Long Cycles in 4-connected Planar Graphs

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Abstract

Let G be a 4-connected planar graph on n vertices. Malkevitch conjectured that if G contains a cycle of length 4, then G contains a cycle of length k for every $k \in \{n, n-1, \ldots, 3\}$. This conjecture is true for every $k \in \{n, n-1, \ldots, n-6\}$ with $k \geq 3$. In this paper, we prove that G also has a cycle of length $n-7$ provided $n \geq 10$.

Keywords: Hamilton cycle; Tutte path; contractible subgraph

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1 Introduction and notation

Whitney [10] proved that every 4-connected planar triangulation contains a Hamilton cycle. Tutte [8] extended Whitney's result to every 4-connected planar graph. Malkevitch [2] conjectured that every 4-connected planar n-vertex graph contains a cycle of length k for every $k \in \{n, n-1, \ldots, 3\}$ if it contains a 4-cycle. Note that the line graph of a cyclically 4-edge-connected cubic planar graph with girth at least 5 contains no cycle of length 4.

Malkevitch's conjecture for $k = n-1$ follows from a theorem of Tutte as observed by Nelson, see [7]. The case for $k = n-2$ was proved by Thomas and Yu [6]. Sanders [5] showed that in any 4-connected planar graph with at least six vertices there are three vertices whose deletion results in a Hamiltonian graph, establishing Malkevitch's conjecture for $k = n-3$. Chen et al. [1] proved Malkevitch's conjecture for $k \in \{n-4, n-5, n-6\}$ with $k \geq 3$. In this paper, we prove the following result.

Theorem 1.1. *Let G be a 4-connected planar graph and let $u \in V(G)$. Then there is a set $X \subseteq V(G)$ such that $u \in X$, $|X| = 6$, and $G - X$ has a Hamilton cycle when $|V(G)| \geq 9$.*

We will show that Theorem 1.1 implies that G contains a cycle of length $n-7$ for all $n \geq 10$ (see Corollary 4.1). The proof of Theorem 1.1 is similar to that in [1], in which the notion of Tutte paths and contractible subgraphs technique are used. Let G be a graph and let $H \subseteq G$. We use G/H to denote the graph obtained from G by contracting H. If H is induced by an edge e, then we write G/e instead of G/H. A subgraph H in a k-connected graph G is said to be *k-contractible* (or *contractible*) if the graph G/H is also k-connected. A graph X is a *minor* of G (or G contains an *X-minor*) if X can be obtained from a subgraph of G by contracting edges. Note that a graph is planar iff it has no K_5-minor or $K_{3,3}$-minor.

Let $X \subseteq E(G)$ (or $X \subseteq V(G)$). We use $G - X$ to denote the graph obtained from G by deleting X (and the edges of G incident with elements of X), and if $X = \{x\}$ then let $G - x := G - \{x\}$. Let P be a path (cycle) in G. A *P-bridge* of G is a subgraph of G which either (1) is induced by an edge of $G - E(P)$ with both incident vertices in $V(P)$ or (2) is induced by the edges in a component D of $G - V(P)$ and all edges between D and P. For a P-bridge B of G, the vertices of $B \cap P$ are the *attachments* of B on P. We say that P is a *Tutte path (cycle)* in G if every P-bridge of G has at most three attachments on P. For any subgraph C of G, P is called a *C-Tutte path (cycle)* in G if P is a Tutte path (cycle) in G and every P-bridge of G containing an edge of C has at most two attachments on P. Note that if P is a Tutte path in a 4-connected graph and $|V(P)| \geq 4$, then P is in fact a Hamilton path.

We consider only simple graphs and use the notation and terminology in [1]. Let G be a graph and let $X \subseteq V(G)$. We use $G[X]$ to denote the subgraph of G induced by X.

Let Z be a set of 2-element subsets of $V(G)$; then we use $G+Z$ to denote the graph with vertex set $V(G)$ and edge set $E(G) \cup Z$, and if $Z = \{\{x,y\}\}$ then let $G+xy := G+Z$. Let $N_G(X) := \{u \in V(G)-X : u \text{ is adjacent to some vertex in } X\}$, and if $X = \{x\}$ then let $N_G(x) := N_G(\{x\})$. For any path P and $x, y \in V(P)$, we use xPy to denote the subpath of P between x and y. Given two distinct vertices x and y on a cycle C in a plane graph, we use xCy to denote the path in C from x to y in clockwise order. It is well known that every face of a 2-connected plane graph is bounded by a cycle.

2 Known results

In this section, we list several results about Tutte paths and contractible subgraphs. The following lemma is shown in [4] and [6].

Lemma 2.1. *Let G be a 2-connected plane graph with a facial cycle C. Let $e, f, g \in E(C)$, and assume that e, f, g occur on C in clockwise order. Then G contains a C-Tutte cycle P through e, f and g.*

A *block* of a graph H is either (1) a maximal 2-connected subgraph of H or (2) a subgraph of H induced by an edge of H not contained in any cycle. An *end block* of a graph H is a block of H containing at most one cut vertex of H. We say that a connected graph H is a *chain of blocks* if H has at most two end blocks. A connected graph H is a *chain of blocks from x to y* if one of the following holds: (1) H is 2-connected and x and y are distinct vertices of H; or (2) H has exactly two end blocks, neither x nor y is a cut vertex of H, and x and y belong to different end blocks of H. Note that if H is not a chain of blocks from x to y, then there exist an end block B of H and a cut vertex b of H such that $b \in V(B)$ and $(V(B) - \{b\}) \cap \{x, y\} = \emptyset$.

Let G be a graph and $\{a_1, \ldots, a_l\} \subseteq V(G)$, where l is a positive integer. We say that $(G, a_1, \ldots, a_l)$ is *planar* if G can be drawn in a closed disc with no pair of edges crossing such that $a_1, \ldots, a_l$ occur on the boundary of the disc in cyclic order. The graph G is called $(4, \{a_1, \ldots, a_l\})$*-connected* if $|V(G)| \geq l+1$ and for any $T \subseteq V(G)$ with $|T| \leq 3$, every component of $G-T$ contains some element of $\{a_1, \ldots, a_l\}$. Note that if G is 4-connected, then G is $(4, S)$-connected for all $S \subseteq V(G)$ with $S \neq V(G)$.

The following four lemmas are proved in [1], using Tutte paths technique.

