A $2n$-point Interpolation Formula with Its Applications to q-Identities

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Abstract

Based on Krattenthaler's determinantal formula and divided difference operators, we give a $2n$-point interpolation formula for a polynomial of degree $\leq n$ in one variable. Several applications of this formula, such as q-identities related to divisor functions, finite forms of the quintuple product identity and a bibasic hypergeometric identity, are discussed. We also give an expansion formula for $\prod_{i=1}^{n}(y-uq^{i-1})$ by using the supersymmetric complete functions and determinant evaluation.

Keywords: Krattenthaler's determinantal formula; divided difference operators; $2n$-point interpolation formula; supersymmetric complete functions; determinant evaluation; q-identities.

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1 Introduction

Let $A = \{a_1, a_2, \ldots\}$ be a set of indeterminates. The n-point Newton interpolation formula for a function $f(x)$ gives the unique polynomial $P(x)$ of degree $n-1$:

$$P(x) = f(x) - f(a_1)\partial_1\partial_2\cdots\partial_n \prod_{i=1}^{n}(x-a_i) = f(a_1) + \sum_{i=1}^{n-1} f(a_1)\partial_1\partial_2\cdots\partial_i \prod_{j=1}^{i}(x-a_j), \quad (1.1)$$

where we take $a_{n+1} = x$ and the divided difference ∂_i $(i \geq 1)$, acting on its left, is defined by

$$f(a_1,\ldots,a_i,a_{i+1},\ldots)\partial_i = \frac{f(a_1,\ldots,a_i,a_{i+1},\ldots) - f(a_1,\ldots,a_{i+1},a_i,\ldots)}{a_i - a_{i+1}}.$$

Unlike Newton's formula (1.1), Lagrange's interpolation formula does not need the knowledge of the difference of a function:

$$P(x) = \sum_{i=1}^{n} f(a_i)\frac{\prod_{j\neq i}(x-a_j)}{\prod_{j\neq i}(a_i-a_j)}. \quad (1.2)$$

Based on the following determinantal formula due to Krattenthaler [7], we shall introduce a $2n$-point interpolation formula which gives a unique polynomial of degree $\leq n$. Given three sets of variables $X = \{x_1, \ldots, x_{n+1}\}$, $A = \{a_1, \ldots, a_n\}$, and $B = \{b_1, \ldots, b_n\}$, Krattenthaler has shown that

$$\det\big((x_i-a_j)\cdots(x_i-a_n)(x_i-b_1)\cdots(x_i-b_{j-1})\big)_{i,j=1}^{n+1} = \prod_{1\leq i<j\leq n+1}(x_i-x_j)\prod_{1\leq i\leq j\leq n}(a_j-b_i). \quad (1.3)$$

Taking $x_{n+1} = y$ and using Laplace's expansion to expand the determinant along the last row, (1.3) can be reduced to the following formula :

$$f(y) = \sum_{k=0}^{n} c_k (y - a_{k+1}) \cdots (y - a_n)(y - b_1) \cdots (y - b_k), \tag{1.4}$$

where $f(y) = (y - x_1) \cdots (y - x_n)$ and c_k is a quotient whose numerator is the minor of the above determinant with respect to the entry $(y - a_{k+1}) \cdots (y - a_n)(y - b_1) \cdots (y - b_k)$ and denominator is the product $(-1)^k \prod_{1 \le i \le j \le n} (a_j - b_i) \prod_{1 \le i < j \le n} (x_i - x_j)$.

Applying the techniques of the divided differences to determine the coefficients c_k, $0 \le k \le n$, we obtain the main result of this paper and will prove it in the next section.

Theorem 1.1 *Suppose $f(y)$ is a polynomial in y with degree $\le n$. Given two sets of points $A = \{a_1, \ldots, a_n\}$ and $B = \{b_1, \ldots, b_n\}$ with $a_i \ne b_j$, $1 \le i, j \le n$, we have the following $2n$-point interpolation formula:*

$$\begin{aligned} f(y) =& f(b_1) \prod_{i=1}^{n} \frac{(y - a_i)}{(b_1 - a_i)} + f(a_1) \prod_{i=1}^{n} \frac{(y - b_i)}{(a_1 - b_i)} \\ &+ \sum_{k=1}^{n-1} \frac{f(b_1)}{\prod_{i=1}^{n-k+1} (b_1 - a_i)} \partial_1 \cdots \partial_k (b_{k+1} - a_{n-k+1}) \prod_{i=1}^{n-k} (y - a_i) \prod_{i=1}^{k} (y - b_i). \end{aligned} \tag{1.5}$$

Theorem 1.1 can be regarded as a terminating case of the Newton type rational interpolation formula for a formal power series $f(y)$ given in [4] :

$$f(y) = f(b_1) + \sum_{n=1}^{\infty} f(b_1) \prod_{i=1}^{n-1} (1 - b_1 c_i) \partial_1 \cdots \partial_n (1 - b_{n+1} c_n) \prod_{i=1}^{n} \frac{y - b_i}{1 - y c_i},$$

where $b_1, b_2, \ldots, c_1, c_2, \ldots$ are complex numbers and the series is convergent when

$$|y| < 1, \quad \lim_{n \to \infty} b_1 \cdots b_n = 0, \quad \text{and} \lim_{n \to \infty} y^n b_1 \cdots b_n = 0.$$

Consider the case $f(y) = 1$ in Theorem 1.1. Comparing with the reminders of Newton's and Lagrange's interpolation formulas for the function $1/(x - y)$, we are led to the following identity.

