Partitions of $\mathbb{Z}_n$ into Arithmetic Progressions

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Abstract

We introduce the notion of arithmetic progression blocks or m-AP-blocks of $\mathbb{Z}_n$, which can be represented as sequences of the form $(x, x+m, x+2m, \ldots, x+(i-1)m)$ (mod n). Then we consider the problem of partitioning $\mathbb{Z}_n$ into m-AP-blocks. We show that subject to a technical condition, the number of partitions of $\mathbb{Z}_n$ into m-AP-blocks of a given type is independent of m, and is equal to the cyclic multinomial coefficient which has occurred in Waring's formula for symmetric functions. The type of such a partition of $\mathbb{Z}_n$ is defined by the type of the underlying set partition. We give a combinatorial proof of this formula and the construction is called the separation algorithm. When we restrict our attention to blocks of sizes one and $p+1$, we are led to a combinatorial interpretation of a formula recently derived by Mansour and Sun as a generalization of the Kaplansky numbers. By using a variant of the cycle lemma, we extend the bijection to deal with an improvement of the technical condition recently given by Guo and Zeng.

Keywords: Kaplansky number, cycle dissection, m-AP-partition, separation algorithm.

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1 Introduction

Let $\mathbb{Z}_n$ be the cyclic group of order n whose elements are written as $1, 2, \ldots, n$. Intuitively, we assume that the elements $1, 2, \ldots, n$ are placed clockwise on a cycle. Thus $\mathbb{Z}_n$ can be viewed as an n-cycle, more specifically, a directed cycle. In his study of the ménages problem, Kaplansky [12] has shown that the number of ways of choosing k elements from $\mathbb{Z}_n$ such that no two elements differ by one modulo n (see also Brauldi [1], Comtet [4], Riordan [20], Ryser [21] and Stanley [22, Lemma 2.3.4]) equals

$$\frac{n}{n-k}\binom{n-k}{k}. \tag{1.1}$$

Moreover, Kaplansky [13] considered the following generalization. Assume that $n \geq pk + 1$. Then the number of k-subsets $\{x_1, x_2, \ldots, x_k\}$ of $\mathbb{Z}_n$ such that

$$x_i - x_j \notin \{1, 2, \ldots, p\} \tag{1.2}$$

for any pair (x_i, x_j) of distinct elements, is given by

$$\frac{n}{n - pk} \binom{n - pk}{k}. \tag{1.3}$$

Here we clarify the meaning of the notation (1.2). Given two elements x and y of $\mathbb{Z}_n$, $x - y$ may be considered as the distance from y to x on the directed cycle $\mathbb{Z}_n$. Therefore, (1.2) says that the distance from any element x_i to any other element x_j on the directed cycle $\mathbb{Z}_n$ is at least $p + 1$.

From a different perspective, Konvalina [15] studied the number of k-subsets $\{x_1, x_2, \ldots, x_k\}$ such that no two elements x_i and x_j are "uni-separated", namely $x_i - x_j \neq 2$ for all x_i and x_j. Remarkably, Konvalina discovered that the answer is also given by the Kaplansky number (1.1) for $n \geq 2k + 1$. Other generalizations and related questions have been investigated by Chu [3], Hwang [9, 10], Hwang, Korner and Wei [11], Moser [17], Munarini and Salvi [18], Prodinger [19] and Kirschenhofer and Prodinger [14]. Recently, Mansour and Sun [16] obtained the following unification of the formulas of Kaplansky and Konvalina.

Theorem 1.1 (Mansour-Sun). *Assume that $m, p, k \geq 1$ and $n \geq mpk + 1$. The number of k-subsets $\{x_1, x_2, \ldots, x_k\}$ of $\mathbb{Z}_n$ such that*

$$x_i - x_j \notin \{m, 2m, \ldots, pm\} \tag{1.4}$$

for any pair (x_i, x_j), is given by the formula (1.3)*, and is independent of m.*

In the spirit of the original approach of Kaplansky, Mansour and Sun first solved the enumeration problem of choosing k-subset from an n-set with elements lying on a line. They established a recurrence relation, and solved the equation by computing the residues of some Laurent series. The case for an n-cycle can be reduced to the case for a line. They raised the question of finding a combinatorial proof of their formula. Guo [7] found a proof by using number theoretic properties and Rothe's identity:

$$\sum_{k=0}^{n} \frac{xy}{(x + kz)(y + (n - k)z)} \binom{x + kz}{k} \binom{y + (n - k)z}{n - k} = \frac{x + y}{x + y + nz} \binom{x + y + nz}{n}.$$

This paper is motivated by the question of Mansour and Sun. We introduce the notion of arithmetic progression blocks or AP-blocks of $\mathbb{Z}_n$. A sequence of the form

$$(x, x + m, x + 2m, \ldots, x + (i - 1)m) \pmod{n}$$

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is called an AP-block, or an m-AP-block, of length i and of difference m. Then we consider partitions of $\mathbb{Z}_n$ into m-AP-blocks $B_1, B_2, \ldots, B_k$ of the same difference m. The type of such a partition is referred to as the type of the multisets of the sizes of the blocks. Our main result shows that subject to a technical condition, the number of partitions of $\mathbb{Z}_n$ into m-AP-blocks of a given type is independent of m and is equal to the multinomial coefficient.

