

# Upper bounds for the rainbow connection numbers of line graphs\*

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## Abstract

A path in an edge-colored graph  $G$ , where adjacent edges may be colored the same, is called a rainbow path if no two edges of it are colored the same. A nontrivial connected graph  $G$  is rainbow connected if for any two vertices of  $G$  there is a rainbow path connecting them. The rainbow connection number of  $G$ , denoted  $rc(G)$ , is defined as the smallest number of colors such that  $G$  is rainbow connected. In this paper, we mainly study the rainbow connection number  $rc(L(G))$  of the line graph  $L(G)$  of a graph  $G$  which contains triangles. We get two sharp upper bounds for  $rc(L(G))$ , in terms of the number of edge-disjoint triangles of  $G$ . We also give results on the iterated line graphs.

**Keywords:** rainbow path, rainbow connection number, (iterated) line graph, edge-disjoint triangles.

**AMS Subject Classification 2000:** 05C15, 05C40

## 1 Introduction

All graphs in this paper are simple, finite and undirected. Let  $G$  be a nontrivial connected graph with an edge coloring  $c : E(G) \rightarrow \{1, 2, \dots, k\}$ ,  $k \in \mathbb{N}$ , where adjacent edges may be colored the same. A path of  $G$  is called *rainbow* if no two edges of it are colored the same. An edge-colored graph  $G$  is *rainbow connected* if for any two vertices there is a rainbow path connecting them. Clearly, if a graph is rainbow connected, it must be connected. Conversely, any connected graph has a trivial edge coloring that makes it rainbow connected, i.e., the coloring such that each edge has a distinct color. Thus, we define the *rainbow connection number* of a connected graph  $G$ , denoted by  $rc(G)$ , as the smallest number of colors for

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which there is an edge coloring of  $G$  such that  $G$  is rainbow connected. An easy observation is that if  $G$  has  $n$  vertices then  $rc(G) \leq n - 1$ , since one may color the edges of a spanning tree with distinct colors, and color the remaining edges with one of the colors already used. Generally, if  $G_1$  is a connected spanning subgraph of  $G$ , then  $rc(G) \leq rc(G_1)$ . We notice the trivial fact that  $rc(G) = 1$  if and only if  $G$  is complete, and the fact that  $rc(G) = n - 1$  if and only if  $G$  is a tree, as well as the easy observation that a cycle  $C_k$  with  $k > 3$  vertices has rainbow connection number  $rc(C_k) = \lceil \frac{k}{2} \rceil$  ([3]). Also notice that, clearly,  $rc(G) \geq diam(G)$  where  $diam(G)$  denotes the diameter of  $G$ .

In [3], Chartrand et al. also determined the precise rainbow connection numbers for other graph classes including wheels and complete bipartite (multipartite) graphs. In [2], Caro et al. studied the rainbow connection number for further graph classes in terms of some graph parameters, such as the minimum degree of a graph.

**Theorem 1.1** ([2]) *If  $G$  is a connected graph with  $n$  vertices and  $\delta(G) \geq 3$ , then  $rc(G) < \frac{5n}{6}$ .* ■

The constant in Theorem 1.1 is not optimal, but it cannot be replaced with a constant smaller than  $\frac{3}{4}$ , since there are 3-regular connected graphs with  $rc(G) = diam(G) = \frac{3n-10}{4}$ , so they conjectured:

**Conjecture 1.2** ([2]) *If  $G$  is a connected graph with  $n$  vertices and  $\delta(G) \geq 3$ , then  $rc(G) < \frac{3}{4}n$ .* ■

Conjecture 1.2 has been recently proved in [8]:

**Theorem 1.3** ([8]) *If  $G$  is a connected graph with  $n$  vertices and  $\delta(G) \geq 3$ , then  $rc(G) \leq \frac{3n-1}{4}$ .* ■

In [2], the authors also derived the following upper bound.

**Theorem 1.4** ([2]) *If  $G$  is a connected graph with  $n$  vertices and minimum degree  $\delta$ , then*

$$rc(G) \leq \min\left\{n \frac{\ln \delta}{\delta} (1 + o_\delta(1)), n \frac{4 \ln \delta + 3}{\delta}\right\}.$$

This, naturally, raised the open problem of determining the true behavior of  $rc(G)$  as a function of  $\delta$ . Krivelevich and Yuster in [5] determined the behavior of  $rc(G)$  as a function of the minimum degree  $\delta(G)$  and the order  $n$ :

**Theorem 1.5** ([5]) *A connected graph  $G$  with  $n$  vertices has  $rc(G) < \frac{20n}{\delta(G)}$ .* ■

Some graph classes, such as line graphs, have many special properties, and by these properties, we can get some interesting results on their rainbow connection numbers in terms of some graph parameters. For example, in [2] the authors got a nice upper bound for the rainbow connection number of a 2-connected graph according to their ear-decomposition and this result is an important ingredient in the proof of Theorem 1.1.

**Proposition 1.6** ([2]) *If  $G$  is a 2-connected graph with  $n$  vertices, then  $rc(G) \leq \frac{2n}{3}$ .*

■

In [7], we studied the rainbow connection numbers of line graphs of triangle-free graphs in the light of particular properties of line graphs of triangle-free graphs shown in [4], and particularly, of 2-connected triangle-free graphs according to their ear decompositions. However, we did not get bounds of the rainbow connection numbers for line graphs of graphs that do contain triangles. In this paper, we aim to investigate the remaining case, i.e., line graphs of graphs that do contain triangles, and give two sharp upper bounds in terms of the number of edge-disjoint triangles of original graphs.

