Rainbow connection numbers of complementary graphs*

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Abstract

A path in an edge-colored graph, where adjacent edges may be colored the same, is a rainbow path if no two edges of it are colored the same. A nontrivial connected graph G is rainbow connected if there is a rainbow path connecting any two vertices, and the rainbow connection number of G, denoted by $rc(G)$, is the minimum number of colors that are needed in order to make G rainbow connected. In this paper, we provide a new approach to investigate the rainbow connection number of a graph G according to some constraints to its complement $\overline{G}$. We first derive that for a connected graph G, if $\overline{G}$ does not belong to the following two cases: (i) $diam(\overline{G}) = 2, 3$, (ii) $\overline{G}$ contains exactly two connected components and one of them is trivial, then $rc(G) \leq 4$, where $diam(G)$ is the diameter of G. Examples are given to show that this bound is best possible. Next we derive that for a connected graph G, if $\overline{G}$ is triangle-free, then $rc(G) \leq 6$.

Keywords: edge-colored graph, rainbow path, rainbow connection number, complement graph, diameter, triangle-free

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1 Introduction

All graphs in this paper are finite, undirected and simple. Let G be a nontrivial connected graph on which an edge-coloring $c : E(G) \rightarrow \{1, 2, \cdots, n\}$, $n \in \mathbb{N}$, is defined, where adjacent edges may be colored the same. A path is *rainbow* if no two edges of it are colored the same. An

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edge-coloring graph G is *rainbow connected* if any two vertices are connected by a rainbow path. Clearly, if a graph is rainbow connected, it must be connected, whereas any connected graph has a trivial edge-coloring that makes it rainbow connected; just color each edge with a distinct color. Thus, in [4] Chartrand et al. defined the *rainbow connection number* of a connected graph G, denoted by $rc(G)$, as the smallest number of colors that are needed in order to make G rainbow connected. If G' is a connected spanning subgraph of G, then $rc(G) \leq rc(G')$. They showed that $rc(G) = 1$ if and only if G is complete, and that $rc(G) = m$ if and only if G is a tree, as well as that a cycle with $k > 3$ vertices has rainbow connection number $\lceil \frac{k}{2} \rceil$, a triangle has rainbow connection number 1. Also notice that, clearly, $rc(G) \geq diam(G)$ where $diam(G)$ denotes the diameter of G. In an edge-colored graph G, we use $c(e)$ to denote the color of an edge e, and for a subgraph H of G, $c(H)$ denotes the set of colors of the edges in H. For a subset X of $V(G)$, we use $E[X]$ to denote the edge set of the induced subgraph $G[X]$. The *distance* between two vertices u and v in a connected graph G, denoted by $dist(u, v)$, is the length of a shortest path between them in G. The *eccentricity* of a vertex v in G is defined as $ecc_G(v) = \max_{x \in V(G)} dist(v, x)$. We follow the notation and terminology of [1].

In this paper, we provide a new approach to investigate the rainbow connection number of a graph G according to some constraints to its complement $\overline{G}$. We give two sufficient conditions to guarantee that $rc(G)$ is bounded by a constant.

One of our main results is:

Theorem 1.1 *For a connected graph G, if $\overline{G}$ does not belong to the following two cases: (i) $diam(\overline{G}) = 2, 3$, (ii) $\overline{G}$ contains exactly two connected components and one of them is trivial, then $rc(G) \leq 4$. Furthermore, this bound is best possible.* ∎

For the remaining cases, $rc(G)$ can be very large as discussed in Section 4. So we add a constraint, i.e., we let $\overline{G}$ be triangle-free. Then G is claw-free, and we can derive our next main result:

Theorem 1.2 *For a connected graph G, if $\overline{G}$ is triangle-free, then $rc(G) \leq 6$.* ∎

2 Preliminaries

We now give a necessary condition for an edge-colored graph to be rainbow connected. If G is rainbow connected under some edge-coloring, then for any two bridges (if two such edges should exist) $e_1 = u_1u_2$, $e_2 =$

v_1v_2, there must exist some $1 \leq i, j \leq 2$, such that any $u_i - v_j$ path must contain edges e_1, e_2. So we have:

Observation 2.1 *If G is rainbow connected under some edge-coloring and e_1 and e_2 are any two cut edges, then $c(e_1) \neq c(e_2)$.* ∎

The following lemma will be useful in our discussion.