Theorem 1.2 *We have*

$$\begin{aligned} &\frac{1}{x - b_1} \prod_{i=1}^{n} \frac{(y - a_i)}{(b_1 - a_i)} + \frac{1}{x - a_1} \left(\prod_{i=1}^{n} \frac{(y - b_i)}{(a_1 - b_i)} - \prod_{i=1}^{n} \frac{(y - b_i)}{(x - b_i)} \right) \\ &+ \sum_{k=1}^{n-1} \frac{1/(x - b_1)}{\prod_{i=1}^{n-k+1} (b_1 - a_i)} \partial_1 \cdots \partial_k (b_{k+1} - a_{n-k+1}) \prod_{i=1}^{n-k} (y - a_i) \prod_{i=1}^{k} (y - b_i) \\ &= \frac{1}{x - b_1} + \sum_{k=1}^{n-1} \frac{1}{x - b_1} \prod_{j=1}^{k} \frac{(y - b_j)}{(x - b_{j+1})} = \sum_{k=1}^{n} \frac{1}{x - b_k} \frac{\prod_{j \ne k} (y - b_j)}{\prod_{j \ne k} (b_k - b_j)}. \end{aligned}$$

The last equality has already appeared in [3] in the proofs of serval q-identities related to divisor functions. A short proof of Theorem 1.2 and some applications of Theorem 1.1, such as q-identities related to divisor functions, finite forms of the quintuple product identity and a bibasic hypergeometric identity, will be discussed in Section 3.

Theorem 1.3 *We have*

$$\prod_{i=1}^{n}(y-uq^{i-1})=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}\prod_{i=1}^{k}\frac{(uq^{n-i}-vq^{1-k})}{(wq^{i}-vq^{1-k})}\prod_{i=1}^{n-k}\frac{(wq-uq^{i-1})}{(wq^{k+1}-vq^{i-n})}\prod_{i=1}^{k}(y-wq^{i})\prod_{i=1}^{n-k}(y-vq^{i-n}), \tag{1.6}$$

where $\begin{bmatrix}n\\k\end{bmatrix}$ *is the* q*-Gauss coefficient defined by*

$$\begin{bmatrix}n\\k\end{bmatrix}=\frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}},$$

where $(a;q)_n$ *is the* q*-shifted factorial defined by*

$$(a;q)_n=(1-a)(1-aq)\cdots(1-aq^{n-1}),\quad n=0,1,\ldots\infty.$$

Theorem 1.3 can be deduced from Theorem 1.1 through the following specializations:

$$f(y)=\prod_{i=1}^{n}(y-uq^{i-1}),\quad A=\{vq^{1-n},\ldots,vq^{-1},v\},\quad B=\{wq,wq^2,\ldots,wq^n\}.$$

Instead of using Theorem 1.1, we shall give an alternative proof of (1.6) in Section 4 by evaluating the numerator of c_k, $0\le k\le n$.

Note that Theorem 1.3 can be considered as a variation of the terminating ${}_6\phi_5$ summation formula [5]. On the other hand, one can view the terminating ${}_6\phi_5$ summation formula as an interpolation formula. Writing wv^{-1} as a, wy^{-1} as bq^{-1} and uv^{-1} as cq^{1-n}, we find

$$\frac{(aq;q)_n(aq/bc;q)_n}{(aq/b;q)_n(aq/c;q)_n}=\sum_{k=0}^{n}\frac{(1-aq^{2k})(a;q)_k(b;q)_k(c;q)_k(q^{-n};q)_k}{(1-a)(q;q)_k(aq/b;q)_k(aq/c;q)_k(aq^{n+1};q)_k}\left(\frac{aq^{n+1}}{bc}\right)^k.$$

2 Proof of Theorem 1.1

In order to prove Theorem 1.1, it suffices to verify the following lemma. The main technique we use in our proof is the following Leibnitz formula [8]:

$$f(x_1)g(x_1)\partial_1\partial_2\cdots\partial_n=\sum_{k=0}^{n}\big(f(x_1)\partial_1\ldots\partial_k\big)\big(g(x_{k+1})\partial_{k+1}\ldots\partial_n\big). \tag{2.1}$$

Lemma 2.1 *We have*

$$c_k=\begin{cases}\frac{f(b_1)}{\prod_{i=1}^{n}(b_1-a_i)}, & \text{if } k=0,\\ \frac{f(b_1)}{\prod_{i=k}^{n}(b_1-a_i)}\partial_1\cdots\partial_k(b_{k+1}-a_k), & \text{if } 1\le k\le n-1,\\ \frac{f(a_n)}{\prod_{i=1}^{n}(a_n-b_i)}, & \text{if } k=n,\end{cases} \tag{2.2}$$

where c_k *and* $f(y)$ *are given as in* (1.4).

Proof. According to (1.3), we have

$$\begin{aligned}
&\det\big((x_i-a_{j+1})\cdots(x_i-a_n)(x_i-b_1)\cdots(x_i-b_j)\big)_{i,j=1}^n\\
&\quad=\prod_{i=1}^n(x_i-b_1)\det\big((x_i-a_{j+1})\cdots(x_i-a_n)(x_i-b_2)\cdots(x_i-b_j)\big)_{i,j=1}^n\\
&\quad=(-1)^n f(b_1)\prod_{1\le i<j\le n}(x_i-x_j)\prod_{2\le i\le j\le n}(a_j-b_i).
\end{aligned}$$

Therefore,

$$c_0=\frac{f(b_1)}{\prod_{i=1}^n(b_1-a_i)}.$$

Similarly, one has

$$c_n=\frac{f(a_n)}{\prod_{i=1}^n(a_n-b_i)}.$$

Specializing y to b_2 in (1.4) to get

$$f(b_2)=\frac{f(b_1)\prod_{i=1}^n(b_2-a_i)}{\prod_{i=1}^n(b_1-a_i)}+c_1(b_2-b_1)\prod_{i=2}^n(b_2-a_i),$$

which implies

$$\begin{aligned}
c_1&=\left(f(b_1)/\prod_{i=1}^n(b_1-a_i)-f(b_2)/\prod_{i=1}^n(b_2-a_i)\right)(b_2-a_1)/(b_1-b_2)\\
&=\frac{f(b_1)}{\prod_{i=1}^n(b_1-a_i)}\partial_1(b_2-a_1).
\end{aligned}$$