We use  $V(G)$ ,  $E(G)$  for the sets of vertices and edges of  $G$ , respectively. For any subset  $X$  of  $V(G)$ , let  $G[X]$  denote the subgraph induced by  $X$ , and  $E[X]$  the edge set of  $G[X]$ ; similarly, for any subset  $E_1$  of  $E(G)$ , let  $G[E_1]$  denote the subgraph induced by  $E_1$ . Let  $\mathcal{G}$  be a set of graphs, then  $V(\mathcal{G}) = \bigcup_{G \in \mathcal{G}} V(G)$ ,  $E(\mathcal{G}) = \bigcup_{G \in \mathcal{G}} E(G)$ . We define a *clique* in a graph  $G$  to be a complete subgraph of  $G$ , and a *maximal clique* is a clique that is not contained in any larger clique. The *clique graph*  $K(G)$  of  $G$  is the intersection graph of the maximal cliques of  $G$ —that is, the vertices of  $K(G)$  correspond to the maximal cliques of  $G$ , and two of these vertices are joined by an edge if and only if the corresponding maximal cliques intersect. Let  $[n] = \{1, \dots, n\}$  denote the set of the first  $n$  natural numbers. For a set  $S$ ,  $|S|$  denotes the cardinality of  $S$ . We follow the notations and terminology of [1] for those not defined here.

## 2 Some basic observations

We first list two observations which were given in [7] and will be used later.

**Observation 2.1** ([7]) *If  $G$  is a connected graph and  $\{E_i\}_{i \in [t]}$  is a partition of the edge set of  $G$  into connected subgraphs  $G_i = G[E_i]$ , then*

$$rc(G) \leq \sum_{i=1}^t rc(G_i).$$

■

Let  $G$  be a connected graph, and  $X$  a proper subset of  $V(G)$ . To *shrink*  $X$  is to delete all the edges between vertices of  $X$  and then identify the vertices of  $X$  into a single vertex, namely  $w$ . We denote the resulting graph by  $G/X$ .

**Observation 2.2** ([7]) *Let  $G'$  and  $G$  be two connected graphs, where  $G'$  is obtained from  $G$  by shrinking a proper subset  $X$  of  $V(G)$ , that is,  $G' = G/X$ , such that any two vertices of  $X$  have no common adjacent vertex in  $V \setminus X$ . Then*

$$rc(G') \leq rc(G).$$

■

Now we introduce two graph operations and two corresponding results which will be used later.

**Operation 1.** As shown in Figure 2.1, for any edge  $e = uv \in G$  with  $\min\{deg_G(u), deg_G(v)\} \geq 2$ , we first subdivide  $e$ , then replace the new vertex with two new vertices  $u_e, v_e$  with  $deg_{G'}(u_e) = deg_{G'}(v_e) = 1$  where  $G'$  is the new graph.

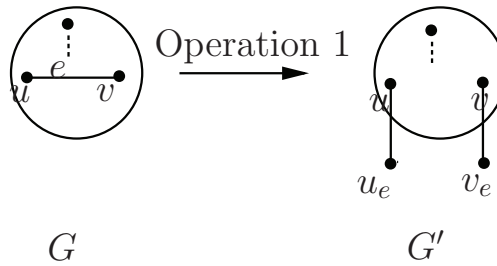


Figure 2.1  $G'$  is obtained from  $G$  by applying Operation 1 to edge  $e$ .

Recall that the *line graph* of a graph  $G$  is the graph  $L(G)$  whose vertex set  $V(L(G)) = E(G)$  and two vertices  $e_1, e_2$  of  $L(G)$  are adjacent if and only if they are adjacent in  $G$ . The star, denoted  $S(v)$ , at a vertex  $v$  of graph  $G$ , is the set of all edges incident to  $v$ . Let  $\langle S(v) \rangle$  be the subgraph of  $L(G)$  induced by  $S(v)$ , clearly, it is a clique of  $L(G)$ .

Since  $deg_{G'}(u_e) = deg_{G'}(v_e) = 1$  in  $G'$ , and by the definition of a line graph, it is easy to show that  $L(G)$  can be obtained from  $L(G')$  by shrinking a vertex set of two nonadjacent vertices (these two vertices correspond to edges  $uu_e, vv_e$ , and belong to cliques  $\langle S(u) \rangle, \langle S(v) \rangle$ , respectively, in  $L(G')$ ). So by Observation 2.2, we have

**Observation 2.3** *If a connected graph  $G'$  is obtained from a connected graph  $G$  by applying Operation 1 at some edge  $e \in G$ , then*

$$rc(L(G)) \leq rc(L(G')).$$

■

**Operation 2.** As shown in Figure 2.2,  $v$  is a common vertex of a set of edge-disjoint triangles in  $G$ . We replace  $v$  by two nonadjacent vertices  $v'$  and  $v''$  such that  $v'$  is the common vertex of some triangles, and  $v''$  is the common vertex of the remaining triangles.

