Nordhaus-Gaddum-type Bounds for the Rainbow Vertex-connection Number of a Graph*

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Abstract

A vertex-colored graph G is rainbow vertex-connected if any pair of distinct vertices are connected by a path whose internal vertices have distinct colors. The rainbow vertex-connection number of G, denoted by $rvc(G)$, is the minimum number of colors that are needed to make G rainbow vertex-connected. In this paper we give a Nordhaus-Gaddum-type result of the rainbow vertex-connection number. We prove that when G and $\overline{G}$ are both connected, then $2 \leq rvc(G) + rvc(\overline{G}) \leq n-1$. Examples are given to show that both the upper bound and the lower bound are best possible for all $n \geq 5$.

Keywords: rainbow vertex-connection number, Nordhaus-Gaddum-type.

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1 Introduction

All graphs considered in this paper are simple, finite and undirected. We follow the notation and terminology of [1]. An edge-colored graph G is rainbow connected if any pair of distinct vertices are connected by a path whose edges have distinct colors. Clearly, if a graph is rainbow edge-connected, then it is also connected. Conversely, any connected graph has trivial edge coloring that makes it rainbow edge-connected; just color each edge with a distinct color. The rainbow connection number of a connected graph G, denoted by $rc(G)$, is the minimum number of colors that are needed in order to make G rainbow connected, which was introduced by Chartrand et al. Obviously, we always have $diam(G) \leq rc(G) \leq n-1$, where $diam(G)$ denotes the diameter of a graph G of order n. Notice that

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$rc(G) = 1$ if and only if G is a complete graph, and that $rc(G) = n - 1$ if and only if G is a tree.

In [3], Krivelevich and Yuster proposed the concept of rainbow vertex-connection. A vertex-colored graph is rainbow vertex-connected if any pair of distinct vertices are connected by a path whose internal vertices have distinct colors. The rainbow vertex-connection of a connected graph G, denoted by $rvc(G)$, is the minimum number of colors that are needed to make G rainbow vertex-connected. An easy observation is that if G is a connected graph with n vertices then $rvc(G) \leq n - 2$. We note the trivial fact that $rvc(G) = 0$ if and only if G is a complete graph. Also, clearly, $rvc(G) \geq diam(G) - 1$ with equality if the diameter is 1 or 2.

A Nordhaus–Gaddum-type result is a (tight) lower or upper bound on the sum or product of a parameter of a graph and its complement. The name Nordhaus–Gaddum-type is used because in 1956 Nordhaus and Gaddum [4] first established the following inequalities for the chromatic numbers of graphs, they proved that if G and $\overline{G}$ are complementary graphs on n vertices whose chromatic numbers are $\chi(G)$, $\chi(\overline{G})$ respectively, then

$$2\sqrt{n} \leq \chi(G) + \chi(\overline{G}) \leq n + 1.$$

Since then, many analogous inequalities of other graph parameters are concerned, such as domination number [6], Wiener index and some other chemical indices [7], and so on.

In [8], the authors considered Nordhaus–Gaddum-type result for the rainbow connection number. In this paper, we are concerned with analogous inequalities involving the rainbow vertex-connection number of graphs. We prove that

$$2 \leq rvc(G) + rvc(\overline{G}) \leq n - 1.$$

The rest of this paper is organized as follows. Section 2 contains the proof of the sharp upper bound. Section 3 contains the proof of the sharp lower bound.

2 Upper bound for $rvc(G) + rvc(\overline{G})$

We begin this section with two lemmas that are needed in order to establish the proof of the upper bound.

Lemma 1 *Let G be a nontrivial connected graph of order n, and $rvc(G) = k$. Add a new vertex v to G, and make v be adjacent to q ($1 \leq q \leq n$) vertices of G, the resulting graph is denoted by G'. Then, if $q \geq n - k$, we have $rvc(G') \leq k$.*