Lemma 2.2. *Let G be a graph and $\{a_1, a_2, a_3, a_4\} \subseteq V(G)$ such that G is $(4, \{a_1, a_2, a_3, a_4\})$-connected. Then $G - \{a_3, a_4\}$ is a chain of blocks from a_1 to a_2.*

Lemma 2.3. *Let H be a graph and $\{a_1, a_2, a_3, a_4\} \subseteq V(H)$. Assume that (H, a_1, a_2, a_3, a_4) is planar, H is $(4, \{a_1, a_2, a_3, a_4\})$-connected, and a_1 has at least two neighbors contained in $V(H) - \{a_1, a_2, a_3, a_4\}$. Then one of the following holds:*

(1) *$H - \{a_2, a_3, a_4\}$ is 2-connected; or*

(2) *both $H - \{a_1, a_3, a_4\}$ and $H - \{a_1, a_2, a_3\}$ are 2-connected.*

Lemma 2.4. *Let G be a graph and $\{a_1, \dots, a_l\} \subseteq V(G)$, where $3 \le l \le 5$. Assume that $(G, a_1, \dots, a_l)$ is planar, G is $(4, \{a_1, \dots, a_l\})$-connected, and $G - \{a_3, \dots, a_l\}$ is a chain of blocks from a_1 to a_2. Then $G - \{a_3, \dots, a_l\}$ has a Hamilton path from a_1 to a_2.*

Lemma 2.5. *Let H be a graph and $\{a_1, a_2, a_3, a_4\} \subseteq V(H)$. Assume that (H, a_1, a_2, a_3, a_4) is planar, H is $(4, \{a_1, a_2, a_3, a_4\})$-connected, and $|V(H)| \ge 6$. Then there is a vertex $z \in V(H) - \{a_1, a_2, a_3, a_4\}$ such that $H - \{z, a_3, a_4\}$ has a Hamilton path from a_1 to a_2.*

We now state some results on contractible subgraphs. Tutte [9] proved that K_4 is the only 3-connected graph with no 3-contractible edges. On the other hand, there are infinitely many 4-connected graphs with no 4-contractible edges. Martinov [3] showed that if G is a 4-connected graph with no contractible edges, then G is either the square of a cycle of length at least 4 or the line graph of a cyclically 4-edge-connected cubic graph. Chen et al. [1] proved the following result which provides information about 4-contractible edges incident with a specific vertex in a 4-connected planar graph.

Theorem 2.6. *Let G be a 4-connected planar graph and let $u \in V(G)$. Then one of the following holds:*

(1) *G has a contractible edge incident with u; or*

(2) *there are two 4-cuts S and T of G such that $1 \le |S \cap T| \le 2$, S contains u and a neighbor of u, T contains u and a neighbor of u, and $G - S$ has a component consisting of only one vertex which is also contained in T.*

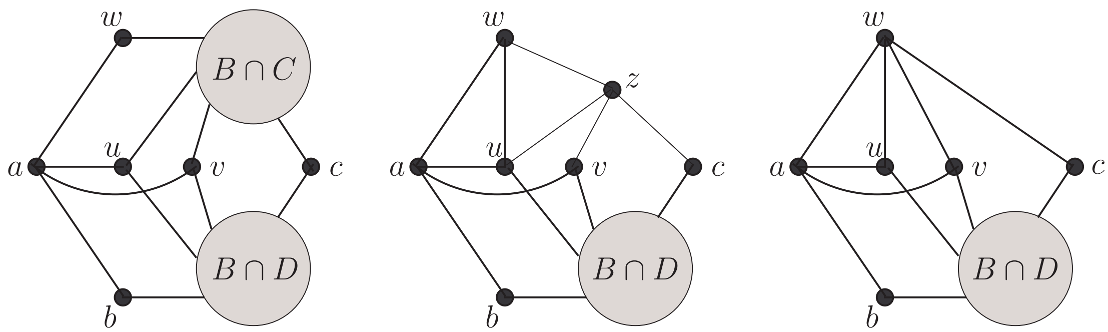
Theorem 2.6 is used in [1] to prove the following result.

Theorem 2.7. *Let G be a 4-connected planar graph and let $u \in V(G)$. Then for each $l \in \{1, \dots, 5\}$ there is a set $X_l \subseteq V(G)$ such that $u \in X_l$, $|X_l| = l$, and $G - X_l$ has a Hamilton cycle when $|V(G)| \ge l + 3$.*

3 A lemma

In this section, we prove the following special case of Theorem 1.1, which deals with a situation in (2) of Theorem 2.6.

Lemma 3.1. *Let G be a 4-connected planar graph and let $u \in V(G)$. Let S, T be two 4-cuts of G such that $|S \cap T| = 2$, $u \in S \cap T$, S has a neighbor of u, and $G - S$ has a component A consisting of only one vertex which is also contained in T. Then there*



$|V(B\cap C)| \geq 2$ $\quad$ $|V(B\cap C)| = 1$ $\quad$ $|V(B\cap C)| = 0$

Figure 1: $|S \cap T| = 2$.

is a set $X \subseteq V(G)$ *such that* $u \in X$*,* $|X| = 6$*, and* $G - X$ *has a Hamilton cycle when* $|V(G)| \geq 9$*.*

Proof. Let a be the only vertex in $V(A)$, and let $B := G - (\{a\} \cup S)$. Let C be a component of $G - T$ and let $D := G - (V(C) \cup T)$. If $S \cap V(C) = \emptyset$, then $B \cap C = C \neq \emptyset$ is a component of $G - (T - \{a\})$, contradicting the assumption that G is 4-connected. Similarly, if $S \cap V(D) = \emptyset$ then $B \cap D = D \neq \emptyset$ is a component of $G - (T - \{a\})$, a contradiction. Hence $S \cap V(C) \neq \emptyset \neq S \cap V(D)$. Therefore $|S \cap V(C)| = 1 = |S \cap V(D)|$. By symmetry, we may assume that $|V(B \cap C)| \leq |V(B \cap D)|$. Let v denote the vertex in $(S \cap T) - \{u\}$, let w denote the vertex in $S \cap V(C)$, let b denote the vertex in $S \cap V(D)$, and let c denote the vertex in $V(B) \cap T$, as shown in Figure 1.

Let $H_1 := G[V(C) \cup \{u, v, c\}]$ and $H_2 := G[V(D) \cup \{u, v, c\}]$. Since $au, av \in E(G)$, in any plane representation of G, a and v are cofacial, and a and u are cofacial. As T is a cut set of G, we see that in any plane representation of G, c and v are cofacial, and c and u are cofacial. Therefore, since a is adjacent to both b and w, (H_1, c, v, w, u) is planar and (H_2, c, v, b, u) is planar. Since G is 4-connected, H_1 is $(4, \{c, v, w, u\})$-connected (if $B \cap C \neq \emptyset$) and H_2 is $(4, \{c, v, b, u\})$-connected (if $B \cap D \neq \emptyset$). Therefore by Lemma 2.2, $H_1 - \{w, u\}$ is a chain of blocks from c to v, and $H_2 - \{b, u\}$ is a chain of blocks from c to v.

Suppose $|V(B \cap C)| \geq 2$. Then by Lemma 2.5, there is a vertex $x \in V(B \cap C)$ such that $H_1 - \{x, w, u\}$ has a Hamilton path P from c to v. Similarly, since $|V(B \cap D)| \geq |V(B \cap C)| \geq 2$, there is a vertex $y \in V(B \cap D)$ such that $H_2 - \{y, b, u\}$ has a Hamilton path Q from c to v. Let $X := \{a, b, u, w, x, y\}$; then $P \cup Q$ is a Hamilton cycle in $G - X$.