Lemma 2.2 ([2]) *If G is a connected graph and $H_1, \cdots, H_k$ is a partition of the vertex set of G into connected subgraphs, then $rc(G) \leq k - 1 + \sum_{i=1}^{k} rc(H_i)$.* ∎

In [4], the authors derived the precise values of the rainbow connection numbers of complete bipartite graph $K_{s,t}(2 \leq s \leq t)$ and complete k-partite graph $(k \geq 3)$.

Theorem 2.3 ([4]) *For integers s and t with $2 \leq s \leq t$,*

$$rc(K_{s,t}) = \min\{\lceil \sqrt[s]{t} \rceil, 4\}.$$

∎

Theorem 2.4 ([4]) *Let $G = K_{n_1,n_2,\ldots,n_k}$ be a complete k-partite graph, where $k \geq 3$ and $n_1 \leq n_2 \leq \ldots \leq n_k$ such that $s = \sum_{i=1}^{k-1} n_i$ and $t = n_k$. Then*

$$rc(G) = \begin{cases} 1 & \text{if } n_k = 1, \\ 2 & \text{if } n_k \geq 2 \text{ and } s > t, \\ \min\{\lceil \sqrt[s]{t} \rceil, 3\} & \text{if } s \leq t. \end{cases}$$

∎

From the above two theorems, we know that $rc(K_{s,t}) \leq 4$ for any $s, t \geq 2$ and $rc(G) \leq 3$ where G is a complete k-partite graph with $k \geq 3$.

The following notions are taken from [3], which will be used later. A dominating set D in a graph G is called a *two-way dominating set* if every pendant vertex of G is included in D. In addition, if $G[D]$ is connected, we call D a *connected two-way dominating set*. Note that if $\delta(G) \geq 2$, then every (*connected*) dominating set in G is a (connected) two-way dominating set. We also need the following result.

Theorem 2.5 ([3]) *If D is a connected two-way dominating set in a graph G, then $rc(G) \leq rc(G[D]) + 3$.* ∎

3 Proof of Theorem 1.1

We first investigate the rainbow connection numbers of connected complement graphs of graphs with diameter at least 4.

Theorem 3.1 *If G is a connected graph with $diam(G) \geq 4$, then $rc(\overline{G}) \leq 4$.*

Proof. First of all, we see that $\overline{G}$ must be connected, since otherwise, $diam(G) \leq 2$, contradicting the condition $diam(G) \geq 4$.

We choose a vertex x with $ecc_G(x) = diam(G) = d \geq 4$. Let $N_G^i(x) = \{v : dist(x, v) = i\}$ where $0 \leq i \leq d$. So $N_G^0(x) = \{x\}, N_G^1(x) = N_G(x)$ as usual. Then $\bigcup_{0 \leq i \leq d} N_G^i(x)$ is a vertex partition of $V(G)$ with $|N_G^i(x)| = n_i$. Let $A = \bigcup_{i \ is \ even} N_G^i(x)$, $B = \bigcup_{i \ is \ odd} N_G^i(x)$. For example, see Figure 3.1, a graph with $diam(G) = 4$.

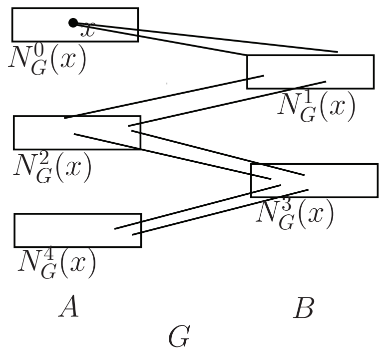

Figure 3.1 Graph for the example with $d = 4$.