Let $g(y)=f(y)/(y-a_1)\cdots(y-a_n)$. Rewrite (1.4) as

$$g(y)=g(b_1)+\sum_{k=1}^{n-1}c_k\frac{(y-b_1)\cdots(y-b_k)}{(y-a_1)\cdots(y-a_k)}+\frac{f(a_n)}{\prod_{i=1}^n(a_n-b_i)}\prod_{i=1}^n\frac{y-b_i}{y-a_i}. \tag{2.3}$$

Multiplying both sides by $(y-a_1)\cdots(y-a_{i-1})$, then applying the operator $\partial_1\cdots\partial_i$, we have

$$\begin{aligned}
&g(y_1)(y_1-a_1)\cdots(y_1-a_{i-1})\partial_1\cdots\partial_i\big|_{y_j=b_j,1\le j\le i+1}\\
&\quad=\sum_{k=1}^{i-1}c_k(y_1-a_{k+1})\cdots(y_1-a_{i-1})(y_1-b_1)\cdots(y_1-b_k)\partial_1\cdots\partial_i\big|_{y_j=b_j,1\le j\le i+1}\\
&\qquad+\sum_{k=i}^{n-1}c_k\frac{(y_1-b_1)\cdots(y_1-b_k)}{(y_1-a_i)\cdots(y_1-a_k)}\partial_1\cdots\partial_i\big|_{y_j=b_j,1\le j\le i+1}.
\end{aligned}\tag{2.4}$$

Since $\partial_1\cdots\partial_i$ decreases degree by i and $(y_1-a_{k+1})\cdots(y_1-a_{i-1})(y_1-b_1)\cdots(y_1-b_k)$ is a polynomial of degree $i-1$, so the first sum on the right side vanishes.

Consider the case $i \le k$. By the Leibnitz type formula (2.1), we find

$$
\begin{aligned}
&\frac{(y_1-b_1)\cdots(y_1-b_k)}{(y_1-a_i)\cdots(y_1-a_k)}\partial_1\cdots\partial_i\Big|_{y_j=b_j,1\le j\le i+1} \\
&\quad= \left((y_1-b_1)\frac{(y_1-b_2)\cdots(y_1-b_k)}{(y_1-a_i)\cdots(y_1-a_k)}\partial_1\cdots\partial_i + \frac{(y_2-b_2)\cdots(y_2-b_k)}{(y_2-a_i)\cdots(y_2-a_k)}\partial_2\cdots\partial_i\right)\Big|_{\substack{y_j=b_j\\1\le j\le i+1}} \\
&\quad= \frac{(y_2-b_2)\cdots(y_2-b_k)}{(y_2-a_i)\cdots(y_2-a_k)}\partial_2\cdots\partial_i\Big|_{\substack{y_j=b_j\\2\le j\le i+1}} \\
&\quad= \cdots = \frac{(y_i-b_i)\cdots(y_i-b_k)}{(y_i-a_i)\cdots(y_i-a_k)}\partial_i\Big|_{\substack{y_i=b_i\\y_{i+1}=b_{i+1}}} = \begin{cases} 0, & i<k, \\ 1/(b_{k+1}-a_k), & i=k. \end{cases}
\end{aligned}
$$

Now (2.4) becomes

$$
g(b_1)(b_1-a_1)\cdots(b_1-a_{k-1})\partial_1\cdots\partial_k = \frac{f(b_1)}{(b_1-a_k)\cdots(b_1-a_n)}\partial_1\cdots\partial_k = \frac{c_k}{(b_{k+1}-a_k)},
$$

as desired. ∎

It is easy to see that Theorem 1.1 can be deduced from (1.4) and Lemma 2.1 by replacing a_i by a_{n-i+1} for $1 \le i \le n$.

There is an obvious symmetry between a_i's and b_i's in Theorem 1.1 :

$$
\begin{aligned}
&\sum_{k=1}^{n-1} \frac{f(b_1)}{\prod_{i=1}^{n-k+1}(b_1-a_i)}\partial_1\cdots\partial_k(b_{k+1}-a_{n-k+1})\prod_{i=1}^{n-k}(y-a_i)\prod_{i=1}^{k}(y-b_i) \\
&\qquad\qquad = \sum_{k=1}^{n-1} \frac{f(a_1)}{\prod_{i=1}^{n-k+1}(a_1-b_i)}\partial_1\cdots\partial_k(a_{k+1}-b_{n-k+1})\prod_{i=1}^{n-k}(y-b_i)\prod_{i=1}^{k}(y-a_i),
\end{aligned}
$$

which implies the following identity.

Corollary 2.2 *For $1 \le k \le n-1$, we have*

$$
\frac{f(b_1)}{\prod_{i=1}^{n-k+1}(b_1-a_i)}\partial_1\cdots\partial_k = -\frac{f(a_1)}{\prod_{i=1}^{k+1}(a_1-b_i)}\partial_1\cdots\partial_{n-k}, \tag{2.5}
$$

where f is a polynomial with degree $\le n$.