Now suppose $|V(B \cap C)| = 1$. Then $|V(B \cap D)| \geq 2$; otherwise, $|V(B \cap D)| = 1$ and $|V(G)| = 8$, and there is nothing to prove. Let z denote the only vertex in $V(B \cap C)$.

We may assume that c has at least two neighbors in $V(B \cap D)$. Otherwise, since G is 4-connected, c is adjacent to at least one element of $\{v, b, w\}$. If c is adjacent to v, then since $|V(B \cap D)| \geq 2$, it follows from Lemma 2.5 that there is a vertex $y \in V(B \cap D)$ such that $H_2 - \{y, b, u\}$ has a Hamilton path Q from c to v. Let $X := \{a, b, u, w, y, z\}$; then $Q + cv$ is a Hamilton cycle in $G - X$. If c is adjacent to b, then by contracting ab, contracting wz, and contracting $B \cap D$ to a single vertex, we produce a minor of G containing $K_{3,3}$, a contradiction. If c is adjacent to w, then by contracting aw, and contracting D to a single vertex, we produce a minor of G containing $K_{3,3}$, again a contradiction.

Hence by Lemma 2.3, there is some $x \in \{v, c\}$ such that $H_2 - (\{v, b, u, c\} - \{x\})$ is 2-connected. Choose a vertex x' of $H_2 - (\{v, b, u, c\} - \{x\})$ such that xx' is an edge and H_2 can be drawn in a closed disc so that that xx' lies on the boundary and $x, x', \{v, b, u, c\} - \{x\}$ occur in cyclic order on the boundary of the disc. By applying Lemma 2.4, we find a Hamilton path R from x to x'. Let $X := \{a, b, u, w, z\} \cup (\{v, c\} - \{x\})$; then $R + xx'$ is a Hamilton cycle in $G - X$.

Therefore we may assume that $|V(B \cap C)| = 0$. Then $|V(B \cap D)| \geq 3$; otherwise, $|V(G)| \leq 8$, and there is nothing to prove. Since H_2 is $(4, \{c, v, b, u\})$-connected and (H_2, c, v, b, u) is planar, $B \cap D$ is connected. We consider two cases according to the connectivity of $B \cap D$.

Case 1. $B \cap D$ is connected but not 2-connected.

Let $J_1, \ldots, J_m$ ($m \geq 2$) be the end blocks of $B \cap D$, and let v_i be the cut vertex of $B \cap D$ contained in $V(J_i)$. We claim that $m = 2$. Otherwise, since H_2 is $(4, \{c, v, b, u\})$-connected, at least three elements of $\{c, v, b, u\}$ have neighbors in $V(J_i) - \{v_i\}$ (for each i), which contradicts the assumption that (H_2, c, v, b, u) is planar.

Let $B_1 := J_1 - v_1$ and $B_2 := (B \cap D) - V(J_1)$. Since H_2 is $(4, \{c, v, b, u\})$-connected and (H_2, c, v, b, u) is planar, either each element of $\{v, u\}$ has neighbors in both $V(B_1)$ and $V(B_2)$ or each element of $\{c, b\}$ has neighbors in both $V(B_1)$ and $V(B_2)$. Moreover, exactly three elements of $\{c, v, b, u\}$ have neighbors in each $V(B_i)$. We only consider the case that each element of $\{v, u\}$ has neighbors in both $V(B_1)$ and $V(B_2)$; the other case can be treated in a similar way (by exchanging the roles of c and v and by exchanging the roles of b and u). Then by planarity, those neighbors of b in $V(B \cap D)$ are contained in $V(B_1)$, and those neighbors of c in $V(B \cap D)$ are contained in $V(B_2)$.

Let $L_1 := G[V(B_1) \cup \{v, v_1, u, b\}]$ and let $L_2 := G[V(B_2) \cup \{v, v_1, u, c\}]$. Note that (L_1, v, v_1, u, b) and (L_2, v, v_1, u, c) are planar. Since H_2 is $(4, \{c, v, b, u\})$-connected, L_1 is $(4, \{v, v_1, u, b\})$-connected and L_2 is $(4, \{v, v_1, u, c\})$-connected. Then by Lemma 2.2, $L_1 - \{u, b\}$ is a chain of blocks from v to v_1, and $L_2 - \{u, c\}$ is a chain of blocks from v to v_1. By applying Lemma 2.4, $L_1 - \{u, b\}$ has a Hamilton path R_1 from v to v_1, and $L_2 - \{u, c\}$ has a Hamilton path R_2 from v to v_1.

Suppose $|V(B \cap D)| = 3$. Then $B \cap D$ is a path $x_1x_2x_3$, where $V(B_1) = \{x_1\}$,

$V(B_2) = \{x_3\}$, and $x_2 = v_1$. Since G is 4-connected and by planarity, x_1 is adjacent to each element of $\{v, u, b\}$, x_2 is adjacent to both v and u, and x_3 is adjacent to each element of $\{v, u, c\}$. Let $X := \{a, b, c, u, w, x_1\}$; then vx_2x_3v is a Hamilton cycle in $G - X$.

So we may assume that $|V(B \cap D)| \geq 4$. If $|V(B_1)| \geq 2$, then by Lemma 2.5, there is a vertex $z_1 \in V(B_1)$ such that $L_1 - \{z_1, u, b\}$ has a Hamilton path R_1' from v to v_1. Let $X := \{a, b, c, u, w, z_1\}$; then $R_1' \cup R_2$ is a Hamilton cycle in $G - X$. Otherwise, if $|V(B_2)| \geq 2$, then there is a vertex $z_2 \in V(B_2)$ such that $L_2 - \{z_2, u, c\}$ has a Hamilton path R_2' from v to v_1. Let $X := \{a, b, c, u, w, z_2\}$; then $R_1 \cup R_2'$ is a Hamilton cycle in $G - X$.

Case 2. $B \cap D$ is 2-connected.

Let F denote the outer cycle of $B \cap D$. Choose $v_1, v_2, v_3, v_4 \in V(F)$ such that v_1, v_2, v_3, v_4 occur on F in clockwise order, $N_G(v) \cap V(F) \subseteq V(v_1Fv_2)$, $N_G(c) \cap V(F) \subseteq V(v_2Fv_3)$, $N_G(u) \cap V(F) \subseteq V(v_3Fv_4)$, and $N_G(b) \cap V(F) \subseteq V(v_4Fv_1)$.