So, if $d = 2k(k \geq 2)$ then $A = \bigcup_{0 \leq i \leq d \ is \ even} N_G^i(x)$, $B = \bigcup_{1 \leq i \leq d-1 \ is \ odd} N_G^i(x)$; if $d = 2k + 1(k \geq 2)$ then $A = \bigcup_{0 \leq i \leq d-1 \ is \ even} N_G^i(x)$, $B = \bigcup_{1 \leq i \leq d \ is \ odd} N_G^i(x)$. Then by the definition of complement graphs, we know that $\overline{G}[A](\overline{G}[B])$ contains a spanning complete k_1-partite subgraph (complete k_2-partite subgraph) where $k_1 = \lceil \frac{d+1}{2} \rceil (k_2 = \lceil \frac{d}{2} \rceil)$. For example, see Figure 3.1, $\overline{G}[A]$ contains a spanning complete tripartite subgraph K_{n_0,n_2,n_4}, $\overline{G}[B]$ contains a spanning complete bipartite subgraph K_{n_1,n_3}.

Case 1. $d \geq 5$. Then $k_1, k_2 \geq 3$. From Theorem 2.4, we have $rc(\overline{G}[A]), rc(\overline{G}[B]) \leq 3$.

We now give $\overline{G}$ an edge-coloring as follows: we first give the subgraph $\overline{G}[A]$ a rainbow edge-coloring using three colors; then give the subgraph $\overline{G}[B]$ a rainbow edge-coloring using the same colors as that of the subgraph

$\overline{G}[A]$; next we give a fresh color to all edges between the subgraph $\overline{G}[A]$ and the subgraph $\overline{G}[B]$.

We will show that this coloring is rainbow. It suffices to show that for any $u \in \overline{G}[A]$, $v \in \overline{G}[B]$, there is a rainbow path connecting them in $\overline{G}$. We first choose an edge uv_1 where $v_1 \in \overline{G}[B]$ (it must exist, without loss of generality, we assume $u \in N_G^2(x)$, then u is adjacent to all vertices in $N_G^5(x)$). Then by adding a rainbow $v_1 - v$ path in $\overline{G}[B]$, we obtain our desired path. So $rc(\overline{G}) \leq 4$ in this case.

Case 2. $d = 4$, that is, $A = N_G^0(x) \cup N_G^2(x) \cup N_G^4(x)$, $B = N_G^1(x) \cup N_G^3(x)$. So $\overline{G}[A](\overline{G}[B])$ contains a spanning complete 3-partite subgraph K_{n_0,n_2,n_4} (complete bipartite subgraph K_{n_1,n_3}). So, from Theorem 2.4 we have $rc(\overline{G}[A]) \leq 3$.

Subcase 2.1. $n_1, n_3 \geq 2$. Since now $\overline{G}[B]$ contains a spanning complete bipartite subgraph K_{n_1,n_3}, from Theorem 2.3 we have $rc(\overline{G}[B]) \leq 4$.

We now give $\overline{G}$ an edge-coloring as follows: we first give the subgraph $\overline{G}[B]$ a rainbow edge-coloring using four colors, say a, b, c, d; then give the subgraph $\overline{G}[A]$ a rainbow edge-coloring using colors a, b, c; next we give the color d to all edges between the subgraph $\overline{G}[A]$ and the $\overline{G}[B]$.

We will show that this coloring is rainbow. It suffices to show that for any $u \in \overline{G}[A]$, $v \in \overline{G}[B]$, there is a rainbow path connecting them in $\overline{G}$. We first choose an edge vu_1 where $u_1 \in \overline{G}[A]$ (it must exist, without loss of generality, we assume $v \in N_G^1(x)$, then v is adjacent to all vertices in $N_G^4(x)$). Then by adding a rainbow $u_1 - u$ path in $\overline{G}[B]$, we obtain our desired path. So $rc(\overline{G}) \leq 4$ in this case.

Subcase 2.2. At least one of n_1, n_3 is 1, say $n_1 = 1$.