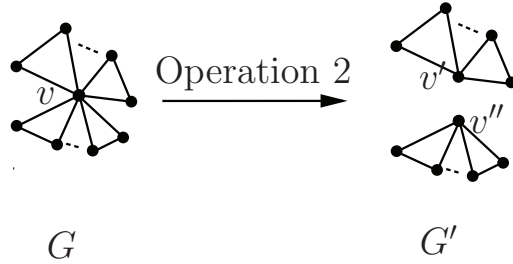


Figure 2.2 Figure of Operation 2.

Since during this procedure, the number of edges does not change, the order of the line graph  $L(G)$  is equal to that of  $L(G')$ . Furthermore, by the definition of a line graph,  $L(G')$  is a spanning subgraph of  $L(G)$ . So we have

**Observation 2.4** *If a connected graph  $G'$  is obtained from a connected graph  $G$  by applying Operation 2 at some vertex  $v \in G$ , then*

$$rc(L(G)) \leq rc(L(G')).$$

■

## 3 Main results

### 3.1 A sharp upper bound

Recall that a *clique decomposition* of  $G$  is a collection  $\mathcal{C}$  of cliques such that each edge of  $G$  occurs in exactly one clique in  $\mathcal{C}$ .

We now introduce a new terminology. For a connected graph  $G$ , we call  $G$  a *clique-tree-structure*, if it satisfies the following condition:

$T_1$ . Each block is a maximal clique.

We call a graph  $H$  a *clique-forest-structure*, if  $H$  is a disjoint union of some clique-tree-structures, that is, each component of a clique-forest-structure is a clique-tree-structure. By condition  $T_1$ , we know that any two maximal cliques of  $G$  have at most one common vertex. Furthermore,  $G$  is formed by its maximal cliques. The *size* of a clique-tree(forest)-structure is the number of its maximal cliques. An example of clique-forest-structure is shown in Figure 3.1.

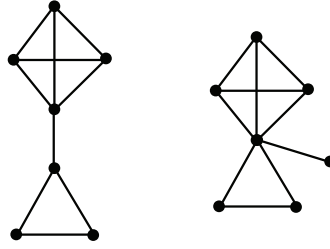


Figure 3.1 A clique-forest-structure with size 6 and 2 components.

If each block of a clique-tree-structure is a triangle, we call it a *triangle-tree-structure*. Let  $\ell$  be the size of a triangle-tree-structure. Then, by definition, it is easy to show that there are  $2\ell + 1$  vertices in it. Similarly, we can give the definition of *triangle-forest-structure* and there are  $2\ell + c$  vertices in a triangle-forest-structure with size  $\ell$  and  $c$  components. A clique-tree-structure  $G$  is called a *clique-path-structure* if the clique graph  $K(G)$  is a path.

For a connected graph  $G$ , we call  $G$  a *clique-cycle-structure*, if it satisfies the following three conditions:

- $C_1$ .  $G$  has at least three maximal cliques;
- $C_2$ . Each edge belongs to exactly one maximal clique;
- $C_3$ . The clique graph is a cycle.

Similarly, the *size* of a clique-cycle-structure is the number of its maximal cliques. In particular, if each maximal clique is a triangle, then it is a *triangle-cycle-structure*. An example of triangle-cycle-structure of size 4 is shown in Figure 3.2.

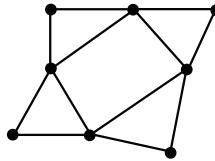


Figure 3.2 An example of triangle-cycle-structure.

Let  $\mathcal{T} = \{T_1, T_2, \dots, T_t\}$  be a set of edge-disjoint triangles of a graph  $G$ . Clearly, by definition, if  $G[E(\mathcal{T})]$  is connected (not connected) and is not a triangle-tree(forest)-structure, then it must contain a triangle-cycle-structure as a subgraph.

An *inner vertex* of a graph is a vertex with degree at least two. For a graph  $G$ , we use  $\overline{V}_2$  to denote the set of all inner vertices of  $G$ . Let  $n_1 = |\{v : \deg_G(v) = 1\}|$ ,  $n_2 = |\overline{V}_2|$ . Recall that  $\langle S(v) \rangle$  is the subgraph of  $L(G)$  induced by  $S(v)$  and it is a clique of  $L(G)$ . Let  $\mathcal{K}_0 = \{\langle S(v) \rangle : v \in V(G)\}$ ,  $\mathcal{K} = \{\langle S(v) \rangle : v \in \overline{V}_2\}$ . It is easy

to show that  $\mathcal{K}_0$  is a clique decomposition of  $L(G)$  ([6]) and each vertex of the line graph belongs to at most two elements of  $\mathcal{K}_0$ . We know that each element  $\langle S(v) \rangle$  of  $\mathcal{K}_0 \setminus \mathcal{K}$ , a single vertex of  $L(G)$ , is contained in the clique induced by  $u$  that is adjacent to  $v$  in  $G$ . So  $\mathcal{K}$  is a clique decomposition of  $L(G)$ .

Let  $X$  and  $Y$  be sets of vertices of a graph  $G$ . We denote by  $E[X, Y]$  the set of the edges of  $G$  with one end in  $X$  and the other end in  $Y$ . If  $Y = X$ , we simply write  $E(X)$  for  $E[X, X]$ . When  $Y = V \setminus X$ , the set  $E[X, Y]$  is called the *edge cut* of  $G$  associated with  $X$ , and is denoted by  $\partial(X)$ .