By Lemma 2.1, we find an F-Tutte cycle H in $B \cap D$ through three edges on F incident with v_2, v_3, v_4, respectively. If H is a Hamilton cycle in $B \cap D$, let $X := \{a, b, c, u, v, w\}$; then H is a Hamilton cycle in $G - X$. So we may assume that H is not a Hamilton cycle in $B \cap D$. Then there is an H-bridge B_1 in $B \cap D$ with $v_1 \in V(B_1 - H)$. Note that $|V(B_1 \cap H)| = 2$. Moreover, each element of $\{b, v\}$ has a neighbor in $V(B_1 - H)$; otherwise, $V(B_1 \cap H) \cup \{v\}$ or $V(B_1 \cap H) \cup \{b\}$ is a 3-cut in G, a contradiction. Let $V(B_1 \cap H) = \{s_1, t_1\}$ such that s_1, v_1, t_1 occur on F in clockwise order.

Similarly, by finding an F-Tutte cycle through three edges on F incident with v_1, v_3, v_4, respectively, we may assume that there exist a 2-cut $\{s_2, t_2\}$ in $B \cap D$ and an $\{s_2, t_2\}$-bridge B_2 in $B \cap D$ with $v_2 \in V(B_2) - \{s_2, t_2\}$ such that each element of $\{v, c\}$ has a neighbor in $V(B_2) - \{s_2, t_2\}$ and s_2, v_2, t_2 occur on F in clockwise order.

By finding an F-Tutte cycle through three edges on F incident with v_1, v_2, v_4, respectively, we may assume that there exist a 2-cut $\{s_3, t_3\}$ in $B \cap D$ and an $\{s_3, t_3\}$-bridge B_3 in $B \cap D$ with $v_3 \in V(B_3) - \{s_3, t_3\}$ such that each element of $\{c, u\}$ has a neighbor in $V(B_3) - \{s_3, t_3\}$ and s_3, v_3, t_3 occur on F in clockwise order.

By finding an F-Tutte cycle through three edges on F incident with v_1, v_2, v_3, respectively, we may further assume that there exist a 2-cut $\{s_4, t_4\}$ in $B \cap D$ and an $\{s_4, t_4\}$-bridge B_4 in $B \cap D$ with $v_4 \in V(B_4) - \{s_4, t_4\}$ such that each element of $\{u, b\}$ has a neighbor in $V(B_4) - \{s_4, t_4\}$ and s_4, v_4, t_4 occur on F in clockwise order.

Therefore each element of $\{c, v, b, u\}$ has at least two neighbors in $V(B \cap D)$.

We claim that $s_1, t_1, \ldots, s_4, t_4$ occur on F in clockwise order. Otherwise, without loss of generality, we may assume that s_1, s_2, t_1, t_2 occur on F in clockwise order, where $s_2 \neq t_1$. In this case, neither c nor b has a neighbor in $V(s_2Ft_1) - \{s_2, t_1\}$. If $V(s_2Ft_1) - \{s_2, t_1\} \neq \emptyset$, then $\{s_2, t_1, v\}$ is a 3-cut in G, contradicting the assumption that H_2 is $(4, \{c, v, b, u\})$-

connected. Therefore $V(s_2Ft_1) = \{s_2, t_1\}$ and $s_2t_1 \in E(G)$. But this implies that $t_1 \notin V(H)$, a contradiction.

Let $J := (B \cap D) - (V(B_1) - \{s_1, t_1\})$; then H is a Hamilton cycle in J and those neighbors of c in $V(B \cap D)$ are contained in $V(J)$. Hence J, $J_1 := G[V(J) \cup \{c\}] + s_1t_1$, and $J_2 := G[V(J) \cup \{c\}]$ are 2-connected. Let F_1 denote the outer cycle of J_1. Then $c, v_4, s_1, t_1 \in V(F_1)$ and $s_1t_1 \in E(F_1)$. By Lemma 2.1, there exists an F_1-Tutte cycle C_1 in J_1 through s_1t_1 and two edges on F_1 incident with c, v_4, respectively. Then C_1 is a Hamilton cycle in J_1. Let $L := G[V(B_1) \cup \{v, b\}]$; then (L, s_1, t_1, v, b) is planar. Since H_2 is $(4, \{c, v, b, u\})$-connected, L is $(4, \{s_1, t_1, v, b\})$-connected. Therefore by Lemma 2.2, B_1 is a chain of blocks from s_1 to t_1.

We may assume that $V(B_i) = \{s_i, t_i, v_i\}$, for $1 \leq i \leq 4$. Otherwise, without loss of generality, we may assume that $V(B_1) \neq \{s_1, t_1, v_1\}$. By applying Lemma 2.5, there is a vertex $z \in V(B_1) - \{s_1, t_1\}$ such that $B_1 - z$ has a Hamilton path P from s_1 to t_1. Let $X := \{a, b, u, v, w, z\}$; then $P \cup t_1C_1s_1$ is a Hamilton cycle in $G - X$.

Let F_2 denote the outer cycle of J_2; then $c, v_4, s_1, t_1 \in V(F_2)$. By Lemma 2.1, we find an F_2-Tutte cycle C_2 in J_2 through three edges on F_2 incident with c, v_4, t_1, respectively. If C_2 is a Hamilton cycle in J_2, let $X := \{a, b, u, v, w, v_1\}$; then C_2 is a Hamilton cycle in $G - X$. So we may assume that C_2 is not a Hamilton cycle in J_2. Then there is a C_2-bridge B'_1 in J_2 such that $s_1 \in V(B'_1 - C_2)$. Let $V(B'_1 \cap C_2) = \{s'_1, t'_1\}$.

Let $B' := (B \cap D) - \{v_1, v_2, v_3, v_4\}$. Then $B'_1 \subseteq B'$ and $\{s'_1, t'_1\}$ is a 2-cut in B'. Since H_2 is $(4, \{c, v, b, u\})$-connected, b has a neighbor in $V(B'_1) - \{s'_1, t'_1\}$.

Similarly, we may assume that there exist a 2-cut $\{s'_2, t'_2\}$ in B' and an $\{s'_2, t'_2\}$-bridge B'_2 in B' such that $s_2 \in V(B'_2) - \{s'_2, t'_2\}$ and v has a neighbor in $V(B'_2) - \{s'_2, t'_2\}$. We may further assume that there exist a 2-cut $\{s'_3, t'_3\}$ in B' and an $\{s'_3, t'_3\}$-bridge B'_3 in B' such that $s_3 \in V(B'_3) - \{s'_3, t'_3\}$ and c has a neighbor in $V(B'_3) - \{s'_3, t'_3\}$.

Then each element of $\{c, v, b\}$ has at least three neighbors in $V(B \cap D)$. Hence it is easy to see that $G - \{a, u, w\}$ is 3-connected. Therefore the triangle L induced by the vertices $\{a, u, w\}$ is a contractible triangle in G. Let u^* denote the vertex of G/L resulting from the contraction of L. Now by Theorem 2.7, there is some $X^* \subseteq V(G/L)$ such that $u^* \in X^*$, $|X^*| = 4$, and $G/L - X^*$ has a Hamilton cycle when $|V(G/L)| \geq 7$. Let $X := (X^* - \{u^*\}) \cup \{a, u, w\}$; then $G - X = G/L - X^*$ has a Hamilton cycle. ∎

4 Proof of main result

We now prove Theorem 1.1.