Proof. Let $c : V(G) \rightarrow \{1, 2, \cdots, k\}$ be a rainbow k-vertex-coloring of G, $X = \{x_1, x_2, \cdots, x_q\}$ be the vertices that are adjacent to v, $V \backslash X = \{y_1, y_2, \cdots, y_{n-q}\}$. We can assume that there exists some y_o such that there is no rainbow vertex-connected-path from v to y_o; otherwise, the result holds obviously. Because G is a rainbow k-vertex-coloring, there is a rainbow vertex-connected-path P_i from x_i to y_o for every x_i, $i \in \{1, 2, \cdots, q\}$. Certainly, $P_i \bigcap P_j$ may not be empty. We claim that no other vertices of $\{x_1, x_2, \cdots, x_q\}$ different from x_i belong to P_i for each $1 \leq i \leq q$. Suppose that is not the case and let x_i' be the last vertex in $\{x_1, x_2, \cdots, x_q\}$ which belongs to P_i, denote P_i by $x_i P_i' x_i' Q_i y_o$, then $v x_i' Q_i y_o$ is a rainbow vertex-connected-path, a contradiction to our assumption. Since v and y_o are not rainbow vertex-connected, for each P_i, there is some y_{k_i} such that $c(x_i) = c(y_{k_i})$. That means that the colors that are assigned to X are among the colors that are assigned to $V \backslash X$. So $rvc(G) = k \leq n - q$. By the hypothesis $q \geq n - k$, we have $rvc(G) = n - q$, that is, all vertices in $V \backslash X$ have distinct colors. Now we construct a new graph $G' = P_1 \bigcup P_2 \bigcup \cdots \bigcup P_q$. To show that for every y_t not in G', there is a $y_s \in G'$ such that $y_t y_s \in E(G)$, suppose that $N(y_t) \subsetneqq \{x_1, x_2, \cdots, x_q\}$. Since G is rainbow k-vertex-connected, there is a rainbow vertex-connected path from y_t to y_o, denoted by $y_t x_k Q y_o$, where $x_k \in N(y_t)$. Thus $v x_k Q y_o$ is a rainbow vertex-connected path, a contradiction. It follows that $G[y_1, y_2, \cdots, y_{n-q}]$ is connected. Certainly, $G[y_1, y_2, \cdots, y_{n-q}]$ has a spanning tree T, and T has at least two pendant vertices. Then there must exist a pendant vertex whose color is different from x_1, and we assign the color to x_1. It is easy to check that G is still rainbow k-vertex-connected, and there is a rainbow vertex-connected path between v and y_o. If there still exists some y_j such that v and y_j are not rainbow vertex-connected, we do the same operation, until v and y_j are rainbow vertex-connected for each $j \in \{1, 2, \cdots, n - q\}$. Thus G' is rainbow vertex-connected. It follows that $rvc(G') \leq k$. ∎

Lemma 2 *Let G be a connected graph of order 5. If $\overline{G}$ is connected, then $rvc(G) + rvc(\overline{G}) \leq 4$.*

Proof. We consider the situations of G.

First, if G is a path, then $rvc(G) = 3$. In this case $diam(\overline{G}) = 2$, and then $rvc(\overline{G}) = 1$.

Second, if G is a tree but not a path, then $rvc(G) < 3$. Since G is a bipartite graph, then $\overline{G}$ consists of a K_2 and a K_3 and two edges between them. So we assign color 1 to the vertices of K_2 and color 2 to the vertices of K_3, and this makes $\overline{G}$ rainbow vertex-connected, that is, $rvc(\overline{G}) \leq 2$.

Finally, if both G and $\overline{G}$ are not trees, then $e(G) = e(\overline{G}) = 5$. If G contains a cycle of length 5, then $G = \overline{G} = C_5$, thus $rvc(G) = rvc(\overline{G}) = 1$. If G contains a cycle of length 4, there is only one graph G which is showed in Figure 1, we can color G and $\overline{G}$ with 2 colors to make them rainbow vertex-connected, see Figure 1. If G contains a cycle of length 3, then G and $\overline{G}$ are showed in Figure 2. By the coloring showed in the graphs, we have $rvc(G) + rvc(\overline{G}) = 4$.

Figure 1: G contains a cycle of length 4.

Figure 2: G contains a cycle of length 3.

By these cases, we have $rvc(G) + rvc(\overline{G}) \leq 4$. ∎

From the above lemmas, we have our first theorem.