We now give $\overline{G}$ an edge-coloring as follows: we give the edges between $N_G^0(x)$ and $N_G^4(x)$ a color a; give the edges between $N_G^0(x)$ and $N_G^2(x)$ a new color b; give the edges between $N_G^2(x)$ and $N_G^4(x)$ a new color c; give the edges between $N_G^1(x)$ and $N_G^4(x)$ a new color d; give the edges between $N_G^0(x)$ and $N_G^3(x)$ the color b; give the edges between $N_G^1(x)$ and $N_G^3(x)$ the color c.

We will show that this coloring is rainbow. We only need to show that there is a rainbow path connecting two vertices $u, v \in N_G^3(x)$, the remaining cases are easy. Let $P := u, x, x_1, x_2, v$ where $x_1 \in N_G^4(x), x_2 \in N_G^1(x)$. Clearly, it is rainbow. So $rc(\overline{G}) \leq 4$ in this case. ∎

With a similar argument to that of Theorem 3.1, we have:

Proposition 3.2 *If G is a tree with $diam(G) \geq 3$, then $rc(\overline{G}) \leq 3$.*

Proof. It is easy to show that if G is a tree with $diam(G) \geq 3$, then $\overline{G}$ is connected. We now use the same terminology as in the argument of Theorem 3.1. Note that A and B are independent sets in G (consider the BFS-tree of G). So, $\overline{G}[A]$ and $\overline{G}[B]$ are two disjoint cliques in $\overline{G}$. Then by Lemma 2.2 we have $rc(\overline{G}) \leq 3$. ∎

Theorem 3.1 is equivalent to the following result.

Theorem 3.3 *For a connected graph G, if $\overline{G}$ is connected and $diam(\overline{G}) \geq 4$, then $rc(G) \leq 4$.* ∎

If G is a graph with $h \geq 2$ connected components, then $\overline{G}$ contains a complete h-partite spanning subgraph, and so we have

Proposition 3.4 *If G is a graph with $h \geq 2$ connected components G_i and $n'_i = n(G_i)(1 \leq i \leq h)$, then $rc(\overline{G}) \leq rc(K_{n'_1,\cdots,n'_h})$.* ∎

Proof of Theorem 1.1. If $\overline{G}$ is connected, since $diam(\overline{G}) \neq 2, 3$ and clearly $diam(\overline{G}) \neq 1$, from Theorem 3.3 we have $rc(G) \leq 4$. If $\overline{G}$ is disconnected, since by the assumption, it has either at least three connected components or exactly two nontrivial components, then from Theorems 2.3 and 2.4 and Proposition 3.4 we have $rc(G) \leq 4$.

Let $\overline{G}$ contain two connected components, one is a clique with $s \geq 2$ vertices, the other is a clique with $t \geq 3^s + 1$ vertices. We have $G = K_{s,t}$, then from Theorem 2.3, $rc(G) = \min\{\lceil \sqrt[s]{t} \rceil, 4\} = 4$, and so the bound is best possible. ∎

4 Proof of Theorem 1.2

For the remaining cases, since the complement of $\overline{G}$ is G itself, we need to investigate $rc(\overline{G})$ in two cases: (i) $diam(G) = 2, 3$, (ii) G contains two connected components and one of them is trivial. We first give some discussion about the case $diam(G) = 3$. We use the same terminology as that of Theorem 3.1.

Theorem 4.1 *For a vertex x of G satisfying $ecc_G(x) = diam(G) = 3$, we have $rc(\overline{G}) \leq 5$ for the three cases (i) $n_1 = n_2 = n_3 = 1$, (ii) $n_1, n_2 = 1, n_3 \geq 2$, and (iii) $n_2 = 1, n_1, n_3 \geq 2$. For the remaining cases, $rc(\overline{G})$ may be very large. Furthermore, if G is triangle-free and $\overline{G}$ is connected, then $rc(\overline{G}) \leq 5$.*

Proof. If $n_1 = n_2 = n_3 = 1$, then G is a 4-path P_4, and so $rc(\overline{G}) = 3$. Thus, we could consider the following three cases.

Case 1. Two of n_1, n_2, n_3 are equal to 1.