We may assume that G contains no contractible edge incident with u. Otherwise, let

$e = uv$ be a contractible edge of G incident with u. Then G/e is also a 4-connected planar graph. Let u^* denote the vertex of G/e resulting from the contraction of e. By Theorem 2.7, there is a set $X^* \subseteq V(G/e)$ such that $u^* \in X^*$, $|X^*| = 5$, and $G/L - X^*$ has a Hamilton cycle when $|V(G/L)| \geq 8$. Let $X := (X^* - \{u^*\}) \cup \{u, v\}$; then $G - X = G/e - X^*$ has a Hamilton cycle.

Let $\mathcal{F}$ denote the set of 4-cuts of G containing u and a neighbor of u. Hence by Theorem 2.6, there are two 4-cuts $S, T \in \mathcal{F}$ such that $1 \leq |S \cap T| \leq 2$ and $G - S$ has a component A consisting of only one vertex which is also contained in T. Let a be the only vertex in $V(A)$, and let $B := G - (\{a\} \cup S)$. Let C be a component of $G - T$ and let $D := G - (V(C) \cup T)$. Hence $S \cap V(C) \neq \emptyset \neq S \cap V(D)$. For if $S \cap V(C) = \emptyset$, then $B \cap C = C \neq \emptyset$ is a component of $G - (T - \{a\})$, contradicting the assumption that G is 4-connected. Similarly, if $S \cap V(D) = \emptyset$ then $B \cap D = D \neq \emptyset$ is a component of $G - (T - \{a\})$, a contradiction.

Then by Lemma 3.1, we may assume that

$(*)$ $S \cap T = \{u\}$ for all choices of S and T from $\mathcal{F}$.

We may choose C, D such that $|S \cap V(C)| = 2$ and $|S \cap V(D)| = 1$. Let v, w denote the vertices in $S \cap V(C)$, let b denote the only vertex in $S \cap V(D)$, and let c, d denote the vertices in $V(B) \cap T$, as shown in Figure 2.

Let $H_1 := G[V(C) \cup \{u, c, d\}]$ and let $H_2 := G[V(D) \cup \{u, c, d\}]$. Since a is adjacent to u and T is a 4-cut of G, c and d are cofacial. Likewise, v and w are cofacial. Without loss of generality, we may assume that (H_1, c, d, u, v, w) is planar. Then (H_2, c, d, u, b) is planar. Since G is 4-connected, H_1 is $(4, \{c, d, u, v, w\})$-connected (if $B \cap C \neq \emptyset$) and H_2 is $(4, \{c, d, u, b\})$-connected (if $B \cap D \neq \emptyset$).

Case 1. $B \cap D \neq \emptyset$.

We claim that $B \cap C \neq \emptyset$. Suppose on the contrary that $B \cap C = \emptyset$. Then one element of $\{v, w\}$ is not adjacent to some element of $\{c, d\}$; otherwise, by contracting $G[V(D) \cup \{u\}]$ to a single vertex, we produce a minor of G containing $K_{3,3}$, a contradiction. If v is not adjacent to some element of $\{c, d\}$, then $T' := N_G(v) \in \mathcal{F}$ and $|S \cap T'| = 2$, which contradicts our assumption $(*)$. Similarly, if w is not adjacent to some element of $\{c, d\}$, then $T' := N_G(w) \in \mathcal{F}$ and $|S \cap T'| = 2$, contradicting $(*)$.

We claim that $H_1 - \{u, v, w\}$ is a chain of blocks from c to d. Otherwise, let K be an end block of $H_1 - \{u, v, w\}$ and r be the cut vertex of $H_1 - \{u, v, w\}$ contained in $V(K)$ such that $(V(K) - \{r\}) \cap \{c, d\} = \emptyset$. As H_1 is $(4, \{c, d, u, v, w\})$-connected, each element of $\{u, v, w\}$ has a neighbor in $V(K) - \{r\}$. Since (H_1, c, d, u, v, w) is planar, $T' := \{a, r, u, w\} \in \mathcal{F}$ and $|S \cap T'| = 2$, contradicting our assumption $(*)$.

Since (H_1, c, d, u, v, w) is planar, by Lemma 2.4, there is a Hamilton path P in $H_1 - \{u, v, w\}$ from c to d.

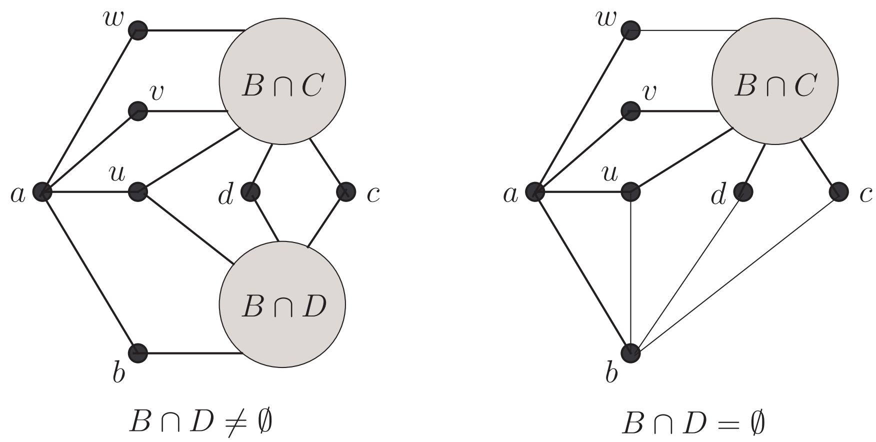


Figure 2: $|S \cap T| = 1$.

Suppose $|V(B \cap D)| \geq 2$. Since H_2 is $(4, \{c, d, u, b\})$-connected and (H_2, c, d, u, b) is planar, it follows from Lemma 2.5 that there is a vertex $y \in V(B \cap D)$ such that $H_2 - \{y, u, b\}$ has a Hamilton path Q from c to d. Let $X := \{a, b, u, v, w, y\}$; then $P \cup Q$ is a Hamilton cycle in $G - X$.

So we may assume that

$(**)$ $|V(B \cap D)| = 1$ for all choices of S, T, A, B, C, D with $|S \cap V(C)| = 2$ and $|S \cap V(D)| = 1$.

Let z denote the only vertex in $V(B \cap D)$. Then we may assume that c is not adjacent to d; otherwise, $P + cd$ is a Hamilton cycle in $G - X$, where $X := \{a, b, u, v, w, z\}$.