3 Interpolation formulas for $f(y) = 1$

As an immediate consequence of Theorem 1.1, we have

$$
\begin{aligned}
1 = &\prod_{i=1}^{n}\frac{y-a_i}{b_1-a_i} + \prod_{i=1}^{n}\frac{y-b_i}{a_1-b_i} \\
&+ \sum_{k=1}^{n-1}\frac{1}{\prod_{i=1}^{n-k+1}(b_1-a_i)}\partial_1\cdots\partial_k(b_{k+1}-a_{n-k+1})\prod_{i=1}^{n-k}(y-a_i)\prod_{j=1}^{k}(y-b_j),
\end{aligned} \tag{3.1}
$$

which is a $2n$-point interpolation formula for $f(y) = 1$. In this section, we shall apply (3.1) to derive several q-identities and a bibasic hypergeometric identity.

3.1 q-identities related to divisor functions

Multiply both sides of (3.1) by $1/(y-a_1)$ and then set $a_1 = x, b_n = y$. Since

$$\frac{1}{y-b_1}\partial_1\cdots\partial_{n-1} = \frac{1}{(y-b_1)\cdots(y-b_n)},$$

the last term of the summation in (3.1) becomes

$$\begin{aligned}
&\frac{1}{(b_1-a_2)(b_1-x)}\partial_1\cdots\partial_{n-1}(y-a_2)\prod_{i=1}^{n-1}(y-b_i)\\
&= \left(\frac{1}{(x-a_2)(a_2-b_1)}+\frac{1}{(a_2-x)(x-b_1)}\right)\partial_1\cdots\partial_{n-1}(y-a_2)\prod_{i=1}^{n-1}(y-b_i)\\
&= \frac{1}{a_2-x}\left(\prod_{i=1}^{n-1}\frac{(y-b_i)}{(a_2-b_i)}-\prod_{i=1}^{n-1}\frac{(y-b_i)}{(x-b_i)}\right)+\frac{1}{y-x}\prod_{i=1}^{n-1}\frac{(y-b_i)}{(x-b_i)}.
\end{aligned}$$

For $1\le i\le n-1$, replacing a_{i+1} by a_i, and multiplying both sides by -1, we find

$$\begin{aligned}
&\frac{1}{x-y}\left(1-\prod_{i=1}^{n-1}\frac{(y-b_i)}{(x-b_i)}\right)\\
&\quad= \frac{1}{x-b_1}\prod_{i=1}^{n-1}\frac{(y-a_i)}{(b_1-a_i)}+\frac{1}{x-a_1}\left(\prod_{i=1}^{n-1}\frac{(y-b_i)}{(a_1-b_i)}-\prod_{i=1}^{n-1}\frac{(y-b_i)}{(x-b_i)}\right)\\
&\quad+\sum_{k=1}^{n-2}\frac{1/(x-b_1)}{\prod_{i=1}^{n-k}(b_1-a_i)}\partial_1\cdots\partial_k(b_{k+1}-a_{n-k})\prod_{j=1}^{n-k-1}(y-a_i)\prod_{i=1}^{k}(y-b_i).
\end{aligned}$$

Since the n-point Newton's formula and the n-point Lagrange's formula for $1/(x-y)$ have the same reminder $\prod_{i=1}^n(y-b_i)\big((x-y)\prod_{i=1}^n(x-b_i)\big)^{-1}$, Theorem 1.2 holds.

Let $A=\{q^{M-1},\ldots,q^{M-n}\}$, $B=\{q^{-1},q^{-2},\ldots,q^{-n}\}$, $x=q^M$, $y=1$. Theorem 1.2 implies

$$\sum_{k=0}^{n}\frac{\left[{M-1\atop k}\right]\left[{n+1\atop n-k}\right]}{\left[{M+n+1-k\atop k+1}\right]\left[{M+n-1-2k\atop n-k}\right]}\frac{q^{k+1}}{1-q^{k+1}}=\sum_{k=1}^{n+1}\frac{q^k}{1-q^k}\Big/\begin{bmatrix}k+M\\k\end{bmatrix}=\sum_{k=1}^{n+1}\begin{bmatrix}n+1\\k\end{bmatrix}\frac{(-1)^{k-1}q^{\binom{k+1}{2}}}{1-q^{M+k}}.$$

Note that the second equality is given by Uchimura [9], see also [3].

3.2 Finite forms of the quintuple product identity

By setting $a_i = vq^{i-n}$ and $b_i = wq^i$ for $1\le i\le n$ in (3.1), one can verify that

$$\frac{1}{\prod_{i=1}^{n-k+1}(b_1-vq^{i-n})}\partial_1\cdots\partial_k=\begin{bmatrix}n\\k\end{bmatrix}\frac{1}{\prod_{j=1}^{n-k+1}(wq^{k+1}-vq^{j-n})\prod_{j=1}^{k}(vq^{1-k}-wq^j)}.$$

Substituting the above relation into (3.1), we obtain

$$1=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}\frac{\prod_{j=1}^{n-k}(y-vq^{j-n})\prod_{j=1}^{k}(y-wq^j)}{\prod_{j=1}^{n-k}(wq^{k+1}-vq^{j-n})\prod_{j=1}^{k}(vq^{1-k}-wq^j)}. \tag{3.2}$$

Setting $v = w^{-1}$ and $y = q$ in (3.2), we reach the following q-identity [2] :

$$1 = \sum_{k=0}^{n} (1 + wq^k) \begin{bmatrix} n \\ k \end{bmatrix} \frac{(w;q)_{n+1}}{(w^2q^k;q)_{n+1}} w^k q^{k^2}.$$

Note that above identity is a finite form of Watson's quintuple product identity [5]:

$$\sum_{k=-\infty}^{\infty} (1 - wq^k) w^{3k} q^{\frac{3k^2-k}{2}} = (q;q)_\infty (w;q)_\infty (q/w;q)_\infty (w^2q;q^2)_\infty (q/w^2;q^2)_\infty.$$