This paper is organized as follows. In Section 2, we give a review of the cycle dissections and make a connection between the Kaplansky numbers and the cyclic multinomial coefficients. We present the main result in Section 3, that is, subject to a technical condition, the number of partitions of $\mathbb{Z}_n$ into m-AP-blocks of a given type equals the multinomial coefficient and does not depend on m. We present a separation algorithm which leads to a bijection between m-AP-partitions and m'-AP-partitions of $\mathbb{Z}_n$. The correspondence between m-AP-partitions and cycle dissections ($m' = 1$) implies the main result Theorem 3.1. For the type $1^{n-(p+1)k}(p+1)^k$ we are led to a combinatorial proof which answers the question of Mansour and Sun. In the last section, we deal with an improvement of our main theorem recently given by Guo and Zeng [8]. It turns out that a variant of the cycle lemma is needed to improve the separation algorithm under the condition of Guo and Zeng.

2 Cycle Dissections

In the combinatorial study of Waring's formula on symmetric functions, Chen, Lih and Yeh [2] introduced the notion of cycle dissections. Recall that a *dissection of an n-cycle* is a partition of the cycle into blocks, which can be viewed by putting cutting bars on some edges of the cycle. Note that there at least one bar to cut a cycle into straight segments. A dissection of an n-cycle is said of *type* $1^{k_1}2^{k_2}\cdots n^{k_n}$ if there are k_i blocks of i elements in it. For instance, Figure 1 gives a 20-cycle dissection of type $1^82^33^2$.
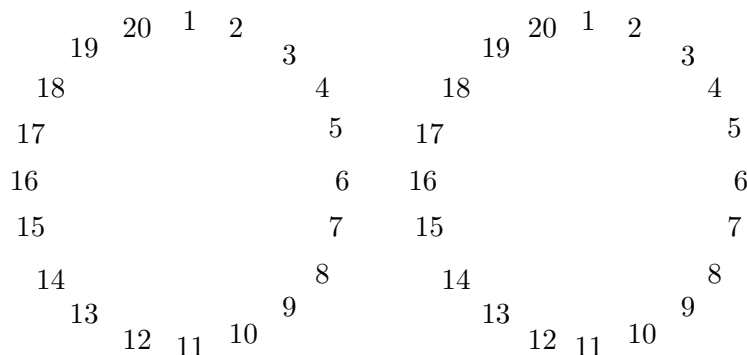

Figure 1: A 20-cycle dissection of type $1^82^33^2$.

The following lemma is due to Chen-Lih-Yeh [2, Lemma 3.1].

Lemma 2.1. *For an n-cycle, the number of dissections of type* $1^{k_1}2^{k_2}\cdots n^{k_n}$ *is given by the* cyclic multinomial coefficients:

$$\frac{n}{k_1 + \cdots + k_n}\binom{k_1 + \cdots + k_n}{k_1, \ldots, k_n}. \tag{2.1}$$

This lemma is easy to prove. Given a dissection, one may pick up any segment as a distinguished segment. This can be done in $k_1 + k_2 + \cdots + k_n$ ways. On the other hand, any of the n elements can serve as the first element of the distinguished segment.

Consider a cycle dissection of type $1^{n-(p+1)k}(p+1)^k$. The set of the first elements of each segment of length $p+1$ corresponds a k-subset of $\mathbb{Z}_n$ satisfying (1.2). Thus the cyclic multinomial coefficient of type $1^{n-(p+1)k}(p+1)^k$ reduces to (1.3) and particularly the cyclic multinomial coefficient of type $1^{n-2k}2^k$ reduces to the Kaplansky number (1.1).

3   Partitions of $\mathbb{Z}_n$ into Arithmetic Progressions

In this section, we present the main result of this paper, namely, a formula for the number of partitions of $\mathbb{Z}_n$ into m-AP-blocks of a given type. The proof is based on a separation algorithm to transform an m-AP-partition to an m'-AP-partition.

We begin with some concepts. First, $\mathbb{Z}_n$ is considered as a directed cycle. An arithmetic progression block, or an AP-block of $\mathbb{Z}_n$, is defined to be a sequence of elements of $\mathbb{Z}_n$ of the following form

$$B = (x, x+m, x+2m, \ldots, x+(i-1)m) \pmod{n},$$

where m is called the *difference* and i is called the *length* of B. An AP-block of difference m is called an m-AP-block. If B contains only one element, then it is called a *singleton*. The first element x is called the *head* of B. An m-AP-partition, or a partition of $\mathbb{Z}_n$ into m-AP-blocks, is a set of m-AP-blocks of $\mathbb{Z}_n$ whose underlying sets form a partition of $\mathbb{Z}_n$. For example,

$$(7, 9, 11), \ (8), \ (10, 12), \ (1), \ (2, 4, 6), \ (3), \ (5) \tag{3.1}$$

is a 2-AP-partition of $\mathbb{Z}_{12}$ with four singletons and three non-singleton heads 7, 10 and 2.