We claim that $H_1 - \{c, d, u\}$ is a chain of blocks from v to w. Otherwise, let K denote an end block of $H_1 - \{c, d, u\}$ and let r be the cut vertex of $H_1 - \{c, d, u\}$ contained in $V(K)$ such that $(V(K) - \{r\}) \cap \{v, w\} = \emptyset$. As H_1 is $(4, \{c, d, u, v, w\})$-connected, each element of $\{c, d, u\}$ has a neighbor in $V(K) - \{r\}$. Since (H_1, c, d, u, v, w) is planar, $T' := \{a, c, r, u\} \in \mathcal{F}$. Let C' be the component of $G - T'$ containing $\{v, w\}$, and let $D' := G - (V(C') \cup T')$. Then $|S \cap V(C')| = 2$, $|S \cap V(D')| = 1$, and $|V(B \cap D')| \geq 2$, contradicting $(**)$.

Since (H_1, c, d, u, v, w) is planar, by Lemma 2.4, there is a Hamilton path R in $H_1 - \{c, d, u\}$ from v to w. Then we may assume that v is not adjacent to w; otherwise, $R + vw$ is a Hamilton cycle in $G - X$, where $X := \{a, b, c, d, u, z\}$.

We may assume that d has at least two neighbors in $V(B \cap C)$. Otherwise, assume that d has at most one neighbor in $V(B \cap C)$. As (H_2, c, d, u, b) is planar, d is not adjacent to b. Since (H_1, c, d, u, v, w) is planar and c is not adjacent to d, d is adjacent to both u and v, u is adjacent to v, u has no neighbor in $V(B \cap C)$, and d has exactly one neighbor in $V(B \cap C)$. Let $H' := H_1 - u$. Then (H', d, v, w, c) is planar and H' is $(4, \{d, v, w, c\})$-connected (since G is 4-connected). Hence by Lemma 2.2, $H' - \{w, c\}$ is a chain of blocks from d to v. By Lemma 2.4, $H' - \{w, c\}$ contains a Hamilton path P' from d to v. Let $X := \{a, b, c, u, w, z\}$; then $P' + dv$ is a Hamilton cycle in $G - X$.

Similarly, by exchanging the roles of d and v and by exchanging the roles of c and w, we may further assume that v has at least two neighbors in $V(B \cap C)$.

We claim that $H_1 - \{c, u, w\}$ is 2-connected. Otherwise, let $J_1, \ldots, J_m$ $(m \geq 2)$ denote the end blocks of $H_1 - \{c, u, w\}$, and let v_i be the cut vertex of $H_1 - \{c, u, w\}$ contained in $V(J_i)$. Then for each $1 \leq i \leq m$, either $v \in V(J_i) - \{v_i\}$ or $d \in V(J_i) - \{v_i\}$; otherwise, since H_1 is $(4, \{c, d, u, v, w\})$-connected, each element of $\{c, u, w\}$ has a neighbor in $V(J_i) - \{v_i\}$, which contradicts the assumption that (H_1, c, d, u, v, w) is planar. Hence $m = 2$, and we may assume that $v \in V(J_1) - \{v_1\}$ and $d \in V(J_2) - \{v_2\}$. As both d and v have at least two neighbors in $V(B \cap C)$, $|V(J_1)| \geq 3$ and $|V(J_2)| \geq 3$. Since H_1 is $(4, \{c, d, u, v, w\})$-connected and (H_1, c, d, u, v, w) is planar, only u, w of $\{c, u, w\}$ have neighbors in $V(J_1) - \{v_1\}$; or only c, u of $\{c, u, w\}$ have neighbors in $V(J_2) - \{v_2\}$. Hence $T' := \{a, u, v_1, w\} \in \mathcal{F}$ or $T'' := \{a, c, u, v_2\} \in \mathcal{F}$. If $T' \in \mathcal{F}$, then $|S \cap T'| = 2$, contradicting our assumption $(*)$. So $T'' \in \mathcal{F}$. Let C' be the component of $G - T''$ containing $\{v, w\}$, and let $D' := G - (V(C') \cup T'')$. Then $|S \cap V(C')| = 2$, $|S \cap V(D')| = 1$, and $|V(B \cap D')| \geq 2$, contradicting $(**)$.

So let F denote the outer cycle of $H_1 - \{c, u, w\}$. Let $y \in V(vFd)$ such that v, y, d occur on F in clockwise order, $N_G(w) \cap V(F) \subseteq V(vFy)$, and $N_G(c) \cap V(F) \subseteq V(yFd)$. By Lemma 2.1, we find an F-Tutte cycle H in $H_1 - \{c, u, w\}$ through three edges on F incident with v, y, d, respectively. Since H_1 is $(4, \{c, d, u, v, w\})$-connected, H is a Hamilton cycle in $H_1 - \{c, u, w\}$. Let $X := \{a, b, c, u, w, z\}$; then H is a Hamilton cycle in $G - X$.

Case 2. $B \cap D = \emptyset$.

In this case, $|V(B \cap C)| \geq 2$. Otherwise, $|V(G)| \leq 8$, and there is nothing to prove. Since H_1 is $(4, \{c, d, u, v, w\})$-connected and (H_1, c, d, u, v, w) is planar, $B \cap C$ is connected. We consider two subcases according to the connectivity of $B \cap C$.

Subcase 2.1. $B \cap C$ is connected but not 2-connected.

Let $J_1, \ldots, J_m$ $(m \geq 2)$ be the end blocks of $B \cap C$, and let v_i be the cut vertex of $B \cap C$ contained in $V(J_i)$. We claim that $m = 2$. Otherwise, since H_1 is $(4, \{c, d, u, v, w\})$-connected, at least three elements of $\{c, d, u, v, w\}$ have neighbors in $V(J_i) - \{v_i\}$ (for each i), contradicting the assumption that (H_1, c, d, u, v, w) is planar.

Let $B_1 := J_1 - v_1$ and $B_2 := (B \cap C) - V(J_1)$. We claim that there is some element $x \in \{c, d, v, w\}$ such that x has neighbors in both $V(B_1)$ and $V(B_2)$. Otherwise, u must have neighbors in both $V(B_1)$ and $V(B_2)$ (since G is 4-connected). Hence we may assume that only u, v, w of $\{c, d, u, v, w\}$ have neighbors in $V(B_1)$, and only c, d, u of $\{c, d, u, v, w\}$ have neighbors in $V(B_2)$. Since (H_1, c, d, u, v, w) is planar, $T' := \{a, u, v_1, w\} \in \mathcal{F}$ and $|S \cap T'| = 2$, contradicting our assumption $(*)$.

By symmetry, we may assume that d has neighbors in both $V(B_1)$ and $V(B_2)$. Since H_1 is $(4, \{c, d, u, v, w\})$-connected and (H_1, c, d, u, v, w) is planar, both u and v have neighbors in $V(B_1)$, both c and w have neighbors in $V(B_2)$, and at most one element of $\{v, w\}$ has neighbors in both $V(B_1)$ and $V(B_2)$. Without loss of generality, we may assume that v has no neighbor in $V(B_2)$; the other case can be treated in the same way.