Subcase 1.1. $n_1, n_2 = 1$. Then it is easy to show that the subgraph $\overline{G}[N_G^0(x) \cup N_G^1(x) \cup N_G^3(x)]$ contains a bipartite spanning subgraph K_{2,n_3}, and so from Lemma 2.2 and Theorem 2.3 we have $rc(\overline{G}) \leq rc(K_{2,n_3}) + 1 \leq 5$.

Subcase 1.2. $n_1, n_3 = 1$. Let $n'_2 = |\{v \in N_G^2(x) : deg_{\overline{G}}(v) = 1\}|$. Then there are n'_2 cut edges in $\overline{G}$, and so from Observation 2.1 we have $rc(\overline{G}) \geq n'_2$.

Furthermore, if G is triangle-free, then $N_G^2(x)$ is a stable set in G, and so a clique in $\overline{G}$, and thus from Lemma 2.2 we have $rc(\overline{G}) \leq 4$.

Subcase 1.3. $n_2, n_3 = 1$. With a similar argument to that of **Subcase 1.2**, we have $rc(\overline{G}) \geq n_1'$ where $n_1' = |\{v \in N_G^1(x) : deg_{\overline{G}}(v) = 1\}|$.

Furthermore, if G is triangle-free, then $N_G^1(x)$ is a stable set in G, and so a clique in $\overline{G}$, and thus from Lemma 2.2 we have $rc(\overline{G}) \leq 4$.

Case 2. One of n_1, n_2, n_3 is equal to 1.

Subcase 2.1. $n_1 = 1$. With a similar argument to that of **Subcase 1.2**, we have $rc(\overline{G}) \geq n_2'$ where $n_2' = |\{v \in N_G^2(x) : deg_{\overline{G}}(v) = 1\}|$.

Furthermore, if G is triangle-free, then $N_G^2(x)$ is a stable set in G, and so a clique in $\overline{G}$. In $\overline{G}$, the subgraph $\overline{G}[N_G^0(x) \cup N_G^1(x) \cup N_G^3(x)]$ contains a spanning bipartite subgraph K_{2,n_3}. So from Theorem 2.3, it needs at most four colors to rainbow it; we then give a new color to the edges between x and $N_G^2(x)$. Clearly, this coloring is rainbow and we have $rc(\overline{G}) \leq 5$.

Subcase 2.2. $n_2 = 1$. Then it is easy to show that the subgraph $\overline{G}[N_G^0(x) \cup N_G^1(x) \cup N_G^3(x)]$ contains a spanning bipartite subgraph K_{1+n_1,n_3}. So from Lemma 2.2 and Theorem 2.3, we have $rc(\overline{G}) \leq rc(K_{1+n_1,n_3}) + 1 \leq 5$.

Subcase 2.3. $n_3 = 1$. Let $N_G^3(x) = \{u\}$. With a similar argument to that of **Subcase 1.2**, we have $rc(\overline{G}) \geq n_1' + n_2'$ where $n_i' = |\{v \in N_G^i(x) : deg_{\overline{G}}(v) = 1\}|$ with $i = 1, 2$.

Furthermore, if G is triangle-free, then $N_G^1(x)$ is a stable set in G, and so a clique in $\overline{G}$. Let V_u be the set of vertices of $N_G^2(x)$ which are adjacent to u in G. So V_u is a stable set in G and a clique in $\overline{G}$. We now give $\overline{G}$ an edge-coloring: We give the edges of the complete graph $\overline{G}[N_G^1(x) \cup \{u\}]$ a color a; give the edge xu a new color b, give the edges (they may not exist, but now $N_G^2(x) = V_u$ is a clique and the procedure is easy) between u and $N_G^2(x) \setminus V_u$ a new color c; the edges between x and $N_G^2(x)$ a new color d. It is easy to check that the coloring is rainbow and $rc(\overline{G}) \leq 4$ in this case.

Case 3. $n_1, n_2, n_3 \geq 2$. With a similar argument to that of **Subcase 1.2**, we have $rc(\overline{G}) \geq n_2'$ where $n_2' = |\{v \in N_G^2(x) : deg_{\overline{G}}(v) = 1\}|$.