Consider the case $v = w^{-1}q^{-1}$ and $y = 1$. Then (3.2) implies another finite form of the quintuple product identity [6]:

$$1 = \sum_{k=0}^{n} (1 - w^2 q^{2k+1}) \begin{bmatrix} n \\ k \end{bmatrix} \frac{(wq;q)_n}{(w^2q^{k+1};q)_{n+1}} w^k q^{k^2}.$$

3.3 A bibasic hypergeometric identity

Let $f(y) = 1$ and $a_i = p^i$, $b_i = q^{-i}$ for $1 \le i \le n$. Then (2.5) implies the following p,q-identity:

$$\sum_{j=0}^{n-k} \frac{(-1)^j p^{\binom{j+1}{2} - (n-k)(j+1)}}{(p;p)_j (p;p)_{n-k-j} (qp^{j+1};q)_{k+1}} = \sum_{j=0}^{k} \frac{(-1)^j q^{\binom{j+1}{2} + (j+1)(n-k)}}{(q;q)_j (q;q)_{k-j} (pq^{j+1};p)_{n-k+1}}. \tag{3.3}$$

Proof. By the partial fraction expansion, we find

$$\frac{q^{\binom{n}{2}}}{\prod_{i=1}^{n}(z - q^i)} = \sum_{j=0}^{n-1} \frac{(-1)^j q^{\binom{n-1-j}{2}}}{(q;q)_j (q;q)_{n-1-j} (z - q^{j+1})}.$$

Thus,

$$\begin{aligned} \frac{1}{(b_1 - p)\cdots(b_1 - p^{n-k+1})} \partial_1 \cdots \partial_k &= \sum_{j=0}^{n-k} \frac{(-1)^j p^{\binom{n-k-j}{2} - \binom{n-k+1}{2}}}{(p;p)_j (p;p)_{n-k-j} (b_1 - p^{j+1})} \partial_1 \cdots \partial_k \\ &= \sum_{j=0}^{n-k} \frac{(-1)^{k+j} p^{\binom{j+1}{2} - (n-k)(j+1)} q^{\binom{k+2}{2}}}{(p;p)_j (p;p)_{n-k-j} (qp^{j+1};q)_{k+1}}. \end{aligned}$$

In the same manner, we deduce that

$$\frac{1}{(a_1 - q^{-1})\cdots(a_1 - q^{-k-1})} \partial_1 \cdots \partial_{n-k} = \sum_{j=0}^{k} \frac{(-1)^{k+j+1} q^{\binom{k+2}{2} + \binom{j+1}{2} + (j+1)(n-k)}}{(q;q)_j (q;q)_{k-j} (pq^{j+1};p)_{n-k+1}}.$$

Now (3.3) immediately follows from (2.5). This completes the proof. ∎

4 An expansion of the q-shifted factorials

In this section, we shall give a different proof of Theorem 1.3 by computing certain minors of Krattenthaler's determinant. Expanding the determinant in (1.3) with respect to the last row, we obtain

$$\prod_{1\le i<j\le n}(x_i-x_j)\prod_{i=1}^{n}(x_i-y)\prod_{1\le i\le j\le n}(a_j-b_i)$$
$$=\sum_{k=1}^{n+1}(-1)^{n+k+1}C_{n,k}(X,A,B)(y-a_k)\cdots(y-a_n)(y-b_1)\cdots(y-b_{k-1}), \quad (4.1)$$

where $C_{n,k}(X,A,B)$ denotes the minor of the determinant with respect to the entry $(y-a_k)\cdots(y-a_n)(y-b_1)\cdots(y-b_{k-1})$.

To present our proof, let us give a quick review of some basic properties of supersymmetric complete functions. Given two sets of indeterminates $X=\{x_1,x_2,\ldots\}$ and $Y=\{y_1,y_2,\ldots\}$, the supersymmetric complete function $h_n(X-Y)$ is defined by

$$h_n(X-Y)=[t^n]\frac{\prod_{y\in Y}(1-yt)}{\prod_{x\in X}(1-xt)}=\sum_{k=0}^{n}(-1)^k e_k(Y)h_{n-k}(X),$$

where $[t^n]f(t)$ stands for the coefficient of t^n in $f(t)$, $e_k(X)$ denotes the k-th elementary symmetric function and $h_k(X)$ denotes the k-th complete symmetric function in X. Clearly, $h_0(X-Y)=1$.

Lemma 4.1 *[8] Let $\{j_1,j_2,\ldots,j_n\}$ be a sequence of integers, and let X_1, …, X_n and Y_1, …, Y_n be sets of indeterminates. The following relation holds*

$$\det\left(h_{j_k+k-l}(X_k-Y_k)\right)_{k,l=1}^{n}=\det\left(h_{j_k+k-l}(X_k-Y_k-D_{k-1})\right)_{k,l=1}^{n},$$

where D_0, D_1, …, D_{n-1} are sets of indeterminates such that the cardinality of D_i is equal to or less than i.