Theorem 1 *$rvc(G) + rvc(\overline{G}) \leq n - 1$ for all $n \geq 5$, and this bound is best possible.*

Proof. We use induction on n. By Lemma 2, the result is evident for $n = 5$. We assume that $rvc(G) + rvc(\overline{G}) \leq n - 1$ holds for complementary graphs on n vertices. To the union of a connected graph G and its $\overline{G}$, which forms the complete graph on these n vertices, we adjoin a vertex v. Let q $(1 \leq q \leq n - 1)$ of the n edges between v and the union be adjoined to G and the remaining $n - q$ edges to $\overline{G}$. If G' and $\overline{G'}$ are the graphs so determined (each of order $n + 1$), then

$$rvc(G') \leq rvc(G) + 1, \quad rvc(\overline{G'}) \leq rvc(\overline{G}) + 1.$$

These inequalities are evident from the fact that if given a rainbow $rvc(G)$-vertex-coloring ($rvc(\overline{G})$-vertex-coloring) of G ($\overline{G}$), we assign a new color to the vertex v and keep other vertices unchanged, the resulting coloring makes G' ($\overline{G'}$) rainbow vertex-connected. Then $rvc(G') + rvc(\overline{G'}) \leq rvc(G) + rvc(\overline{G}) + 2 \leq n + 1$. And $rvc(G') + rvc(\overline{G'}) \leq n$ except possibly when

$$rvc(G') = rvc(G) + 1, \quad rvc(\overline{G'}) = rvc(\overline{G}) + 1.$$

In this case, by Lemma 1, $q \leq n - rvc(G) - 1, n - q \leq n - rvc(\overline{G}) - 1$, thus $rvc(G) + rvc(\overline{G}) \leq n - 2$, from which $rvc(G') + rvc(\overline{G'}) \leq n$. This completes the induction.

The following example shows that the bound established is sharp for all $n \geq 5$: If G be a path of order n, then $rvc(G) = n - 2$. It is easy to obtain $\overline{G}$, and check that $diam(\overline{G}) = 2$. Then $rvc(\overline{G}) = 1$, and so we have $rvc(G) + rvc(\overline{G}) = n - 1$. ∎

Remark: For $n \leq 4$, note that P_4, the path on 4 vertices, is the only connected graph with fewer than 5 vertices that has a connected complement, and $rvc(P_4) = 2$. So, the sum of the rainbow vertex-connection numbers of P_4 and its complement P_4 is 4.

3 Lower bound for $rvc(G) + rvc(\overline{G})$

As we note that $rvc(G) = 0$ if and only if G is a complete graph. Thus if we want both G and $\overline{G}$ are connected, and so $rvc(G) \geq 1, rvc(\overline{G}) \geq 1$. Then $rvc(G) + rvc(\overline{G}) \geq 2$. Our next theorem shows that the lower bound is sharp for all $n \geq 5$.

Theorem 2 *For $n \geq 5$, the lower bound of $rvc(G) + rvc(\overline{G}) \geq 2$ is best possible, that is, there are graphs G and $\overline{G}$ with n vertices, such that $rvc(G) = rvc(\overline{G}) = 1$.*

Proof. We only need to prove that for $n \geq 5$, there are graphs G and $\overline{G}$ with n vertices, such that $diam(G) = diam(\overline{G}) = 2$.

We construct G as follows: if $n = 2k + 1$,

$$V(G) = \{v, v_1, v_2, \cdots, v_k, u_1, u_2, \cdots, u_k\}$$

$$E(G) = \{vv_i | 1 \leq i \leq k\} \bigcup \{v_i u_i | 1 \leq i \leq k\} \bigcup \{u_i u_j | 1 \leq i, j \leq k\};$$

if $n = 2k$,

$$V(G) = \{v, v_1, v_2, \cdots, v_k, u_1, u_2, \cdots, u_{k-1}\}$$

$E(G) = \{vv_i | 1 \leq i \leq k\} \bigcup \{v_i u_i | 1 \leq i < k\} \bigcup \{v_k u_{k-1}\} \bigcup \{u_i u_j | 1 \leq i, j \leq k - 1\}$.

We can easily check that $diam(G) = diam(\overline{G}) = 2$. ∎

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