A Newton Type Rational Interpolation Formula

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Abstract. We give a Newton type rational interpolation formula (Theorem 2.2). It contains as a special case the original Newton interpolation, as well as the interpolation formula of Liu, which allows to recover many important classical q-series identities. We show in particular that some bibasic identities are a consequence of our formula.

Keywords: Newton type rational interpolation formula, bibasic identities.

1. Introduction and Notation

As usual, $(a;q)_n$ (resp. $(a;p)_n$) denotes

$$\prod_{j=0}^{n-1}(1-aq^j)\Big(\text{resp. } \prod_{j=0}^{n-1}(1-ap^j)\Big), \quad n=0,1,2,\ldots,\infty.$$

Newton obtained the following interpolation formula:

$$f(x) = f(x_1) + f\,\partial_1(x-x_1) + f\,\partial_1\partial_2(x-x_1)(x-x_2) + \cdots ,$$

where ∂_i is the divided difference which will be defined below.

Special cases of Newton's interpolation are the Taylor formula and the q-Taylor formula (c.f. [5]), with derivatives or q-derivatives instead of divided differences.

Using q-derivatives, Liu [7] gave an interpolation formula involving rational functions in x as coefficients, instead of only polynomials in x as in the q-Taylor formula:

$$f(x) = \sum_{n=0}^{\infty} \frac{(1-aq^{2n})(aq/x;q)_n x^n}{(q;q)_n (x;q)_n} [D_q f(x)(x;q)_{n-1}]\big|_{x=aq}, \tag{1.1}$$

D_q being defined by

$$D_q f(x) = \frac{f(x) - f(xq)}{x}.$$

Let us remark that Carlitz's q-analog of a special case of the Lagrange inversion formula is the limit for $a \to 0$ of (1.1):

$$f(x) = \sum_{n=0}^{\infty} \frac{x^n}{(q;q)_n (x;q)_n} [D_q f(x)(x;q)_{n-1}]|_{x=0}.$$

Our formula involves two sets of indeterminate $\mathcal{X}$ and $\mathcal{C}$. Newton interpolation is the case when

$$\mathcal{C} = \{0, 0, \ldots\},$$

and Liu's expansion is the case when

$$\mathcal{X} = \{aq^1, aq^2, \ldots\},\ \mathcal{C} = \{q^0, q^1, q^2, \ldots\}.$$

2. Rational Interpolation

By convenience, we denote

$$[x;\mathcal{X}]_n = (x-x_1)(x-x_2)\cdots(x-x_n)$$

and

$$(x;\mathcal{C})_n = (1-xc_1)(1-xc_2)\cdots(1-xc_n).$$

The divided difference ∂_i (acting on its left), $i = 1, 2, 3, \ldots$, is defined by

$$f(x_1, \ldots, x_i, x_{i+1}, \ldots)\,\partial_i = \frac{f(x_1, \ldots, x_i, x_{i+1}, \ldots) - f(x_1, \ldots, x_{i+1}, x_i, \ldots)}{x_i - x_{i+1}}.$$

Divided differences satisfy a Leibnitz type formula:

$$(f(x_1)g(x_1))\partial_1 = f(x_1)\left(g(x_1)\partial_1\right) + \left(f(x_1)\partial_1\right)g(x_2).$$

By induction, one obtains

$$f(x_1)g(x_1)\,\partial_1\partial_2\cdots\partial_n = \sum_{k=0}^{n}\left(f(x_1)\partial_1\cdots\partial_k\right)\left(g(x_{k+1})\partial_{k+1}\cdots\partial_n\right).$$

Lemma 2.1 *Letting i, n be two nonnegative integers, one has*

$$[b_1;\mathcal{X}]_n\,\partial_1\partial_2\cdots\partial_i\big|_{\mathcal{B}=\mathcal{X}} = \begin{cases} 0, & i \neq n; \\ 1, & i = n, \end{cases}$$

where $\big|_{\mathcal{B}=\mathcal{X}}$ denotes the specialization $b_1 = x_1, b_2 = x_2, \ldots,$ and the divided differences are relative to $b_1, b_2, \ldots$.