It should be noted that different AP-blocks may correspond to the same underlying set. For example, $(1, 3)$ and $(3, 1)$ are regarded as different AP-blocks of $\mathbb{Z}_4$, but they have the same underlying set $\{1, 3\}$. For example, given the difference $m = 3$, the AP-block $(12, 15, 2, 5, 8)$ of $\mathbb{Z}_{16}$ is uniquely determined by the underlying set $\{2, 5, 8, 12, 15\}$ since there is only one way to order these five elements to form an arithmetic progression of difference 3 modulo 16.

For an m-AP-partition π, the *type* of π is defined by the type of the multisets of the sizes of the blocks. Usually, we use the notation $1^{k_1}2^{k_2}\cdots n^{k_n}$ to denote a type for which there are k_1 blocks of size one, k_2 blocks of size two, etc. However, for the sake of presentation, we find it more convenient to ignore the zero exponents and express a type in the form $i_1^{k_1}i_2^{k_2}\cdots i_r^{k_r}$, where $1 \le i_1 < i_2 < \cdots < i_r$ and all $k_j \ge 1$. For example, the AP-partition (3.1) is of type $1^42^13^2$.

Throughout this paper, we restrict our attention to m-AP-partitions with at least one singleton block and also at least one non-singleton block, namely, $i_1 = 1$ and $r \ge 2$ in the above notation of types. Here is the aforementioned condition:

$$\left\lceil \frac{k_1}{k_2 + \cdots + k_r} \right\rceil \ge (m-1)(i_r - 1), \tag{3.2}$$

where the notation $\lceil x \rceil$ for a real number x stands for the smallest integer that is larger than or equal to x. Obviously, the condition (3.2) holds for $m = 1$. For $m \ge 2$, (3.2) is equivalent to the relation

$$k_1 \ge (k_2 + \cdots + k_r)\big[(m-1)(i_r - 1) - 1\big] + 1. \tag{3.3}$$

We prefer the form (3.2) for a reason that will become clear in the combinatorial argument in the proof of Theorem 3.1. In fact on an n-cycle dissection, the $\sum_{j=2}^{r} k_j$ non-singleton heads divide the k_1 singletons into $\sum_{j=2}^{r} k_j$ segments. By virtue of the pigeonhole principle, there exists a segment containing at least $(m-1)(i_r-1)$ singletons.

For example in the AP-partition (3.1), the three non-singleton heads divide the four singletons into three segments and therefore there exists one segment containing at least 2 singletons. In this particular partition it is the path from 2 to 7 that contains two singletons 3 and 5, see the right cycle in Figure 2.

Theorem 3.1. *Given a type $1^{k_1}i_2^{k_2}\cdots i_r^{k_r}$ satisfying the condition (3.2), the number of m-AP-partitions of $\mathbb{Z}_n$ does not depend on m, and is equal to the cyclic multinomial coefficient*

$$\frac{n}{k_1 + \cdots + k_r}\binom{k_1 + \cdots + k_r}{k_1, \ldots, k_r}. \tag{3.4}$$

In fact, Theorem 3.1 reduces to Theorem 1.1 when we specialize the type to $1^{n-(p+1)k}(p+1)^k$. In this case the condition (3.2) becomes $n \ge kmp + 1$. The heads of the k AP-blocks of length $p+1$ satisfy the condition (1.4). Conversely, any k-subset of $\mathbb{Z}_n$ satisfying (1.4) determines an m-AP-partition of the given type. The cyclic multinomial coefficient (3.4) agrees with the formula (1.3) of Theorem 1.1. For example, given the type $1^42^13^2$ and difference 2, the AP-partition (3.1) is determined by the selection of $\{7, 10, 2\}$ as heads from $\mathbb{Z}_{12}$.

Note that the cyclic multinomial coefficient (3.4) has occurred in Lemma 2.1. Indeed, Lemma 2.1 is the special case of Theorem 3.1 for $m = 1$. We proceed to describe an algorithm, called the *separation algorithm*, to transform m-AP-partitions

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to m'-AP-partitions of the same type $T = i_1^{k_1} i_2^{k_2} \cdots i_r^{k_r}$, assuming the following condition holds:

$$\left\lceil \frac{k_1}{k_2 + \cdots + k_r} \right\rceil \geq (\max\{m, m'\} - 1)(i_r - 1). \qquad (3.5)$$

The separation algorithm enables us to verify Theorem 3.1. We will state our algorithm for m-AP-partitions and m'-AP-partitions, instead of restricting m' to one, because it is more convenient to present the proof by exchanging the role of m and m'.

Given a type $T = 1^{k_1} i_2^{k_2} \cdots i_r^{k_r}$, let $\mathcal{P}_m$ be the set of m-AP-partitions of type T. To prove Theorem 3.1, it suffices to show that there is a bijection between $\mathcal{P}_m$ and $\mathcal{P}'_m$ under the condition (3.5).