**Theorem 3.1** *For any set  $\mathcal{T}$  of  $t$  edge-disjoint triangles of a connected graph  $G$ , if the subgraph induced by the edge set  $E(\mathcal{T})$  is a triangle-forest-structure, then*

$$rc(L(G)) \leq n_2 - t.$$

*Moreover, the bound is sharp.*

*Proof.* Let  $\mathcal{T} = \bigcup_{i=1}^c \mathcal{T}_i = \bigcup_{i=1}^c \{T_{i,j_i} \text{ is a triangle of } G : 1 \leq j_i \leq t_i\}$  ( $\sum_{i=1}^c t_i = t$ ) be a set of  $t$  edge-disjoint triangles of  $G$  such that the subgraph of  $G$ ,  $G[E(\mathcal{T}_i)]$ , induced by each  $E(\mathcal{T}_i)$  is a component of the subgraph  $G[E(\mathcal{T})]$ , that is, a triangle-tree-structure of size  $t_i$ .

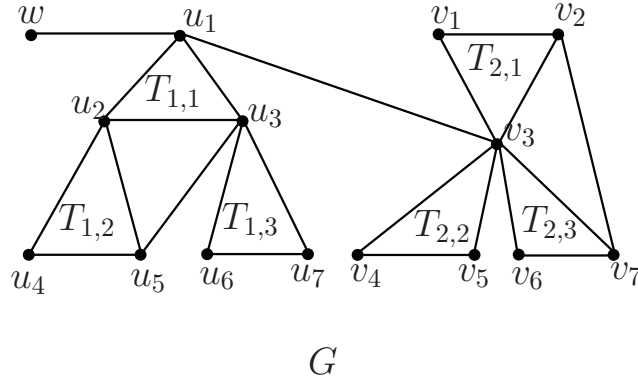


Figure 3.3 An example of Theorem 3.1.

In  $G$ , for each  $1 \leq i \leq c$ , let  $G_i = G[E(\mathcal{T}_i)]$ ,  $V_i = V(G_i)$ ,  $E_i = E(\mathcal{T}_i)$ ;  $E_i^0 = E(V_i) \cup \partial(V_i) \supseteq E_i$ , and  $G_i^0 = G[E_i^0]$ . We obtain a new graph  $G'$  from  $G$  by applying Operation 1 at each edge  $e \in E(V_i) \setminus E_i$  for  $1 \leq i \leq c$  (Note that here any edge  $e \in E(V_i) \setminus E_i$  cannot be a bridge, so the graph obtained from  $G$  by applying Operation 1 at a such edge is still connected), and we denote by  $G'_i$  the new subgraph (of  $G'$ ) corresponding to  $G_i^0$ . Applying Observation 2.3 repeatedly, we have  $rc(L(G)) \leq rc(L(G'))$ .

For example, see Figure 3.3, here  $c = 2$ ,  $t_1 = t_2 = 3$ ,  $t = t_1 + t_2 = 6$ ,  $\mathcal{T} = \bigcup_{i=1}^2 \mathcal{T}_i = \bigcup_{i=1}^2 \{T_{i,j_i} : 1 \leq j_i \leq 3\}$  is a set of 6 edge-disjoint triangles

of  $G$ . Clearly, the subgraph  $G[E(\mathcal{T}_i)]$  of  $G$  induced by each  $E(\mathcal{T}_i)$  is a triangle-tree-structure of size 3, so  $G[E(\mathcal{T})]$  is a triangle-forest-structure with size 6 and 2 components.  $G_1 = G[E(\mathcal{T}_1)]$ ,  $V_1 = V(G_1) = \{u_i : 1 \leq i \leq 7\}$ ,  $E_1 = E(\mathcal{T}_1) = \{u_1u_2, u_2u_3, u_1u_3, u_2u_4, u_4u_5, u_2u_5, u_3u_6, u_6u_7, u_3u_7\}$ ,  $G_1^0 = G[E_1^0]$  where  $E_1^0 = E(V_1) \cup \partial(V_1) = E_1 \cup \{u_3u_5\} \cup \{wu_1, u_1v_3\}$  with  $E(V_1) = E_1 \cup \{u_3u_5\}$  and  $\partial(V_1) = \{wu_1, u_1v_3\}$ . The edge  $u_3u_5 \in E(V_1) \setminus E_1$  cannot be a bridge, similarly, neither is  $v_2v_7 \in E(V_2) \setminus E_2$ . We obtain  $G'$  from  $G$  by applying Operation 1 at edges  $u_3u_5$  and  $v_2v_7$ , the new subgraph  $G'_i$  (of  $G'$ ) corresponding to  $G_i^0$  contains exactly  $t_i = 3$  (edge-disjoint) triangles  $\{T_{i,j_i}\}$  with  $i = 1, 2$  and  $1 \leq j_i \leq 3$ .

