set D such that $|D| \leq 6\frac{n-|N^2(D)|}{\sigma_2+2}$.

Proof of Claim 1. Let $u \in V(G)$ and $d(u) = \delta$. Then for any $v \in V(G) \setminus N[u]$, we get $d(v) \geq \lceil \frac{\sigma_2}{2} \rceil$. Let $D = \{u\}$. If $N^3(D) \neq \emptyset$ for any $v \in N^3(D)$, let $P = vv_2v_1v_0$ be a shortest $v - D$ path, where $v_2 \in N^2(D), v_1 \in N^1(D), v_0 \in D$. Let $D = D \cup \{v, v_2, v_1\}$. While $N^3(D) = \emptyset$, let t_1 be the number of iterations executed in the above procedure, we can get $t_1 \leq \frac{|D \cup N^1(D)| - (\delta+1)}{\lceil \frac{\sigma_2}{2} \rceil + 1} < \frac{|D \cup N^1(D)|}{\lceil \frac{\sigma_2}{2} \rceil + 1} \leq \frac{|D \cup N^1(D)|}{\frac{\sigma_2}{2} + 1} = 2\frac{|D \cup N^1(D)|}{\sigma_2+2} = 2\frac{n-|N^2(D)|}{\sigma_2+2}$. Hence, $|D| = 1 + 3t_1 < 1 + 6\frac{n-|N^2(D)|}{\sigma_2+2}$. Therefore, Claim 1 is true. ∎

Claim 2. Every connected graph G of order n with at most one pendant vertex has a connected two-way two-step dominating set D of size at most $\frac{6n}{\sigma_2+2} + 3$.

Proof of Claim 2. We consider the connected two-step dominating set D which was obtained in Claim 1. While there exist two vertices $u, v \in N^2(D)$ with $uv \notin E(G)$ and $d(u) \geq d(v)$ such that u has only one neighbor in $N^1(D)$ and v has only one neighbor in $N^1(D)$, we set $D = D \cup \{u, u_1\}$ where uu_1u_0 is a shortest $u - D$ path with $u_0 \in D$. Clearly, D remains a connected two-step dominating set. The procedure ends only when $N^2(D)$ can be partitioned into two parts $N_1^2(D)$ and $N_2^2(D)$ such that for any $v \in N_1^2(D)$, v has at least two neighbors in $N^1(D)$, and for any $v \in N_2^2(D)$, v has only a neighbor in $N^1(D)$, and $G[N_2^2(D)]$ is a complete subgraph, where $|N_1^2(D)| \geq 0, |N_2^2(D)| \geq 0$. ∎

Let k_2 be the number of iterations executed, we add to D a vertex which has at least $\frac{\sigma_2}{2} - 1$ neighbors in $N^2(D)$. Then $|N^2(D)|$ reduces by at least $\frac{\sigma_2}{2}$ in every iteration. Since we start with $|N^2(D)|$ vertices, then $k_2 \leq \frac{|N^2(D)|}{\frac{\sigma_2}{2}}$. Since we add two vertices to D in each iteration, then $|D| = |D| + 2k_2$ and so $|D| \leq 6\frac{n-|N^2(D)|}{\sigma_2+2} + 4\frac{|N^2(D)|}{\sigma_2} \leq \frac{6n}{\sigma_2+2}$. Then we get that $|D| \leq \frac{6n}{\sigma_2+2}$.

Take a vertex $w \in N_2^2(D)$, let $ww_1w_0, w_0 \in D$ be a shortest $w - D$ path, and let $D = D \cup \{w, w_1\}$. It is obvious that D also remains connected, and $|D| \leq \frac{6n}{\sigma_2+2} + 2$.

If G has no pendant vertex, then D is exactly the two-way two-step dominating set and so Claim 2 follows. If u is the pendant vertex of G and u is in D, then D is exactly the two-way two-step dominating set and Claim 2 also follows. If u is the pendant vertex of G and u is in $N^1(D)$, then we add u to D and let $D = D \cup \{u\}$. We get that D is exactly the two-way two-step dominating set of size at most $\frac{6n}{\sigma_2+2} + 3$ and so Claim 2 is true.

We know that if D is a connected two-way two-step dominating set in a graph G, then $rc(G) \leq rc(G[D]) + 6$, and by Claim 2 we know that $|D| \leq \frac{6n}{\sigma_2+2} + 3$. Hence, we can get that $rc(G) \leq rc(G[D]) + 6 \leq \frac{6n}{\sigma_2+2} + 3 - 1 + 6 = \frac{6n}{\sigma_2+2} + 8$. This completes the proof of Theorem 1. ∎

Acknowledgement. The authors would like to thank the referees for their helpful comments and suggestions which helped to improve the original submission.

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