Lemma 4.2 *For $1\le k\le n+1$, set*

$$Y_j=\begin{cases}\{a_j,\ldots,a_n,b_1,\ldots,b_{j-1}\}, & 1\le j<k;\\ \{a_{j+1},\ldots,a_n,b_1,\ldots,b_j\}, & k\le j\le n,\\ \{x_1,x_2,\ldots,x_n\} & j=n+1.\end{cases}$$

Then we have

$$C_{n,k}(X,A,B)=\prod_{1\le i<j\le n}(x_i-x_j)\det(h_i(X-Y_j))_{i,j=1}^{n} \quad (4.2)$$
$$=(-1)^{\binom{n}{2}}\prod_{1\le i<j\le n}(x_i-x_j)\det(e_{i-1}(Y_j))_{i,j=1}^{n+1}. \quad (4.3)$$

Proof. The identity (4.2) easily follows from the definition of the supersymmetric complete function and Lemma 4.1. Since

$$h_0(X-X)=1,\quad h_n(X-X)=\sum_{k=0}^{n}(-1)^k e_k(X)h_{n-k}(X)=0,\quad n\ne 0,$$

we deduce that

$$\begin{aligned}\det(h_i(X-Y_j))_{i,j=1}^n &= (-1)^n \det(h_{i-1}(X-Y_j))_{i,j=1}^{n+1} \\ &= (-1)^n \det(h_{i-j}(X))_{i,j=1}^{n+1} \det((-1)^{i-1}e_{i-1}(Y_j))_{i,j=1}^{n+1} \\ &= (-1)^{\binom{n}{2}} \det(e_{i-1}(Y_j))_{i,j=1}^{n+1}.\end{aligned}$$

This completes the proof. ∎

Setting $X=\{u,uq,\ldots,uq^{n-1},y\}$, $A=\{v,vq^{-1},\ldots,vq^{1-n}\}$ and $B=\{wq,wq^2,\ldots,wq^n\}$ in (4.1), we find

$$\prod_{i=1}^{n}(y-uq^{i-1}) \prod_{1\le i\le j\le n}(vq^{1-j}-wq^i) \\ = \sum_{k=1}^{n+1}(-1)^{k+1}D_{n,k}(X,A,B)\prod_{i=k}^{n}(y-vq^{1-i})\prod_{i=1}^{k-1}(y-wq^i),$$

where

$$D_{n,k}(X,A,B)=C_{n,k}(X,A,B)/\prod_{1\le i<j\le n}(uq^{i-1}-uq^{j-1}).$$

To prove (1.6), it suffices to establish the following theorem.

Theorem 4.3 *For $1\le k\le n+1$, we have*

$$D_{n,k}(X,A,B)=(-1)^{n+1-k}\begin{bmatrix} n \\ k-1\end{bmatrix} \\ \prod_{i=0}^{n-k}(wq-uq^i)\prod_{i=1}^{k-1}(uq^{n-i}-vq^{2-k})\prod_{\substack{i=1\\ i\ne k}}^{n}\prod_{\substack{j=i\\ j\ne k-1}}^{n}(vq^{1-j}-wq^i). \quad (4.4)$$

It is convenient for us to present the proof of Theorem 4.3 via two steps. In the first step (Lemma 4.4) we shall evaluate a special case of $D_{n,k}(X,A,B)$, where $u=wq^2$. The second step (Lemma 4.5) is to show that $D_{n,k}(X,A,B)$ is a polynomial in u with n roots $wq,w,\ldots,wq^{1+k-n},vq^{1-n},\ldots,vq^{3-k-n}$.

Lemma 4.4 *Let $A'=\{v,vq^{-1},\ldots,vq^{1-n}\}$ and $B'=\{wq,wq^2,wq^3,\ldots,wq^{n+1}\}$. For $1\le k\le n$, we have*

$$D'_{n,k}(A',B')=(-1)^{\binom{n+1}{2}}w^{n-k+1}q^{n-k+1}(q;q)_{n-k+1} \\ \begin{bmatrix} n \\ k-1\end{bmatrix}\prod_{\substack{i=1\\ i\ne k}}^{n}\prod_{\substack{j=i\\ j\ne k-1}}^{n}(vq^{1-j}-wq^i)\prod_{i=1}^{k-1}(vq^{2-k}-wq^{n-i+2}). \quad (4.5)$$

where

$$D'_{n,k}(A',B')=(-1)^{\binom{n}{2}}D_{n,k}(X,A,B)\Big|_{u=wq^2}.$$

Proof. From Lemma 4.2 it follows that

$$D'_{n,k}(A',B') = \det(e_{i-1}(Y_j))_{i,j=1}^{n+1}, \tag{4.6}$$

where

$$Y_j = \begin{cases} \{vq^{1-j}, \dots, vq^{1-n}, wq, \dots, wq^{j-1}\}, & 1 \le j < k; \\ \{vq^{-j}, \dots, vq^{1-n}, wq, \dots, wq^{j}\}, & k \le j \le n, \\ \{wq^2, \dots, wq^{n+1}\} & j = n+1. \end{cases}$$

We shall proceed by induction on n. When $n = 1$ and $k = 1, 2$, we have $D'_{1,1}(\{v\}, \{wq, wq^2\}) = -wq(1-q)$, $D'_{1,2}(\{v\}, \{wq, wq^2\}) = -(v - wq^2)$, which are in accordance with the right hand side of (4.5). We now assume that (4.5) holds for $1 \le n \le m-1$, where $m \ge 2$. Since

$$e_j(Y_k) - e_j(Y_{k+1}) = (vq^{1-k} - wq^k)e_{j-1}(vq^{-k}, \dots, wq^{k-1})$$

and

$$e_j(Y_m) - e_j(Y_{m+1}) = (wq - wq^{m+1})e_{j-1}(wq^2, \dots, wq^m),$$

it is easy to verify that

$$D'_{m,1}(A',B') = (-1)^{\binom{m+1}{2}} w^m q^m (q;q)_m \prod_{\substack{i=1 \\ i \ne 1}}^{m} \prod_{j=i}^{m} (vq^{1-j} - wq^i),$$

which is equal to the right side of (4.5).