Furthermore, if G is triangle-free, then $N_G^1(x)$ is a stable set in G, and so a clique in $\overline{G}$. If every vertex in $N_G^3(x)$ is adjacent to all vertices of $N_G^2(x)$ in G, then both $N_G^2(x)$ and $N_G^3(x)$ are stable sets in G, and so cliques in $\overline{G}$, since G is triangle-free. Then in $\overline{G}$, $\overline{G}[N_G^0(x) \cup N_G^3(x)]$, $\overline{G}[N_G^2(x)]$, $\overline{G}[N_G^1(x)]$ are complete graphs. So from Lemma 2.2 we have $rc(\overline{G}) \leq rc(\overline{G}[N_G^0(x) \cup N_G^3(x)]) + rc(\overline{G}[N_G^2(x)]) + rc(\overline{G}[N_G^1(x)]) + 2 = 5$. Thus we choose a vertex $u \in N_G^3(x)$ with $V_u \neq \emptyset, N_G^2(x)$, where V_u denotes the set of neighbors of u in $N_G^2(x)$ in G, and so it is a stable set in G and a clique in $\overline{G}$.

We now give $\overline{G}$ an edge-coloring: We give a new color a to the edges of $\overline{G}[N_G^1(x)]$; for every vertex w of $N_G^3(x) \setminus \{u\}$, since w is adjacent to all vertices of $N_G^1(x)$ in $\overline{G}$, we give a new color b to an edge between w and

$N_G^1(x)$, give a new color c to the remaining edges between w and $N_G^1(x)$; give color a to the edges between x and $N_G^3(x)\backslash\{u\}$; give the edge xu a new color d; give a new color e to the edges between x and $N_G^2(x)$; give the color b to the edges between u and $N_G^2(x)\backslash V_u$. It is easy to check that the above coloring is rainbow and $rc(\overline{G}) \leq 5$ in this case.

From the above discussion, we know that $rc(\overline{G}) \leq 5$ for the three cases (i) $n_1 = n_2 = n_3 = 1$, (ii) $n_1, n_2 = 1, n_3 \geq 2$, and (iii) $n_2 = 1, n_1, n_3 \geq 2$. For the remaining cases, $rc(\overline{G})$ can be very large if $n_i'(i = 1, 2)$ is sufficiently large. Furthermore, if G is triangle-free, then $rc(\overline{G}) \leq 5$. ∎

The following corollary clearly holds.

Corollary 4.2 *For a connected graph G, if $\overline{G}$ is triangle-free and $diam(\overline{G}) = 3$, then $rc(G) \leq 5$.* ∎

For a graph G with $diam(G) = 2$, let x be a vertex satisfying $ecc_G(x) = diam(G)$. Then, the two cases: (i) $n_1 = n_2 = n_3 = 1$ and (ii) $n_1 = 1, n_2 \geq 2$ do not hold, since in both cases $\overline{G}$ are disconnected and $rc(\overline{G})$ are undefined. For the remaining two cases, that is, $n_1 \geq 2, n_2 = 1$, $n_1, n_2 \geq 2$, with a similar argument to that of Theorem 4.1, we have $rc(\overline{G}) \geq n_1'$, $rc(\overline{G}) \geq n_2'$, respectively. So $rc(\overline{G})$ can be very large if $n_i'(i = 1, 2)$ is sufficiently large. So we add an additional constraint, i.e., we let G be triangle-free.

Proposition 4.3 *Let G be a triangle-free graph with $diam(G) = 2$. If $\overline{G}$ is connected, then $rc(\overline{G}) \leq 5$.*

Proof. We choose a vertex x with $ecc_G(x) = diam(G) = 2$, and we use the same terminology as that of Theorem 3.1. By the above discussion, we only need to consider the following two cases.

Case 1. $n_1 \geq 2, n_2 = 1$. Since G is triangle-free, $N_G^1(x)$ is a stable set in G and so a clique in $\overline{G}$. Thus, $rc(\overline{G}) \leq 3$.