Next we will show  $rc(L(G')) \leq n_2 - t$ . By previous discussion, we know that

$$\mathcal{K} = \{\langle S(v) \rangle : v \in \overline{V_2}\} = \bigcup_{i=1}^c \{\langle S(v) \rangle : v \in V_i\} \cup \{\langle S(v) \rangle : v \in \overline{V_2} \setminus \bigcup_{i=1}^c V_i\}$$

is a clique decomposition of  $L(G)$ . So

$$\{E(\langle S(v) \rangle) : v \in V_i\}_{i=1}^c \cup \{E(\langle S(v) \rangle) : v \in \overline{V_2} \setminus \bigcup_{i=1}^c V_i\},$$

that is,

$$\{E(L(G_i^0))\}_{i=1}^c \cup \{E(\langle S(v) \rangle) : v \in \overline{V_2} \setminus \bigcup_{i=1}^c V_i\}$$

is an edge partition of  $L(G)$ . So

$$\{E(L(G'_i))\}_{i=1}^c \cup \{E(\langle S(v) \rangle) : v \in \overline{V_2} \setminus \bigcup_{i=1}^c V_i\}$$

is an edge partition of  $L(G')$ . By Observation 2.1, we have

$$rc(L(G')) \leq \sum_{i=1}^c rc(L(G'_i)) + \sum_{v \in \overline{V_2} \setminus \bigcup_{i=1}^c V_i} rc(\langle S(v) \rangle).$$

We know  $|V_i| = 2t_i + 1$ , since the triangle-tree-structure  $G_i$  has size  $t_i$ . So

$$rc(L(G')) \leq \sum_{i=1}^c rc(L(G'_i)) + (n_2 - 2t - c).$$

In order to get  $rc(L(G')) \leq n_2 - t$ , we need to show  $rc(L(G'_i)) \leq t_i + 1$ .

**Claim.**  $rc(L(G'_i)) \leq t_i + 1$ .

**Proof of the Claim.** Since the graph  $G'_i$  is obtained from  $G_i$  by applying Operation 1 at each edge  $e \in E(V_i) \setminus E_i$ ,  $G'_i$  contains exactly  $t_i$  triangles:  $\{T_{i,j_i} \in \mathcal{T}_i : 1 \leq j_i \leq t_i\}$ . We will show that there is a  $(t_i + 1)$ -rainbow coloring of  $L(G'_i)$  by induction on  $t_i$ . For  $t_i = 1$ ,  $G'_i$  contains exactly one triangle, and we give its line



graph a 2-rainbow coloring as shown in Figure 3.4. We give color 1 to the edges of  $\langle S(u) \rangle$  incident with vertex  $e_1$ , edges of  $\langle S(v) \rangle$  incident with vertex  $e_2$ , and edges of  $\langle S(w) \rangle$  incident with vertex  $e_3$ ; We then give color 2 to the edges of  $\langle S(u) \rangle$  incident with vertex  $e_3$ , edges of  $\langle S(v) \rangle$  incident with vertex  $e_1$ , and edges of  $\langle S(w) \rangle$  incident with vertex  $e_2$ ; Finally, give color 2 to the rest of the edges. It is easy to show that this is a rainbow coloring. So, the above conclusion holds for the case  $t_i = 1$ . We

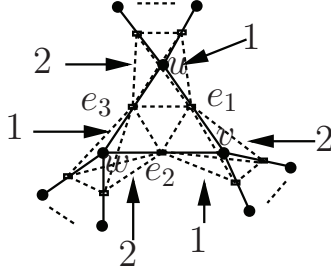


Figure 3.4 2-rainbow coloring of line graph of graph with exactly one triangle.

assume that the conclusion holds for the case  $t_i = h - 1 (h \geq 2)$ , and now show that it holds for the case  $t_i = h$ . By the definition of triangle-tree-structure, there must exist one triangle:  $T = \{u, v, w\}$ , which has exactly one common vertex, namely  $u$ , with other triangles and  $v, w$  do not belong to any other triangle. We now obtain a new graph  $\overline{G}'_i$  from  $G'_i$  by doing Operation 1 at edges  $e_1$  and  $e_2$ , this produces two subgraphs  $H_1$  and  $H_2$  (see Figure 3.5). It is clear that  $L(\overline{G}'_i)$  can be obtained from

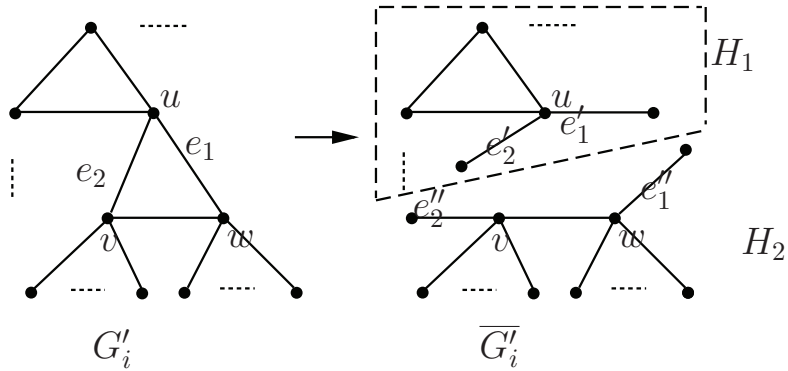


Figure 3.5  $\overline{G}'_i$  is obtained from  $G'_i$  by applying Operation 1 to edges  $e_1$  and  $e_2$ .