Let $\pi \in \mathcal{P}_m$. Denote by $H(\pi)$ the set of heads in π. For each head h of π, we consider the nearest non-singleton head in the counterclockwise direction, denoted h^*. Then we denote by $g(h)$ the number of singletons lying on the path from h^* to h under the convention that h is not counted by $g(h)$. For example, for the AP-partition π' on the right of Figure 2, we have $H(\pi') = \{1, 2, 3, 5, 7, 8, 10\}$, $g(1) = g(3) = g(8) = 0$, $g(2) = g(5) = g(10) = 1$ and $g(7) = 2$. The values $g(h)$ will be needed in the separation algorithm.
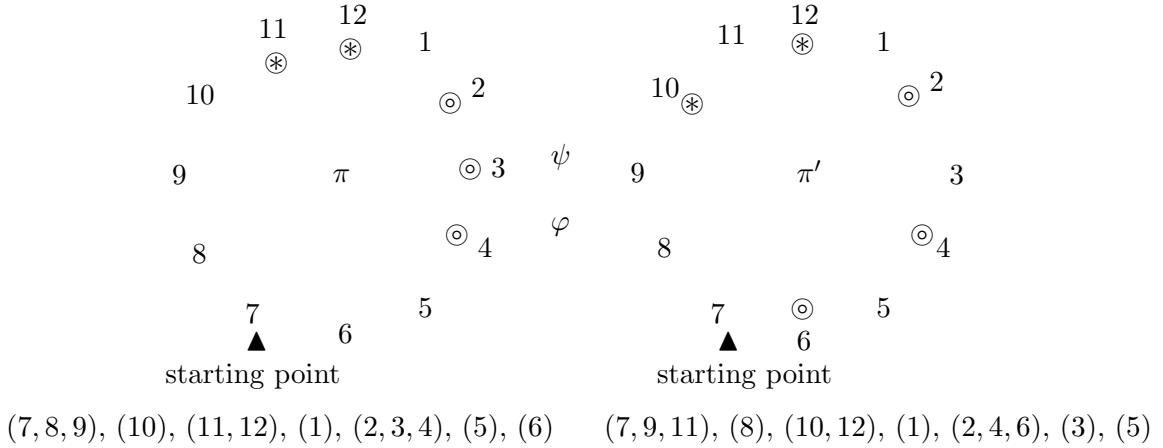

$$(7, 8, 9), (10), (11, 12), (1), (2, 3, 4), (5), (6) \qquad (7, 9, 11), (8), (10, 12), (1), (2, 4, 6), (3), (5)$$

Figure 2: The algorithms ψ and φ for $T = 1^4 2^1 3^2$, $m = 1$ and $m' = 2$.

The Separation Algorithm. Let π be an m-AP-partition of type T. As the first step, we choose a head h_1 of π, called the *starting point*, such that $g(h_1)$ is the maximum. Then we impose a linear order on the elements of $\mathbb{Z}_n$ with respect to the choice of h_1:

$$h_1 < h_1 + 1 < h_1 + 2 < \cdots < h_1 - 1 \qquad (\text{mod } n). \qquad (3.6)$$

In accordance with the above order, we denote the heads of π by $h_1 < h_2 < \cdots < h_t$, where $t = \sum_{i=1}^{r} k_i$. The m-AP-block of π with head h_i is denoted by B_i. Let l_i be the length of B_i, and so $\sum_{i=1}^{t} l_i = n$.

We now aim to construct m'-AP-blocks $B'_1, B'_2, \ldots, B'_t$ such that B'_i has the same number of elements as B_i. We begin with B'_1 by setting $h'_1 = h_1$ and letting B'_1 be the m'-AP-block of length l_1, namely,

$$B'_1 = (h'_1, \ h'_1 + m', \ \ldots, \ h'_1 + (l_1 - 1)m') \,.$$

Among the remaining elements, namely, those that are not in B'_1, we choose the smallest element with respect to (3.6), denoted by h'_2, and let B'_2 be the m'-AP-block of length l_2 with head h'_2. Repeating the above procedure, as will be justified later, after t steps we obtain an m'-AP-partition, denoted $\psi(\pi)$, of type T with blocks $B'_1, B'_2, \ldots, B'_t$.

Figure 2 illustrates the separation algorithm from a 1-AP-partition π to a 2-AP-partition π' of the same type $T = 1^4 2^1 3^2$ and vice versa. The solid dots stand for singletons, whereas the other symbols represent different AP-blocks.

We remark that, as indicated by the example, the starting point can never be a singleton. In fact, if s is a singleton and h is a non-singleton head such that all the heads lying on the path from s to h are singletons, then we have the relation $g(h) > g(s)$. Since $g(h_1)$ is maximum, we see that the starting point is always a non-singleton head.

Clearly, it is necessary to demonstrate that the above algorithm ψ is valid, namely, we need to justify that underlying sets of the blocks $B'_1, B'_2, \ldots, B'_t$ are disjoint.