Case 2. $n_1, n_2 \geq 2$. Since G is triangle-free, $N_G^1(x)$ is a stable set in G and so a clique in $\overline{G}$. Since $\overline{G}$ is connected, there exist $u \in N_G^1(x), v \in N_G^2(x)$ such that $uv \in E(\overline{G})$.

If there exists some vertex $w \in N_G^2(x)$ with $deg_G(w) = n - 2$, then w is adjacent to the remaining vertices except x in G. Since $diam(G) = 2$, there exists $w_1w_2 \in E(G)$ with $w_1 \in N_G^1(x), w_2 \neq w \in N_G^2(x)$. So $\{w, w_1, w_2\}$ is a triangle in G, this produces a contradiction. So $deg_G(w) < n - 2$ for all $w \in N_G^2(x)$, and $deg_{\overline{G}}(w) \geq 2$ for all $w \in N_G^2(x)$. Let $D = \{x, v\} \cup N_G^1(x)$. Then D is a connected two-way dominated set in $\overline{G}$. So from Theorem 2.5, we have $rc(\overline{G}) \leq rc(\overline{G}[D]) + 3 \leq 5$. ∎

If G contains two connected components, say G_1, G_2. Let $n_1' = |\{v \in G_2 : deg_G(v) = n - 2\}|$. Then in $\overline{G}$, there are n_1' pendant vertices and so there are n_1' cut edges. From Observation 2.1, we have $rc(\overline{G}) \geq n_1'$. So in this case, $rc(\overline{G})$ can be very large if n'_1 is sufficiently large. So we also add an additional constraint, i.e., we let G be triangle-free.

Proposition 4.4 *If G is triangle-free and contains two connected components one of which is trivial, then $rc(\overline{G}) \leq 6$.*

Proof. Suppose that G contains two components, one is trivial, the other is not trivial. Since G is triangle-free, then $\overline{G}$ is claw-free. Let u be the isolated vertex in G, so it is adjacent to any other vertex in $\overline{G}$, and so $diam(\overline{G}) = 2$. We will consider two cases according to the value of $\delta_{\overline{G}}$ where $\delta_{\overline{G}}$ denotes the minimum degree of $\overline{G}$.

Case 1. $\delta_{\overline{G}} = 1$. Let $deg_{\overline{G}}(v_1) = \delta_{\overline{G}}$ and $v_1v_2 \in \overline{G}$ ($v_2 = u$). Since $\overline{G}$ is claw-free, the subgraph $\overline{G}[V \backslash \{v_1\}]$ is a complete graph, so $rc(\overline{G}) = 2$.

Case 2. $\delta_{\overline{G}} \geq 2$. Let $deg_{\overline{G}}(v_1) = \delta_{\overline{G}}$. Then $u \in N^1_{\overline{G}}(v_1)$ and is adjacent to any other vertex in $\overline{G}$. So the subgraph $\overline{G}[D]$ contains a spanning bipartite subgraph $K_{2,\delta_{\overline{G}}-1}$ where $D = \{v_1\} \cup N^1_{\overline{G}}(v_1)$. Clearly, D is a connected two-way dominating set. We give the edge uv_1 a color a, give the edges between v_1 and $N^1_{\overline{G}}(v_1) \backslash \{u\}$ a new color b, give the edges between u and $N^1_{\overline{G}}(v_1) \backslash \{u\}$ a new color c. It is easy to check that this coloring is rainbow. From Theorems 2.5 and 2.3, we have $rc(\overline{G}) \leq rc(\overline{G}[D]) + 3 \leq 6$. ∎

From Theorem 1.1, Corollary 4.2, and Propositions 4.3 and 4.4, our next main result can be derived.

Proof of Theorem 1.2. We consider two cases:

Case 1. $\overline{G}$ is connected. The result holds for the case $diam(\overline{G}) \geq 4$ from Theorem 1.1, the case $diam(\overline{G}) = 3$ from Corollary 4.2 and the case $diam(\overline{G}) = 2$ from Proposition 4.3.

Case 2. $\overline{G}$ is disconnected. The result holds for the case that $\overline{G}$ contains two connected components with one of them trivial from Proposition 4.4, and holds for the remaining case from Theorem 1.1. ∎

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