$L(G'_i)$  by subdividing vertex  $e_1$  into two vertices  $\{e'_1, e''_1\}$  and  $e_2$  into two vertices  $\{e'_2, e''_2\}$ . Since  $H_1$  has  $h - 1$  (edge-disjoint) triangles, by induction hypothesis,  $rc(L(H_1)) \leq h$ . As the subgraph of  $L(G'_i)$  induced by  $\{\langle S(v) \rangle : v \in V_i \setminus \{v, w\}\}$  is isomorphic to  $L(H_1)$ , so it also has a rainbow  $h$ -coloring. The remaining edges of  $L(G'_i)$  belong to  $\langle S(v) \rangle$  and  $\langle S(w) \rangle$ , we color them as follows: give a new color to the edges of  $\langle S(w) \rangle$  incident with vertex  $e_1$ , and edges of  $\langle S(v) \rangle$  incident with vertex  $e_2$ . Let  $e_3 = vw$ , we then give any one color, say  $c_1$ , of the former  $h$  colors to the edges

of  $\langle S(w) \rangle$  incident with vertex  $e_3$ , and give a distinct color, say  $c_2$ , of the former  $h$  colors to the edges of  $\langle S(v) \rangle$  incident with vertex  $e_3$ . It is easy to show that, with the above coloring,  $L(G'_i)$  is rainbow connected, and so  $L(G'_i) \leq t_i + 1$  holds for  $t_i = h$ .

For the sharpness of the upper bound, see Example 3.3. ■

We call a set of triangles *independent* if any two of them are vertex-disjoint. Since each single triangle is a triangle-tree-structure, applying Theorem 3.1, we have the following corollary, and for the sharpness of the upper bound, see Example 3.3:

**Corollary 3.2** *If  $G$  is a connected graph with  $t'$  independent triangles, then*

$$rc(L(G)) \leq n_2 - t'.$$

*Moreover, the bound is sharp.* ■

**Example 3.3** *Let  $G$  consist of  $t$  (independent) triangles and  $t - 1$  edges which do not belong to any triangles, as shown in Figure 3.6. Since  $G$  has  $3t$  inner vertices, by Theorem 3.1(Corollary 3.2), we know  $rc(L(G)) \leq 2t$ ; on the other hand, it is easy to show that the diameter of the line graph  $L(G)$  is  $2t$ , and so we have  $rc(L(G)) = 2t$ . Then the bound of the above theorem(corollary) is sharp.* ■

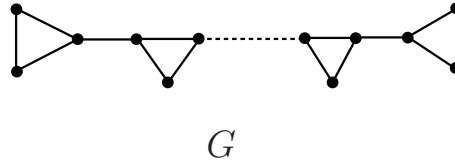


Figure 3.6 Figure of Example 3.3.

## 3.2 Another sharp upper bound

Now we give another upper bound which is derived from Theorem 3.1 and, of course, it is sharp.

**Theorem 3.4** *If  $G$  is a connected graph,  $\mathcal{T}$  is a set of  $t$  edge-disjoint triangles that cover all but  $n'_2$  inner vertices of  $G$  and  $c$  is the number of components of the subgraph  $G[E(\mathcal{T})]$ , then*

$$rc(L(G)) \leq t + n'_2 + c.$$

*Moreover, the bound is sharp.* ■

*Proof.* Let  $\mathcal{T} = \bigcup_{i=1}^c \mathcal{T}_i = \bigcup_{i=1}^c \{T_{i,j_i} \text{ is a triangle of } G : 1 \leq j_i \leq t_i\}$  ( $\sum_{i=1}^c t_i = t$ ) be a set of  $t$  edge-disjoint triangles of  $G$  such that the subgraph of  $G$ ,  $G[E(\mathcal{T})]$ ,

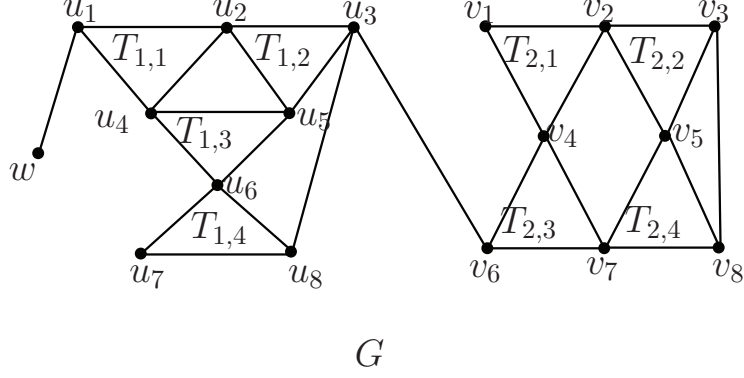


Figure 3.7 Figure of Example of Theorem 3.4.

induced by each  $E(\mathcal{T}_i)$  is a connected component of the subgraph  $G[E(\mathcal{T})]$ . If each  $G[E(\mathcal{T}_i)]$  is a triangle-tree-structure, then the number of inner vertices covered by all these  $t$  triangles is  $2t + c$ , that is,  $n'_2 = n_2 - (2t + c)$ . So by Theorem 3.1, in this case we have  $rc(L(G)) \leq n_2 - t = n'_2 + (2t + c) - t = t + n'_2 + c$ . Now we consider the case that there exists at least one  $i$  such that the subgraph  $G[E(\mathcal{T}_i)]$  is not a triangle-tree-structure, that is, it may contain a triangle-cycle-structure as a subgraph. An example is shown in Figure 3.7, the subgraph induced by the edges of triangles:  $\{T_{1,1}, T_{1,2}, T_{1,3}\}$  is a triangle-cycle-structure of size 3.