Proof. If $i \leq n$, using Leibnitz formula, we have

$$\begin{aligned}
&[b_1;\mathcal{X}]_n\,\partial_1\partial_2\cdots\partial_i\big|_{\mathcal{B}=\mathcal{X}} \\
&= \prod_{k=2}^{n}(b_1 - x_k)\,\partial_1\cdots\partial_i(b_1 - x_1)\big|_{\mathcal{B}=\mathcal{X}} + \prod_{k=2}^{n}(b_2 - x_k)\,\partial_2\cdots\partial_i(b_1 - x_1)\,\partial_1\big|_{\mathcal{B}=\mathcal{X}} \\
&= \prod_{k=2}^{n}(b_2 - x_k)\,\partial_2\cdots\partial_i\big|_{\mathcal{B}=\mathcal{X}} = \cdots = \prod_{k=i}^{n}(b_i - x_k)\,\partial_i\big|_{\mathcal{B}=\mathcal{X}} \\
&= \begin{cases} \prod_{k=i+1}^{n}(b_{i+1} - x_k)\big|_{\mathcal{B}=\mathcal{X}} = 0, & i < n; \\ (b_n - x_n)\,\partial_n\big|_{\mathcal{B}=\mathcal{X}} = 1, & i = n. \end{cases}
\end{aligned}$$

In the case $i > n$, nullity comes from the fact that each ∂_i decreases degree by 1. ∎

Theorem 2.2 *For any formal series $f(x)$ in x, we have the following identity in the ring of formal series in $x, x_1, x_2, \ldots$:*

$$\begin{aligned} f(x) &= f(x_1) + f(x_1)\,\partial_1(1-x_2c_1)\frac{[x;\mathcal{X}]_1}{(x;\mathcal{C})_1} \\ &\quad + f(x_1)(1-x_1c_1)\,\partial_1\partial_2(1-x_3c_2)\frac{[x;\mathcal{X}]_2}{(x;\mathcal{C})_2} + \cdots \\ &\quad + f(x_1)(x_1;\mathcal{C})_{n-1}\,\partial_1\cdots\partial_n(1-x_{n+1}c_n)\frac{[x;\mathcal{X}]_n}{(x;\mathcal{C})_n} + \cdots . \end{aligned} \tag{2.2}$$

Proof. Let

$$f(b) = \sum_{n=0}^{\infty} A_n \frac{[b;\mathcal{X}]_n}{(b;\mathcal{C})_n}.$$

Specializing b to x_1 or x_2, one gets the following coefficients:

$$A_0 = f(x_1), A_1 = f(x_1)\,\partial_1(1-x_2c_1).$$

Now we have to check

$$\frac{[b_1;\mathcal{X}]_n}{(b_1;\mathcal{C})_n}(b_1;\mathcal{C})_{k-1}\,\partial_1\partial_2\cdots\partial_k\big|_{\mathcal{B}=\mathcal{X}} = \begin{cases} 0, & k \neq n; \\ \frac{1}{1-x_{n+1}c_n}, & k = n. \end{cases} \tag{2.3}$$

If $k > n$, $\dfrac{[b;\mathcal{X}]_n}{(b;\mathcal{C})_n}(b;\mathcal{C})_{k-1}$ is a polynomial of degree $k-1$, and therefore annihilated by a product of k divided differences.

If $k < n$, from Leibnitz formula, we get

$$\begin{aligned} &\frac{[b_1;\mathcal{X}]_n}{(b_1;\mathcal{C})_n}(b_1;\mathcal{C})_{k-1}\,\partial_1\partial_2\cdots\partial_k\big|_{\mathcal{B}=\mathcal{X}} \\ &\quad = \frac{[b_1;\mathcal{X}]_n}{\prod_{p=k}^{n}(1-b_1c_p)}\,\partial_1\partial_2\cdots\partial_k\big|_{\mathcal{B}=\mathcal{X}} \\ &\quad = \sum_{i=0}^{k}\frac{1}{\prod_{p=k}^{n}(1-b_{i+1}c_p)}\,\partial_{i+1}\cdots\partial_k[b_1;\mathcal{X}]_n\,\partial_1\cdots\partial_i\big|_{\mathcal{B}=\mathcal{X}}, \end{aligned}$$

and Lemma 2.1 shows that this function is equal to 0.