We now consider that case $1 \le k \le m-1$. Since

$$\begin{aligned} e_j(Y_k) - e_j(Y_{k+2}) =& (vq^{1-k} - wq^k)e_{j-1}(vq^{-k}, \dots, wq^{k-1}) \\ &+ (vq^{-k} - wq^{k+1})e_{j-1}(vq^{-1-k}, \dots, wq^k), \end{aligned}$$

it follows that

$$\begin{aligned} D'_{m,k+1}(A',B') =& (-1)^m \prod_{i=1}^{m} (vq^{1-i} - wq^i)(wq - wq^{m+1}) \\ & \left(\frac{D'_{m-1,k}(A' \setminus \{v\}, B' \setminus \{wq^{m+1}\})}{vq^{-k+1} - wq^k} + \frac{D'_{m-1,k+1}(A' \setminus \{v\}, B' \setminus \{wq^{m+1}\})}{vq^{-k} - wq^{k+1}} \right), \end{aligned}$$

where $A \setminus B$ denotes the set difference of A and B.

By the inductive hypothesis, we get

$$\begin{aligned} D'_{m-1,k}(A' \setminus \{v\}, B' \setminus \{wq^{m+1}\}) =& (-1)^{\binom{m}{2}} w^{m-k} q^{m-k} (q;q)_{m-k} \\ & \begin{bmatrix} m-1 \\ k-1 \end{bmatrix} \prod_{\substack{i=1 \\ i \ne k}}^{m-1} \prod_{\substack{j=i \\ j \ne k-1}}^{m-1} (vq^{-j} - wq^i) \prod_{i=1}^{k-1} (vq^{1-k} - wq^{m-i+1}) \end{aligned}$$

and

$$\begin{aligned} D'_{m-1,k+1}(A' \setminus \{v\}, B' \setminus \{wq^{m+1}\}) =& (-1)^{\binom{m}{2}} w^{m-k-1} q^{m-k-1} (q;q)_{m-k-1} \\ & \begin{bmatrix} m-1 \\ k \end{bmatrix} \prod_{\substack{i=1 \\ i \ne k+1}}^{m-1} \prod_{\substack{j=i \\ j \ne k}}^{m-1} (vq^{-j} - wq^i) \prod_{i=1}^{k} (vq^{-k} - wq^{m-i+1}). \end{aligned}$$

Therefore,

$$\begin{aligned} D'_{m,k+1}(A',B') &= (-1)^{\binom{m+1}{2}} w^{m-k} q^{m-k} (q;q)_{m-k} \begin{bmatrix} m \\ k \end{bmatrix} \prod_{\substack{i=1 \\ i\neq k+1}}^{m} \prod_{\substack{j=i \\ j\neq k}}^{m} (vq^{1-j} - wq^i) \\ & \prod_{i=1}^{k} (vq^{1-k} - wq^{m-i+2}) \left(\frac{w(1-q^k)}{vq^{-k} - wq^k} + \frac{(v - wq^k)q^{-k}}{vq^{-k} - wq^k} \right), \end{aligned}$$

as desired.

Finally, we are left with the case $k = m+1$. Since, by (1.3),

$$\begin{aligned} & D_{m,m+1}(X,A,B) \prod_{1\leq i<j\leq m} (uq^{i-1} - uq^{j-1}) \\ &= \det \Big((uq^{i-1} - vq^{1-j}) \cdots (uq^{i-1} - wq^{j-1}) \Big)_{i,j=1}^{m} \\ &= \prod_{i=1}^{m} (uq^{i-1} - vq^{1-m}) \prod_{1\leq i<j\leq m} (uq^{i-1} - uq^{j-1}) \prod_{i=1}^{m-1} \prod_{j=i}^{m-1} (vq^{1-j} - wq^i), \end{aligned}$$

we have

$$\begin{aligned} D'_{m,k}(A',B') &= (-1)^{\binom{m}{2}} D_{m,k}(X,A,B) \Big|_{u=wq^2} \\ &= (-1)^{\binom{m}{2}} \prod_{i=1}^{m} (uq^{i-1} - vq^{1-m}) \prod_{i=1}^{m-1} \prod_{j=i}^{m-1} (vq^{1-j} - wq^i) \Big|_{u=wq^2} \\ &= (-1)^{\binom{m+1}{2}} \prod_{i=1}^{m} (vq^{1-m} - wq^{i+1}) \prod_{i=1}^{m-1} \prod_{j=i}^{m-1} (vq^{1-j} - wq^i), \end{aligned}$$

which is equal to the right hand side of (4.5). This completes the proof. ∎

Lemma 4.5 *We have*

$$D_{n,k}(X,A,B) = C \prod_{i=0}^{n-k} (uq^i - wq) \prod_{i=1}^{k-1} (uq^{n-i} - vq^{2-k}),$$

where C is independent of u.

Proof. We view $D_{n,k}(X,A,B)$ as a polynomial in u of degree n with coefficients depending on v and w. The essence of the proof is to show that $wq, w, \ldots, wq^{1+k-n}, vq^{1-n}, \ldots, vq^{3-k-n}$ are the roots of the polynomial.

For $u = wq^{2-i}$, where $1 \leq i \leq n-k+1$, let

$$D_0 = \emptyset, \quad D_1 = \{wq^{2-i}\}, \ldots, D_{i-1} = \{wq^{2-i}, \ldots, w\},$$
$$D_i = \{wq^{2-i}, \ldots, w, wq^{1-i+n}\}, \ldots, D_{n-1} = \{wq^{2-i}, \ldots, w, wq^{1-i+n}, \ldots, wq^2\}.$$

Clearly, $e_k(X) = 0$ if the cardinality of X is less than k. In view of Lemma 4.1, $D_{n,k}(X,A,B)$ can be transformed into a determinant whose (i,j)-th entry is equal to 0 if

$$(i,j) \in \{(i,j)\colon 2 \leq j \leq k-1 \text{ and } n-j+2 \leq i \leq n, \text{ or } k \leq j \leq n \text{ and } n-k+1 \leq i \leq n\}.$$

Thus $D_{n,k}(X,A,B)\big|_{u=wq^{2-i}} = 0$.