In  $G$ , for each  $1 \leq i \leq c$ , let  $G_i = G[E(\mathcal{T}_i)]$ ,  $V_i = V(G_i)$ ,  $E_i = E(\mathcal{T}_i)$ ;  $E_i^0 = E(V_i) \cup \partial(V_i) \supseteq E_i$ , and  $G_i^0 = G[E_i^0]$ . We obtain a new graph  $G'$  from  $G$  by applying Operation 1 at each edge  $e \in E(V_i) \setminus E_i$  for  $1 \leq i \leq c$  (Note that here any edge  $e \in E(V_i) \setminus E_i$  cannot be a bridge, so the graph obtained from  $G$  by applying Operation 1 at one such edge is still connected), and we denote by  $G'_i$  the new subgraph (of  $G'$ ) corresponding to  $G_i^0$ . Applying Observation 2.3 repeatedly, we have  $rc(L(G)) \leq rc(L(G'))$ . Now each  $G'_i$  contains exactly  $t_i$  triangles:  $T_{i,j_i}$  where  $1 \leq j_i \leq t_i$ . We now obtain a new graph  $G''$  from  $G'$  by applying Operation 2 to those  $G'_i$ s which contain triangle-cycle-structures such that each subgraph (of  $G''$ )  $G''_i$  corresponding to  $G'_i$  contains no triangle-cycle-structure (During this procedure, we can make sure that  $G''$  is connected).

For example, see Figure 3.7, here  $c = 2$ ,  $t_1 = t_2 = 4$ ,  $t = t_1 + t_2 = 8$ ,  $\mathcal{T} = \bigcup_{i=1}^2 \mathcal{T}_i = \bigcup_{i=1}^2 \{T_{i,j_i} : 1 \leq j_i \leq 4\}$  is a set of 8 edge-disjoint triangles of  $G$ . Clearly, the subgraph of  $G$ ,  $G[E(\mathcal{T}_i)]$ , induced by each  $E(\mathcal{T}_i)$  is not a triangle-tree-structure,  $G[E(\mathcal{T}_1)]$  contains a triangle-cycle-structure of size 3 and  $G[E(\mathcal{T}_2)]$  is a triangle-cycle-structure of size 4.  $G_1 = G[E(\mathcal{T}_1)]$ ,  $V_1 = V(G_1) = \{u_i : 1 \leq i \leq 8\}$ ,  $E_1 = E(\mathcal{T}_1) = \{u_1u_2, u_2u_4, u_1u_4, u_2u_3, u_3u_5, u_2u_5, u_4u_5, u_4u_6, u_6u_5, u_6u_7, u_6u_8, u_7u_8\}$ ,  $G_1^0 = G[E_1^0]$  where  $E_1^0 = E(V_1) \cup \partial(V_1) = E_1 \cup \{u_3u_8\} \cup \{wu_1, u_3v_6\}$  with  $E(V_1) = E_1 \cup \{u_3u_8\}$  and  $\partial(V_1) = \{wu_1, u_3v_6\}$ . Edge  $u_3u_8 \in E(V_1) \setminus E_1$  cannot be a bridge, similarly,  $v_3v_8 \in E(V_2) \setminus E_2$ . We obtain  $G'$  from  $G$  by applying Operation 1 at edges  $u_3u_8$  and  $v_3v_8$ , the new subgraph  $G'_i$  (of  $G'$ ) corresponding to  $G_i^0$  contains exactly  $t_i = 4$  (edge-disjoint) triangles:  $\{T_{i,j_i}\}$  with  $i = 1, 2$  and  $1 \leq j_i \leq 4$ . We now

obtain a new graph  $G''$  from  $G'$  by applying Operation 2 at vertex  $u_4$  (or  $u_5$ ) and  $v_4$  (or  $v_5$ ) such that  $G''_i$  contains no triangle-cycle-structure. Clearly, during the above procedure,  $G''$  is connected.

Let  $op(G')$  be the minimum times of doing Operation 2 needed during the above procedure. Clearly,  $op(G') = op(G[E(\mathcal{T})])$  (minimum times of applying Operation 2 needed for  $G[E(\mathcal{T})]$  such that the resulting graph is connected and contains no triangle-cycle-structure). For example, in the above example,  $op(G') = op(G[E(\mathcal{T})]) = 2$ . As the number of inner vertices increases 1 after we applied Operation 2 once, there are  $op(G')$  new inner vertices totally produced, then by Observation 2.4 and the discussion of Theorem 3.1(before the Claim), we have  $rc(L(G)) \leq rc(L(G')) \leq rc(L(G'')) \leq \sum_{i=1}^c rc(L(G''_i)) + (n_2 + op(G') - 2t - c) \leq \sum_{i=1}^c (t_i + 1) + (n_2 + op(G[E(\mathcal{T})]) - 2t - c) = n_2 + op(G[E(\mathcal{T})]) - t$ , that is, our following claim holds:

**Claim.** For any set  $\mathcal{T}$  of  $t$  edge-disjoint triangles of a connected graph  $G$  with  $n_2$  inner vertices, we have  $rc(L(G)) \leq n_2 + op(G[E(\mathcal{T})]) - t$ .