Proposition 3.2. *The mapping ψ is well-defined, and for any $\pi \in \mathcal{P}_m$, we have $\psi(\pi) \in \mathcal{P}_{m'}$.*

Proof. Let $\pi \in \mathcal{P}_m$ with AP-blocks $B_1, B_2, \ldots, B_t$. Without loss of generality, we may assume that $h_1, h_2, \ldots, h_t$ are the heads of $B_1, B_2, \ldots, B_t$, where h_1 is the starting point for the mapping ψ and $h'_1, h'_2, \ldots, h'_t$ are the corresponding heads generated by ψ. Let l_i be the length of B_i. Suppose to the contrary that there exist two heads h_i and h_j $(i < j)$ such that

$$h'_i + am' \equiv h'_j + bm' \pmod{n},$$

where $0 \leq a \leq l_i - 1$ and $0 \leq b \leq l_j - 1$.

If $a \geq b$, then $0 \leq a - b \leq l_i - 1$ and $h'_j \equiv h'_i + (a - b)m' \pmod{n}$. But the point $h'_i + (a - b)m'$ is in B'_i, contradicting the choice of h'_j. This yields $a < b$ and thus $0 \leq b - a \leq l_j - 1$.

We claim that the starting point h_1 lies on the path from h'_j to h'_i. In fact, when the Algorithm ψ is at the j-th step to deal with the head h_j, all the points smaller than h'_i lie in one of the blocks $B'_1, B'_2, \ldots, B'_i$. Then we see that $h'_j > h'_i$. Meanwhile, there are $n - l_1 - l_2 - \cdots - l_{j-1} > 0$ points which are not contained in $B'_1, B'_2, \ldots, B'_{j-1}$. But the head h'_j is chosen to be the smallest point not in $B'_1, B'_2, \ldots, B'_{j-1}$, we find that h'_j lies on the path from h'_i to h_1.

In addition to h'_i and h'_j, we assume that there are N points on the path from h'_j to h'_i. Since $h'_i \equiv h'_j + (b - a)m' \pmod{n}$ and $1 \leq b - a \leq l_j - 1$, we obtain

$N = (b - a)m' - 1$. On the other hand, at the j-th step, in addition to the point h'_j, there are at least $l_j - 1$ points not contained in $B'_1, B'_2, \ldots, B'_{j-1}$. Similarly, the choice of h_1 and the condition (3.5) yield that the largest $(\max\{m, m'\} - 1)(i_r - 1)$ heads with respect to the order (3.6) are all singletons by the pigeonhole principle. Therefore, there are at least $(\max\{m, m'\} - 1)(i_r - 1)$ points not contained in $B'_1, B'_2, \ldots, B'_{j-1}$. It follows that

$$N \geq (\max\{m, m'\} - 1)(i_r - 1) + (l_j - 1). \tag{3.7}$$

Since $N = (b - a)m' - 1$ and $1 \leq b - a \leq l_j - 1$, we deduce that

$$(m' - 1)(i_r - 1) + (l_j - 1) \leq (b - a)m' - 1 \leq (l_j - 1)m' - 1,$$

leading to the contradiction $l_j > i_r$. This completes the proof. ∎

Proposition 3.3. *Given an m-AP-partition of $\mathbb{Z}_n$, the separation algorithm ψ generates the same m'-AP-partition regardless of the choice of the starting point subject to the maximum property.*

Proof. Let π be an m-AP-partition of $\mathbb{Z}_n$. Suppose that $u_1, u_2, \ldots, u_s$ ($s \geq 2$) are all the heads such that $g(u_1) = g(u_2) = \cdots = g(u_s)$ is the maximum on π. Let u_1 be the starting point and $u_1 < u_2 < \cdots < u_s$ with respect to (3.6).

It suffices to show that when the Algorithm ψ processes u_i ($1 \leq i \leq s$), the m'-AP-blocks which have been generated consist of all the elements smaller than u_i. By induction we assume that this statement holds up to u_{j-1}.

Let $v_q, v_{q-1}, \ldots, v_1, u_j$ be all heads lying on the path Q from u_{j-1} to u_j such that $u_{j-1} = v_q < v_{q-1} < \cdots < v_1 < u_j$. Let B_i be the m-AP-block containing v_i. Let l_i be the length of B_i and

$$B'_i = (v'_i, v'_i + m', \ldots, v'_i + (l_i - 1)m')$$

be the corresponding m'-AP-blocks generated by the Algorithm ψ. It suffices to show that the path Q consists of the elements of $B'_q, B'_{q-1}, \ldots, B'_1$.

Suppose that $v_1, v_2, \ldots, v_p$ are all singletons, but v_{p+1} is not a singleton. Then $p \leq q - 1$ since u_{j-1} is always a non-singleton head. The condition (3.5) yields that

$$p \geq (\max\{m, m'\} - 1)(i_r - 1).$$

We now wish to show that for any $1 \leq i \leq q$, the block B_i lies entirely on the path Q. If $i \leq p$, then $B_i = (v_i)$ is a singleton block lying on Q. Otherwise, we have $i \geq p + 1$ and

$$B_i = (v_i, v_i + m, \ldots, v_i + (l_i - 1)m).$$

But the total number of points between any two consecutive elements of B_i is

$$(l_i - 1)(m - 1) \leq (\max\{m, m'\} - 1)(i_r - 1) \leq p.$$

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Intuitively, all these points can be fulfilled by the singletons $v_p, v_{p-1}, \ldots, v_1$. Since $u_j > v_1$, the largest element $v_i + (l_i - 1)m$ in the block B_i is smaller than u_j. Hence the block B_i ($i = 1, 2, \ldots, q$) lies entirely on Q.