Similarly, for $u = vq^{2-n-i}$, $1 \le i \le k-1$, we take

$$D_0 = \emptyset, \quad D_1 = \{vq^{2-n-i}\}, \ldots, D_{i-1} = \{vq^{2-n-i}, \ldots, vq^{-n}\},$$
$$D_i = \{vq^{2-n-i}, \ldots, vq^{-n}, vq^{1-i}\}, D_{n-1} = \{vq^{2-n-i}, \ldots, vq^{2-n}, vq^{1-i}, \ldots, vq^{2-n}\}.$$

Now $D_{n,k}(X,A,B)$ becomes a determinant with the (i,j)-th entry being 0 if

$$(i,j) \in \{(i,j)\colon k \le j \le n-1 \text{ and } j+1 \le i \le n, \text{ or } 1 \le j \le k-1 \text{ and } k-1 \le i \le n\}.$$

So we deduce that $D_{n,k}(X,A,B)\big|_{u=vq^{2-n-i}} = 0$. This completes the proof. ∎

We are now ready to complete the proof of Theorem 4.3.

Proof of Theorem 4.3:

In Lemma 4.5, we have established that

$$D_{n,k}(X,A,B) = C \prod_{i=0}^{n-k} (uq^i - wq) \prod_{i=1}^{k-1} (uq^{n-i} - vq^{2-k}),$$

where C is independent of u.

To determine C, we set $u = wq^2$. Applying Lemma 4.2 and Lemma 4.4, we obtain

$$C \prod_{i=0}^{n-k} (wq^{2+i} - wq) \prod_{i=1}^{k-1} (uq^{n-i} - vq^{2-k}) = (-1)^n w^{n-k+1} q^{n-k+1} (q;q)_{n-k+1}$$
$$\begin{bmatrix} n \\ k-1 \end{bmatrix} \prod_{\substack{i=1 \\ i \ne k}}^{n} \prod_{\substack{j=i \\ j \ne k-1}}^{n} (vq^{1-j} - wq^i) \prod_{i=1}^{k-1} (vq^{2-k} - wq^{n-i+2}).$$

This implies that

$$C = \begin{bmatrix} n \\ k-1 \end{bmatrix} \prod_{\substack{i=1 \\ i \ne k}}^{n} \prod_{\substack{j=i \\ j \ne k-1}}^{n} (vq^{1-j} - wq^i).$$

Therefore,

$$D_{n,k}(X,A,B) = (-1)^{n-k+1} \begin{bmatrix} n \\ k-1 \end{bmatrix}$$
$$\prod_{i=0}^{n-k} (wq - uq^i) \prod_{i=1}^{k-1} (uq^{n-i} - vq^{2-k}) \prod_{\substack{j=1 \\ j \ne k}}^{n} \prod_{\substack{i=j \\ i \ne k-1}}^{n} (vq^{1-i} - wq^j), \quad (4.7)$$

as desired. This completes the proof. ∎

To conclude this paper, we give two special cases of (1.6). Bailey [1] found the following two identities as the q-analogues of Dixon's theorem for the cubic-sums of binomial coefficients :

$$\sum_{k=-n}^{n} (-1)^k \begin{bmatrix} 2n \\ n+k \end{bmatrix}^3 q^{k(3k+1)/2} = \frac{(q;q)_{3n}}{(q;q)_n^3}, \quad (4.8)$$

and

$$\sum_{k=-n-1}^{n}(-1)^k \begin{bmatrix} 2n+1 \\ n+k+1 \end{bmatrix}^3 q^{k(3k+1)/2} = \frac{(q;q)_{3n+1}}{(q;q)_n^3}. \tag{4.9}$$

Replacing y by uq^{3n}, v and w by uq^{2n-1} in (1.6), we obtain

$$\begin{aligned}
(q^{2n+1};q)_n &= \sum_{k=0}^{n}(-1)^k \begin{bmatrix} n \\ k \end{bmatrix} q^{3k(k-1)/2} \frac{(q^{n-k+1};q)_k^2}{(q^k;q)_k} \frac{(q^{n+k+1};q)_{n-k}^2}{(q^{2k+1};q)_{n-k}} \\
&= \frac{(q^{n+1};q)_n^2}{(q;q)_n} + \sum_{k=1}^{n}(-1)^k(1+q^k)q^{3k(k-1)/2} \frac{(q;q)_n(q^{n-k+1};q)_k^2(q^{n+k+1};q)_{n-k}^2}{(q;q)_{n-k}(q;q)_{n+k}}.
\end{aligned}$$

Multiplying both sides by $(q;q)_{2n}/(q;q)_n^3$ gives (4.8).

Taking $y = uq^{3n+1}$, $v = uq^{2n-1}$ and $w = uq^{2n}$ in (1.6), we find

$$\begin{aligned}
(q^{2n+2};q)_n &= \sum_{k=0}^{n}(-1)^k \begin{bmatrix} n \\ k \end{bmatrix} q^{k(3k+1)/2} \frac{(q^{n-k+1};q)_k^2}{(q^{k+1};q)_k} \frac{(q^{n+k+2};q)_{n-k}^2}{(q^{2k+2};q)_{n-k}} \\
&= \sum_{k=0}^{n}(-1)^k(1-q^{2k+1})q^{k(3k+1)/2} \frac{(q;q)_n(q^{n-k+1};q)_k^2(q^{n+k+2};q)_{n-k}^2}{(q;q)_{n-k}(q;q)_{n+k+1}}.
\end{aligned}$$

Multiplying both sides by $(q;q)_{2n+1}/(q;q)_n^3$, we arrive at (4.9).

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