Recall that a triangle-forest-structure of size  $\ell$  contains  $2\ell + c$  (inner) vertices where  $c$  is the number of components of it. Operation 2 does not change the number of edge-disjoint triangles, but the number of inner vertices increases 1 after we applied Operation 2 once. Then it is easy to show that after applying Operation 2  $op(G[E(\mathcal{T})])$  times, the number of inner vertices of the new graph  $G''$  is  $op(G[E(\mathcal{T})]) + n_2 = 2t + c + n'_2$  where  $n'_2$  denotes the number of inner vertices not covered by the original  $t$  edge-disjoint triangles. So, by the above claim, we have  $rc(L(G)) \leq n_2 + op(G[E(\mathcal{T})]) - t = 2t + c + n'_2 - t = n'_2 + t + c$ . For the sharpness of the bound, we can see Example 3.5. ■

**Example 3.5** Let  $G$  be a graph shown in Figure 3.8. The set  $\mathcal{T} = \{u_i, v_i, u_{i+1}\}_{i=1}^{k-1}$  is a set of  $k - 1$  edge-disjoint triangles,  $n'_2 = 1$  and  $c = 1$ . By Theorem 3.4, we have  $rc(L(G)) \leq k + 1$ ; on the other hand, it is easy to show that the diameter of  $L(G)$  is  $k + 1$ , and so  $rc(L(G)) = k + 1$ . ■

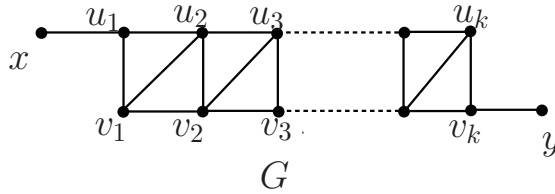


Figure 3.8 Figure of Example 3.5.

## 4 Bounds for iterated line graphs

Recall that the *iterated line graph* of a graph  $G$ , denoted by  $L^2(G)$ , is the line graph of the graph  $L(G)$ . The following corollary deduced from Theorem 3.4 is an upper bound of the rainbow connection numbers of iterated line graphs of connected cubic graphs.

**Corollary 4.1** *If  $G$  is a connected cubic graph with  $n$  vertices, then*

$$rc(L^2(G)) \leq n + 1.$$

*Proof.* Since  $G$  is a connected cubic graph, each vertex is an inner vertex and the clique  $\langle S(v) \rangle$  in  $L(G)$  corresponding to each vertex  $v$  is a triangle. We know that  $\mathcal{K} = \{\langle S(v) \rangle : v \in \overline{V_2}\} = \{\langle S(v) \rangle : v \in V\}$  is a clique decomposition of  $L(G)$ . Let  $\mathcal{T} = \mathcal{K}$ . Then  $\mathcal{T}$  is a set of  $n$  edge-disjoint triangles that cover all vertices of  $L(G)$  and  $L(G) = L(G)[E(\mathcal{T})]$ . So  $n'_2 = 0$  and  $c = 1$ , by Theorem 3.4, the conclusion holds.  $\blacksquare$

In a graph  $G$ , we call a path of length  $k$  a *pendant  $k$ -length path* if one of its end vertex has degree 1 and all inner vertices has degree 2. By definition, a pendant  $k$ -length path contains a pendant  $\ell$ -length path ( $1 \leq \ell \leq k$ ). A pendant 1-length path is a *pendant edge*.

**Theorem 4.2** *Let  $G$  be a connected graph with  $m$  edges and  $m_1$  pendant 2-length paths. Then  $rc(L^2(G)) \leq m - m_1$ , the equality holds if and only if  $G$  is a path of length at least 3.*

*Proof.* Now  $L(G)$  is a graph with  $m$  vertices and  $m_1$  pendant edges. Then it has  $m - m_1$  inner vertices. By the discussion before Theorem 3.1, we give each clique  $\langle S(v) \rangle$  in  $L^2(G)$  a fresh color, where  $v$  is an inner vertex of  $L(G)$ . It is easy to show that this gives a rainbow  $(m - m_1)$ -edge-coloring of  $L^2(G)$ , and so  $rc(L^2(G)) \leq m - m_1$ .

If  $G$  is a path of length at least 3, then the equality clearly holds.

If  $G$  contains a cycle, then  $L(G)$  clearly contains a minimal cycle  $C : v_1, \dots, v_\ell$ . So  $L^2(G)$  contains a clique-cycle-structure of size  $\ell$  which is formed by cliques  $\{\langle S(v_i) \rangle\}_{i=1}^\ell$ . By a result in [7], we know that this structure needs at most  $\lceil \frac{\ell+1}{2} \rceil$  colors to make sure that it is rainbow connected. The rest part of  $L^2(G)$  is formed by cliques  $\{\langle S(v) \rangle : v \in V^2 \setminus \{v_i\}_{i=1}^\ell\}$  where  $V^2$  is the set of inner vertices of  $L(G)$ . We give each such  $\langle S(v) \rangle$  a fresh color, by Observation 2.1,  $rc(L^2(G)) \leq \lceil \frac{\ell+1}{2} \rceil + (m - m_1 - \ell) < m - m_1$ , a contradiction.

If  $G$  is a tree with a vertex of degree at least 3, then  $L(G)$  contains a cycle, with a similar argument, we can derive that  $rc(L^2(G)) < m - m_1$ . This also produces a contradiction

So  $G$  is a tree with maximum degree 2, and hence it must be a path of length at least 3. ■

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