Therefore, the total number of elements in $B_q, B_{q-1}, \ldots, B_1$ equals the length $u_j - u_{j-1}$ of the path Q. Since B'_i has the same number of elements as B_i, the total number of elements in $B'_q, B'_{q-1}, \ldots, B'_1$ also equals $u_j - u_{j-1}$.

Moreover, it can be shown that the block B'_i also lies entirely on the path Q for any $1 \leq i \leq q$. If $i \leq p$, the block $B'_i = (v'_i)$ is a singleton given by the separation algorithm. Since the total number of elements in $B'_q, B'_{q-1}, \ldots, B'_{i+1}$ is smaller than $u_j - u_{j-1}$ and v'_i is chosen to be the smallest element which is not in $B'_q, B'_{q-1}, \ldots, B'_{i+1}$, we see the relation $v'_i < u_j$. Otherwise, we have $i \geq p + 1$ and the total number of points between any two consecutive elements of B'_i equals

$$(l_i - 1)(m' - 1) \leq (\max\{m, m'\} - 1)(i_r - 1) \leq p.$$

Intuitively, all these points can be fulfilled by the singletons $v'_p, v'_{p-1}, \ldots, v'_1$. Since $u_j > v'_1$, the largest element $v'_i + (l_i - 1)m'$ in the block B'_i is smaller than u_j. Consequently, the block B'_i lies entirely on Q.

In summary, the total number of elements in $B'_q, B'_{q-1}, \ldots, B'_1$ which lie on the path Q coincides with the length of Q. Hence the path Q consists of the elements of $B'_q, B'_{q-1}, \ldots, B'_1$. This completes the proof. ∎

Theorem 3.4. *Let T be a type as given before. The separation algorithm induces a bijection between $\mathcal{P}_m$ and $\mathcal{P}_{m'}$ under the condition (3.5).*

Proof. We may employ the separation algorithm by interchanging the roles of m and m' to construct an m-AP-partition from an m'-AP-partition, and we denote this map by φ. We aim to show that φ is indeed the inverse map of ψ, namely, $\varphi(\psi(\pi)) = \pi$ for any $\pi \in \mathcal{P}_m$.

Let $h_1, h_2, \ldots, h_t$ be the heads of π for the map ψ, where h_1 is the starting point. Assume that π has AP-blocks $B_1, B_2, \ldots, B_t$ with h_i being the head of B_i. Let l_i be the length of B_i. By the construction of ψ, the generated heads $h'_1 = h_1, h'_2, \ldots, h'_t$ have the order $h'_1 < h'_2 < \cdots < h'_t$ in accordance with $h_1 < h_2 < \cdots < h_t$. It follows that $g(h'_1)$ is the maximum considering all heads of the AP-partition $\psi(\pi)$.

We now apply the map φ on the m'-AP-partition $\psi(\pi)$ and choose h'_1 as the starting point. Let $h''_1, h''_2, \ldots, h''_t$ be the heads generated by φ respectively. In light of the construction of φ, we have $h''_1 = h'_1 = h_1$ and $h''_1 < h''_2 < \cdots < h''_t$.

For any i, the separation algorithm has the property that the length of the m-AP-block in $\varphi(\psi(\pi))$ containing h''_i is l_i, which is the length of the m-AP-block in π containing h_i.

Note that both $\varphi(\psi(\pi))$ and π are m-AP-partitions. They have the same starting point $h''_1 = h_1$ and the same length sequence $(l_1, l_2, \ldots, l_t)$. Thus for any $i =$

$2, 3, \ldots, t$, the head h_i'' is the smallest point which is not contained in the m-AP-blocks $B_1, B_2, \ldots, B_{i-1}$, and so does h_i. Hence we conclude that $h_i'' = h_i$ and $\varphi(\psi(\pi)) = \pi$. This completes the proof. ∎

4   An Improved Condition and the Cycle Lemma

After the completion of an earlier version of this paper, Guo and Zeng [8] found that the Condition (3.2) in Theorem 3.1 can be replaced by $m|n$ and

$$\Delta = n - m(n - k_1 - k_2 - \cdots - k_r) > 0. \tag{4.1}$$

It should be noted that for $m = 1$ the above condition is automatically satisfied. Guo and Zeng obtained the following improvement of Theorem 3.1.