If $k = n$, we have

$$\begin{aligned}
&\frac{[b_1;\mathcal{X}]_n}{1-b_1c_n}\,\partial_1\partial_2\cdots\partial_n\Big|_{\mathcal{B}=\mathcal{X}} \\
&\quad = \sum_{i=0}^{n} \frac{1}{1-b_{i+1}c_n}\,\partial_{i+1}\cdots\partial_n[b_1;\mathcal{X}]_n\,\partial_1\cdots\partial_i\Big|_{\mathcal{B}=\mathcal{X}} \\
&\quad = \frac{1}{1-b_{n+1}c_n}[b_1;\mathcal{X}]_n\,\partial_1\cdots\partial_n\Big|_{\mathcal{B}=\mathcal{X}} \\
&\quad = \frac{1}{1-x_{n+1}c_n}.
\end{aligned}$$

Formula (2.3) thus implies

$$A_n = f(x_1)(x_1;\mathcal{C})_{n-1}\partial_1\cdots\partial_n(1-x_{n+1}c_n),$$

and the theorem. ∎

3. Bibasic summation formulas

Proposition 3.3 *Taking*

$$f(x) = \frac{1-c_0x}{1-vx}$$

and

$$\mathcal{X} = \{x_1, x_2, \ldots\},\ \mathcal{C} = \{c_0, c_1, c_2, \ldots\},$$

we have

$$f(x) = \sum_{k=0}^{\infty} \frac{[v;\mathcal{C}]_k}{(v;\mathcal{X})_{k+1}}\frac{[x;\mathcal{X}]_k}{(x;\mathcal{C})_k}(1-x_{k+1}c_k). \tag{3.4}$$

The proposition is a direct application of Theorem 2.2 and the following lemma.

Lemma 3.4

$$\frac{(x_1;\mathcal{C})_k}{1-vx_1}\partial_1\partial_2\cdots\partial_k = [v;\mathcal{C}]_k/(v;\mathcal{X})_{k+1}. \tag{3.5}$$

We first need to recall some facts about symmetric functions [8]. The generating functions for the elementary symmetric function $e_i(x_1, x_2, \ldots)$, and the complete symmetric function $h_i(x_1, x_2, \ldots)$ are

$$\sum_{i\geq 0} e_i(x_1, x_2, \ldots)t^i = \prod_{i\geq 0}(1 + x_i t),$$

and

$$\sum_{i\geq 0} h_i(x_1, x_2, \ldots)t^i = \prod_{i\geq 0}(1 - x_i t)^{-1}.$$

We shall need a slightly more general notion than usual, for a Schur function. Given $\lambda \in \mathbb{N}^n$, and n sets of variables $A_1, \ldots, A_n$, then the multi-Schur function $s_\lambda(A_1, \ldots, A_n)$ is equal to $\left|h_{\lambda_j+j-i}(A_j)\right|_{1\leq i,j\leq n}$.

One has the following identity [6]:

$$s_\lambda(x_2, x_3, \cdots)x_1^r = s_{\lambda,r}(\mathcal{X}, x_1), \tag{3.6}$$

where one uses complete functions of x_1 in the last column of the determinant $s_{\lambda,r}(\mathcal{X}, x_1)$, and complete functions of $\mathcal{X}$ elsewhere.

Proof of Lemma 3.4 Multiply the denominator of $(x_1; \mathcal{C})_k/(1 - vx_1)$ by the symmetrical factor $(v; \mathcal{X})_{k+1}$, which commutes with $\partial_1 \cdots \partial_k$. Let $\mathcal{X}_k = \{x_1, x_2, \ldots, x_{k+1}\}$. One has

$$\begin{aligned}
&\prod_{i=0}^{k-1}(1 - x_1 c_i)\prod_{j=2}^{k+1}(1 - vx_j)\\
&\quad= \sum_{i=0}^{k}\sum_{j=0}^{k}(-1)^i(-v)^j e_i(c_0, c_1, \ldots, c_{k-1})e_j(x_2, x_3, \ldots, x_{k+1})x_1^i\\
&\quad= \sum_{i=0}^{k}\sum_{j=0}^{k}(-1)^i(-v)^j e_i(c_0, c_1, \ldots, c_{k-1})s_{1^j,i}(\underbrace{\mathcal{X}_k, \ldots, \mathcal{X}_k}_{j}, x_1),
\end{aligned}$$

thanks to (3.6), and to the fact that for every j, $e_j(\mathcal{X}) = s_{1^j}(\underbrace{\mathcal{X}, \ldots, \mathcal{X}}_{j})$.