Let $L_1 := G[V(B_1) \cup \{d, v_1, w, v, u\}]$ and let $L_2 := G[V(B_2) \cup \{d, v_1, w, c\}]$. Then (L_1, d, v_1, w, v, u) and (L_2, d, v_1, w, c) are planar. Since H_1 is $(4, \{c, d, u, v, w\})$-connected, L_1 is $(4, \{d, v_1, w, v, u\})$-connected and L_2 is $(4, \{d, v_1, w, c\})$-connected. Therefore $L_1 - \{w, v, u\}$ is a chain of blocks from d to v_1, and $L_2 - \{w, c\}$ is a chain of blocks from d to v_1. Hence by Lemma 2.4, $L_1 - \{w, v, u\}$ has a Hamilton path P_1 from d to v_1, and $L_2 - \{w, c\}$ has a Hamilton path P_2 from d to v_1. Now let $X := \{a, b, c, d, u, v\}$; then $P_1 \cup P_2$ is a Hamilton cycle in $G - X$.

Subcase 2.2. $B \cap C$ is 2-connected.

Let F denote the outer cycle of $B \cap C$. Choose $v_1, v_2, v_3, v_4, v_5 \in V(F)$ such that v_1, v_2, v_3, v_4, v_5 occur on F in clockwise order, $N_G(c) \cap V(F) \subseteq V(v_1 F v_2)$, $N_G(d) \cap V(F) \subseteq V(v_2 F v_3)$, $N_G(u) \cap V(F) \subseteq V(v_3 F v_4)$, $N_G(v) \cap V(F) \subseteq V(v_4 F v_5)$, and $N_G(w) \cap V(F) \subseteq V(v_5 F v_1)$.

We may assume that v has at least two neighbors in $V(B \cap C)$. Suppose on the contrary that v has at most one neighbor in $V(B \cap C)$. If u has no neighbor in $V(B \cap C)$, then let $H' := H_1 - u$. So (H', c, d, v, w) is planar and H' is $(4, \{c, d, v, w\})$-connected (since G is 4-connected). Hence by Lemma 2.2, $H' - \{c, d\}$ is a chain of blocks from v to w, and $H' - \{c, w\}$ is a chain of blocks from v to d. Since $|V(B \cap C)| \geq 2$, it follows from Lemma 2.5 that there is a vertex $z_1 \in V(B \cap C)$ such that $H' - \{z_1, c, d\}$ has a Hamilton path P_1 from v to w. Similarly, there is a vertex $z_2 \in V(B \cap C)$ such that $H' - \{z_2, c, w\}$ has a Hamilton path P_2 from v to d. If v is adjacent to w, let $X := \{a, b, c, d, u, z_1\}$; then $P_1 + vw$ is a Hamilton cycle in $G - X$. If v is adjacent to d, let $X := \{a, b, c, u, w, z_2\}$; then $P_2 + vd$ is a Hamilton cycle in $G - X$. So assume that v is adjacent to neither w nor d. But this contradicts the assumption that G is 4-connected. Therefore we may assume that u has a neighbor in $V(B \cap C)$. If w has no neighbor in $V(B \cap C)$, then $T' := \{a, c, u, v\} \in \mathcal{F}$ and $|S \cap T'| = 2$, contradicting our assumption $(*)$. Hence we may further assume that w has a neighbor in $V(B \cap C)$. Since (H_1, c, d, u, v, w) is planar, v is adjacent to neither c nor d. Since G is 4-connected and by planarity, v is adjacent to both u and w. Then $T' := N_G(v) \in \mathcal{F}$ and $|S \cap T'| = 2$, contradicting $(*)$.

Hence $T_1 := G[(V(B \cap C)) \cup \{v\}]$ is 2-connected. Let D_1 denote the outer cycle of

T_1. Then $v, v_i \in V(D_1)$ $(1 \le i \le 5)$. By Lemma 2.1, we find a D_1-Tutte cycle H in T_1 through three edges on D_1 incident with v, v_2, v_3, respectively. If H is a Hamilton cycle in T_1, let $X := \{a, b, c, d, u, w\}$; then H is a Hamilton cycle in $G - X$. So we may assume that H is not a Hamilton cycle in T_1. Then there is an H-bridge B_1 in T_1 such that $v_1 \in V(B_1 - H)$. Note that $|V(B_1 \cap H)| = 2$. Since $B \cap C$ is 2-connected, $B_1 \subseteq B \cap C$. Moreover, each element of $\{w, c\}$ has a neighbor in $V(B_1 - H)$; otherwise, $V(B_1 \cap H) \cup \{w\}$ or $V(B_1 \cap H) \cup \{c\}$ is a 3-cut in G, a contradiction. Let $V(B_1 \cap H) = \{s_1, t_1\}$ such that s_1, v_1, t_1 occur on D_1 (also on F) in clockwise order.

Similarly, by finding a D_1-Tutte cycle through three edges on D_1 incident with v, v_1, v_3, respectively, we may assume that there exist a 2-cut $\{s_2, t_2\}$ in $B \cap C$ and an $\{s_2, t_2\}$-bridge B_2 in $B \cap C$ with $v_2 \in V(B_2) - \{s_2, t_2\}$ such that each element of $\{c, d\}$ has a neighbor in $V(B_2) - \{s_2, t_2\}$ and s_2, v_2, t_2 occur on D_1 (also on F) in clockwise order.

By finding a D_1-Tutte cycle through three edges on D_1 incident with v, v_1, v_2, respectively, we may assume that there exist a 2-cut $\{s_3, t_3\}$ in $B \cap C$ and an $\{s_3, t_3\}$-bridge B_3 in $B \cap C$ with $v_3 \in V(B_3) - \{s_3, t_3\}$ such that each element of $\{d, u\}$ has a neighbor in $V(B_3) - \{s_3, t_3\}$ and s_3, v_3, t_3 occur on D_1 (also on F) in clockwise order.

So each element of $\{c, d\}$ has at least two neighbors in $V(B \cap C)$. Hence $T_2 := G[(V(B \cap C)) \cup \{c\}]$ is 2-connected. Let D_2 denote the outer cycle of T_2. Then $c, v_i \in V(D_2)$ $(1 \le i \le 5)$. As before, by finding a D_2-Tutte cycle through three edges on D_2 incident with c, v_3, v_5, respectively, we may assume that there exist a 2-cut $\{s_4, t_4\}$ in $B \cap C$ and an $\{s_4, t_4\}$-bridge B_4 in $B \cap C$ with $v_4 \in V(B_4) - \{s_4, t_4\}$ such that each element of $\{u, v\}$ has a neighbor in $V(B_4) - \{s_4, t_4\}$ and s_4, v_4, t_4 occur on D_2 (also on F) in clockwise order.

By finding a D_2-Tutte cycle through three edges on D_2 incident with c, v_3, v_4, respectively, we may further assume that there exist a 2-cut $\{s_5, t_5\}$ in $B \cap C$ and an $\{s_5, t_5\}$-bridge B_5 in $B \cap C$ with $v_5 \in V(B_5) - \{s_5, t_5\}$ such that each element of $\{v, w\}$ has a neighbor in $V(B_5) - \{s_5, t_5\}$ and s_5, v_5, t_5 occur on D_2 (also on F) in clockwise order.