Theorem 4.1. *Let m, n be positive integers, and let $k_1, k_2, \ldots, k_r$ be non-negative integers such that $n = k_1 + 2k_2 + \cdots + rk_r$. Let $m|n$ and $\Delta = n - m(n - k_1 - \cdots - k_r)$. Then the number of partitions of $\mathbb{Z}_n$ into m-AP-blocks of type $1^{k_1} 2^{k_2} \cdots r^{k_r}$ is given by the cyclic multinomial coefficient (3.4) if $\Delta > 0$.*

More importantly, their Condition (4.1) is not only simpler, but also the best possible. They noted that the number of AP-partitions of $\mathbb{Z}_n$ of a given type for which $\Delta = 0$ or $\Delta = -m$ does not equal the cyclic multinomial coefficient.

The proof of Guo and Zeng is based on a number theoretical argument and Raney-Mohanty's identity. It is natural to consider the question whether the combinatorial treatment in the preceding section can be adopted to deal with this more general case. It turns out that a variant of the cycle lemma will be needed to construct an improved algorithm under the new condition.

The idea of proving Theorem 4.1 is to construct a bijection between $\mathcal{P}_m$ and $\mathcal{P}_{m'}$ under the Condition (4.1), where $\mathcal{P}_m$ and $\mathcal{P}_{m'}$ are defined as in the previous section, via an improved algorithm. To be precise, let $T = 1^{k_1} i_2^{k_2} \cdots i_r^{k_r}$ such that $\Delta > 0$ holds. Let $\mathcal{P}_m$ be the set of m-AP-partitions of type T. Moreover, let

$$\Delta' = n - m'(n - k_1 - k_2 - \cdots - k_r).$$

We assume that $\Delta' > 0$. Similarly, $\mathcal{P}_{m'}$ denotes the set of m'-AP-partitions of the same type T.

First, we may assume that both m and m' divide n. Suppose d is the greatest common divisor of m and n. Guo and Zeng have shown that the number of partitions of m-AP-partitions of $\mathbb{Z}_n$ of a given type equals the number of partitions of d-AP-partitions of $\mathbb{Z}_n$ of the given type. Therefore, we may consider d-AP-partitions when

m is not a divisor of n. For this reason, the condition $m|n$ in Theorem 4.1 is not really a restriction.

Let us denote the the improved algorithm by ψ'. It uses the same strategy as the previous algorithm ψ. The only difference lies in the choice of the starting point. Let $h_1, h_2, \ldots, h_{t'}$ be the non-singleton heads clockwise on the cycle $\mathbb{Z}_n$, where $t' = t - k_1$. For $i = 1, 2, \ldots, t'$, let l_i denote the length of the block containing h_i, and let g_i be the number of singletons on the path from h_i to h_{i+1} with the convention that $h_{t'+1} = h_1$. For $i = 1, 2, \ldots, t'$, put

$$x_i = g_i - (l_i - 1)\max(m, m') + l_i. \tag{4.2}$$

Note that the values of x_i may be negative. The starting point h_j is chosen subject to the following conditions:

$$\begin{cases} x_{j-1} > 0, \\ x_{j-2} + x_{j-1} > 0, \\ \ldots \\ x_1 + x_2 + \cdots + x_{j-1} > 0, \\ x_{t'} + x_1 + x_2 + \cdots + x_{j-1} > 0, \\ \ldots \\ x_j + x_{j+1} + \cdots + x_{t'} + x_1 + \cdots + x_{j-1} > 0. \end{cases} \tag{4.3}$$

First of all, it is necessary to verify the existence of the index j subject to the conditions (4.3). It turns out that a variant of the cycle lemma (see Dershowitz and Zaks [5], Dvoretzky and Motzkin [6]) will play a crucial role in justifying the choice of the starting point. For the sake of completeness, a proof will be provided.

Lemma 4.2. *Let $x_1, x_2, \ldots, x_n$ be integers. Let $x_{k+n} = x_k$ and write the cyclic permutation $(x_k, x_{k+1}, \ldots, x_{k+n})$ as $x(k)$ where $k = 1, 2, \ldots, n$. If $\sum_{i=1}^n x_i > 0$, then there exists a cyclic permutation $x(j)$ such that for any $p = j, j+1, \ldots, j+n-1$, the sum $x_j + x_{j+1} + \cdots + x_p$ is positive.*

Proof. Let $y_i = x_1 + x_2 + \cdots + x_i$ for $1 \le i \le n$. Let w be the minimum value of y_i for $i = 1, 2, \ldots, n$. Furthermore, let j' be the maximum number such that $1 \le j' \le n$ and $y_{j'}$ equals the minimum value w. We assume that elements $x_1, x_2, \ldots, x_n$ are arranged on a cycle, that is, $x(n+1) = x(1)$. Let $j = j' + 1$. It is easy to check that the cyclic permutation $x(j)$ meets the requirement of the lemma, that is, the sum $x_j + x_{j+1} + \cdots + x_p$ is positive for any $p = j, j+1, \ldots, j+n-1$. In fact, if $j = n+1$, or equivalently, $j = 1$, then by definition y_n equals the minimum value w and we see that for any $p = 1, 2, \ldots, n$,

$$x_1 + x_2 + \cdots + x_p = y_p \ge y_n > 0,$$

because of the condition $y_n = x_1 + x_2 + \cdots + x_n > 0$. Otherwise, we may assume that $2 \le j \le n$. If $j \le p \le n$, we have