The image of a power of x_1 under $\partial_1 \cdots \partial_k$ is a complete symmetric function in $\mathcal{X}$ [6]. Therefore,

$$s_{1^j,i}(\underbrace{\mathcal{X}_k, \ldots, \mathcal{X}_k}_{j}, x_1)\,\partial_1 \cdots \partial_k = s_{1^j,i-k}(\underbrace{\mathcal{X}_k, \ldots, \mathcal{X}_k}_{j+1}).$$

This determinant is equal to 0 (because it has two identical columns), except for $i+j=k$, in which case it is equal to $s_{0^{j+1}}(\mathcal{X}) = (-1)^j$.

Now

$$\frac{(x_1;\mathcal{C})_k}{1-vx_1}(v;\mathcal{X})_{k+1}\,\partial_1\partial_2\cdots\partial_k = \sum_{i+j=k}(-1)^i v^j e_i(c_0,c_1,\ldots,c_{k-1}) = [v;\mathcal{C}]_k,$$

thus (3.5) is true. ∎

In [2], Gasper obtained the following identity:

$$\sum_{k=0}^{\infty}\frac{1-ap^kq^k}{1-a}\frac{(a;p)_k(b^{-1};q)_k b^k}{(q;q)_k(abp;p)_k} = 0. \tag{3.7}$$

We also prove an identity due to Gosper (c.f. [3]):

$$\sum_{k=0}^{n}\frac{1-ap^kq^k}{1-a}\frac{(a;p)_k(c;q)_k c^{-k}}{(q;q)_k,(ap/c;p)_k} = \frac{(ap;p)_n(cq;q)_n c^{-n}}{(q;q)_n(ap/c;p)_n},$$

or equivalently,

$$\sum_{k=0}^{n}\frac{(1-ap^{n-k}q^{n-k})(q^{n-k+1};q)_k(ap^{n-k+1}/c;p)_k}{(cq^{n-k};q)_{k+1}(ap^{n-k};p)_{k+1}}c^k = \frac{1}{1-c}. \tag{3.8}$$

In fact, (3.7) and (3.8) are special cases of Proposition 3.3.

Taking $c_0 = 0$ in (3.4), we get

$$\frac{1}{1-vx} = \frac{1}{1-vx_1} + \sum_{k=1}^{\infty}\frac{[v;\mathcal{C}]_k}{(v;\mathcal{X})_{k+1}}\frac{[x;\mathcal{X}]_k}{(x;\mathcal{C})_k}(1-x_{k+1}c_k). \tag{3.9}$$

Multiplying both sides of (3.9) by $(1-vx_1)$, one has

$$\frac{1-vx_1}{1-vx} = 1 + \sum_{k=1}^{\infty}\frac{[v;\mathcal{C}]_k}{(v;\mathcal{X}/x_1)_k}\frac{[x;\mathcal{X}]_k}{(x;\mathcal{C})_k}(1-x_{k+1}c_k),$$

where $\mathcal{X}/x_1 = \{x_2, x_3, \ldots\}$.

Taking

$$\mathcal{X} = \{q^0, q^1, q^2, \ldots\},\ \mathcal{C} = \{ap^1, ap^2, \ldots\},\ v = 1,\ x = b,$$

we get (3.7).

In (3.4), taking

$$\mathcal{X} = \{p^{-n}/a, p^{-n+1}/a, \ldots\},\ \mathcal{C} = \{q^{-n+1}, q^{-n+2}, \ldots\},$$

$c_0 = q^{-n}$, $v = 1$, $x = c^{-1}$, we get (3.8).

Acknowledgments. This work was supported by the 973 Project, the PCSIRT Project of the Ministry of Education, the Ministry of Science and Technology, and the National Science Foundation of China.

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