Therefore each element of $\{c, d, u, v, w\}$ has at least two neighbors in $V(B \cap C)$.

We claim that $s_1, t_1, \ldots, s_5, t_5$ occur on F in clockwise order. Otherwise, without loss of generality, we may assume that s_1, s_2, t_1, t_2 occur on F in clockwise order, where $s_2 \ne t_1$. Then neither w nor d has a neighbor in $V(s_2 F t_1) - \{s_2, t_1\}$. If $V(s_2 F t_1) - \{s_2, t_1\} \ne \emptyset$, then $\{s_2, t_1, c\}$ is a 3-cut in G, which contradicts the assumption that H_1 is $(4, \{c, d, u, v, w\})$-connected. Therefore $V(s_2 F t_1) = \{s_2, t_1\}$ and $s_2 t_1 \in E(G)$. But this implies that $t_1 \notin V(H)$, a contradiction.

Let $J := T_1 - (V(B_1) - \{s_1, t_1\})$; then H is a Hamilton cycle in J and those neighbors of d in $V(T_1)$ are contained in $V(J)$. Hence J, $J_1 := G[V(J) \cup \{d\}] + s_1 t_1$, and $J_2 := G[V(J) \cup \{d\}]$ are 2-connected. Let F_1 denote the outer cycle of J_1. Then $d, v, s_1, t_1 \in V(F_1)$ and $s_1 t_1 \in E(F_1)$. By applying Lemma 2.1, there exists an F_1-Tutte cycle C_1 in J_1 through $s_1 t_1$ and two edges on F_1 incident with d, v, respectively. Then C_1 is a

Hamilton cycle in J_1. Let $L := G[V(B_1) \cup \{c, w\}]$; then (L, s_1, t_1, c, w) is planar. Since H_1 is $(4, \{c, d, u, v, w\})$-connected, L is $(4, \{s_1, t_1, c, w\})$-connected. Therefore by Lemma 2.2, B_1 is a chain of blocks from s_1 to t_1.

We may assume that $V(B_i) = \{s_i, t_i, v_i\}$, for $i = 1, 3, 4$. Otherwise, without loss of generality, we may assume that $V(B_1) \neq \{s_1, t_1, v_1\}$. By Lemma 2.5, there is a vertex $z \in V(B_1) - \{s_1, t_1\}$ such that $B_1 - z$ has a Hamilton path P from s_1 to t_1. Let $X := \{a, b, c, u, w, z\}$; then $P \cup t_1 C_1 s_1$ is a Hamilton cycle in $G - X$.

Let F_2 denote the outer cycle of J_2; then $d, v, s_1, t_1 \in V(F_2)$. By Lemma 2.1, we find an F_2-Tutte cycle C_2 in J_2 through three edges on F_2 incident with d, v, t_1, respectively. If C_2 is a Hamilton cycle in J_2, let $X := \{a, b, c, u, w, v_1\}$; then C_2 is a Hamilton cycle in $G - X$. So we may assume that C_2 is not a Hamilton cycle in J_2. Then there is a C_2-bridge B_1' in J_2 with $s_1 \in V(B_1' - C_2)$. Let $V(B_1' \cap C_2) = \{s_1', t_1'\}$.

Let $B' := (B \cap C) - (\cup_{i=1}^{5}(V(B_i) - \{s_i, t_i\}))$. Then $B_1' \subseteq B'$ and $\{s_1', t_1'\}$ is a 2-cut in B'. Since H_1 is $(4, \{c, d, u, v, w\})$-connected, w has a neighbor in $V(B_1') - \{s_1', t_1'\}$.

Similarly, we may assume that there exist a 2-cut $\{s_2', t_2'\}$ in B' and an $\{s_2', t_2'\}$-bridge B_2' in B' such that $t_1 \in V(B_2') - \{s_2', t_2'\}$ and c has a neighbor in $V(B_2') - \{s_2', t_2'\}$. We may assume that there exist a 2-cut $\{s_3', t_3'\}$ in B' and an $\{s_3', t_3'\}$-bridge B_3' in B' such that $s_3 \in V(B_3') - \{s_3', t_3'\}$ and d has a neighbor in $V(B_3') - \{s_3', t_3'\}$. We may further assume that there exist a 2-cut $\{s_4', t_4'\}$ in B' and an $\{s_4', t_4'\}$-bridge B_4' in B' such that $t_4 \in V(B_4') - \{s_4', t_4'\}$ and v has a neighbor in $V(B_4') - \{s_4', t_4'\}$.

Then each element of $\{c, d, v, w\}$ has at least three neighbors in $V(B \cap C)$. Hence it is easy to see that $G - \{a, b, u\}$ is 3-connected. Therefore the triangle L induced by the vertices $\{a, b, u\}$ is a contractible triangle in G. Let u^* denote the vertex of G/L resulting from the contraction of L. Now by Theorem 2.7, there is some $X^* \subseteq V(G/L)$ such that $u^* \in X^*$, $|X^*| = 4$, and $G/L - X^*$ has a Hamilton cycle when $|V(G/L)| \geq 7$. Let $X := (X^* - \{u^*\}) \cup \{a, b, u\}$; then $G - X = G/L - X^*$ has a Hamilton cycle. ∎

We now use Theorem 1.1 to prove the following result.

Corollary 4.1. *Let G be a 4-connected planar graph on n vertices. Then G contains a cycle of length $n - 7$ for all $n \geq 10$.*

Proof. Suppose this is not true and let G be a counter example. If G contains a contractible edge e, we consider G/e. Let u be the vertex resulting from the contraction of e. By applying Theorem 1.1, there is some $X \subseteq V(G/e)$ such that $u \in X$, $|X| = 6$, and $G/e - X$ has a Hamilton cycle when $|V(G/e)| \geq 9$. Hence, if $n \geq 10$ then G has a cycle of length $n - 7$, a contradiction.

So G contains no contractible edge. Then G is either the square of a cycle of length at least 4 or the line graph of a cyclically 4-edge-connected cubic graph. It is not hard to see that if G is the square of a cycle, then G has cycles of length k for all $3 \leq k \leq n$. Since G is a counter example, G is the line graph of a cyclically 4-edge-connected cubic graph. Therefore G is 4-regular, every vertex is contained in exactly two triangles, and no two triangles share an edge. Using these properties and by planarity, we can see that every triangle T in G is contractible. Let u denote the vertex resulting from the contraction of T. Now by Theorem 2.7, there is some $X^* \subseteq V(G/T)$ such that $u \in X^*$, $|X^*| = 5$, and $G/T - X^*$ has a Hamilton cycle when $|V(G/T)| \geq 8$. Hence G has a cycle of length $n - 7$ for all $n \geq 10$, a contradiction. ∎

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