$$x_j + x_{j+1} + \cdots + x_p = y_p - y_{j-1} = y_p - y_{j'} > 0$$

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since j' is chosen to be the maximum index such that $y_{j'} = w$. We now consider the case $n + 1 \le p \le j + n - 1$. Let $q = p - n$. Clearly, $1 \le q \le j - 1$. Then we have

$$
\begin{aligned}
x_j + x_{j+1} + \cdots + x_p &= x_j + \cdots + x_n + x_1 + \cdots + x_q. \\
&= \sum_{i=1}^{n} x_i - (x_{q+1} + x_{q+2} + \cdots + x_{j-1}) \\
&= y_n - (y_{j-1} - y_p) > 0,
\end{aligned}
$$

since $y_n > 0$ and $y_{j-1} - y_p = y_{j'} - y_p \le 0$. This completes the proof. ∎

We are now ready to show that the starting point h_j can be found by the above algorithm. Since $\Delta, \Delta' > 0$, by the definition (4.2) of x_i, we see that

$$
\begin{aligned}
\sum_{i=1}^{t'} x_i &= \sum_{i=1}^{t'} [g_i - (l_i - 1) \max(m, m') + l_i] \\
&= k_1 - (n - t' - k_1) \max(m, m') + (n - k_1) \\
&= n - \max(m, m')(n - k_1 - \cdots - k_r) > 0.
\end{aligned}
$$

Since the sum of x_i's is positive, we can apply the cycle lemma to the sequence $x_{t'}, x_{t'-1}, \ldots, x_1$ and deduce that there exists an index j satisfying the conditions (4.3). Without loss of generality, we can write such a starting point as h_1.

It remains to show that the improved Algorithm ψ' is well-defined and it induces a bijection between $\mathcal{P}_m$ and $\mathcal{P}_{m'}$. We will only sketch the proofs, since they are similar to those in the previous section.

Proposition 4.3. *For any $\pi \in \mathcal{P}_m$, we have $\psi'(\pi) \in \mathcal{P}_{m'}$.*

Proof. As in the proof of Proposition 3.2, let $p \in \{1, 2, \ldots, t'\}$. There are $\sum_{i=p}^{t'} g_i$ singletons which will be placed on the path from h_p to h_1. Thus by the conditions (4.3), we have

$$
\sum_{i=p}^{t'} g_i \ge \sum_{i=p}^{t'} [(l_i - 1)m' - l_i] + 1 \ge (l_p - 1)m' - l_p + 1 = (l_p - 1)(m' - 1).
$$

It follows that the point $h'_p + (l_p - 1)m'$ lies on the path from h_p to h_1 clockwise. This completes the proof. ∎

Proposition 4.4. *Given an m-AP-partition of $\mathbb{Z}_n$, the improved algorithm ψ' generates the same m'-AP-partition regardless of the choice of the starting point subject to the conditions (4.3).*

Proof. Suppose that both h_1 and h_j $(2 \leq j \leq t')$ can be chosen as the starting point of the Algorithm ψ' subject to the conditions (4.3). Now we take h_1 as the starting point and apply the improved Algorithm ψ'. We claim that it is the same bijection if we choose h_j as the starting point instead. For h_j being the starting point, we need the condition $x_1 + x_2 + \cdots + x_{j-1} > 0$, or equivalently,

$$\sum_{i=1}^{j-1} g_i > \sum_{i=1}^{j-1} [(l_i - 1)m' - l_i].$$

Intuitively, the number of singletons are large enough to fulfill the gaps generated by any m'-AP-block with head h'_i $(1 \leq i \leq j-1)$. So if we apply the improved algorithm starting with h_1, the image h'_j of the point h_j coincides with the point h_j itself. So in the remaining process of applying the improved algorithm ψ', everything is the same as using the algorithm starting with h_j. This completes the proof. ∎

We now arrive at the last theorem of this paper.

Theorem 4.5. *The improved algorithm ψ' induces a bijection between $\mathcal{P}_m$ and $\mathcal{P}_{m'}$ under the conditions $\Delta > 0$ and $\Delta' > 0$.*

Proof. We may employ the improved algorithm by interchanging the roles of m and m' to construct an m-AP-partition from an m'-AP-partition, and we denote this map by φ'. Let h_1 be the starting point of ψ'. In view of the conditions (4.3), we find that

$$\sum_{i=p}^{t'} g_i > \sum_{i=p}^{t'} [(l_i - 1) \max(m, m') - l_i], \quad \forall p = 1, 2, \ldots, t'.$$

Let g'_i be the number of singletons on the path from h'_i to h'_{i+1}. Since the improved algorithm ψ' does not change the relative positions of the heads, it follows that

$$\sum_{i=p}^{t'} g'_i > \sum_{i=p}^{t'} [(l_i - 1) \max(m, m') - l_i], \quad \forall p = 1, 2, \ldots, t'.$$

So we can apply the algorithm φ' on the m'-AP-partition $\psi'(\pi)$ with starting point h'_1. The remaining of this proof is similar to that of Theorem 3.4. ∎

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