Bipartite rainbow numbers of matchings*

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Abstract

Given two graphs G and H, let $f(G,H)$ denote the maximum number c for which there is a way to color the edges of G with c colors such that every subgraph H of G has at least two edges of the same color. Equivalently, any edge-coloring of G with at least $rb(G,H)=f(G,H)+1$ colors contains a rainbow copy of H, where a rainbow subgraph of an edge-colored graph is such that no two edges of it have the same color. The number $rb(G,H)$ is called the *rainbow number of H with respect to G*, and simply called the *bipartite rainbow number of H* if G is the complete bipartite graph $K_{m,n}$. Erdős, Simonovits and Sós showed that $rb(K_n,K_3)=n$. In 2004, Schiermeyer determined the rainbow numbers $rb(K_n,K_k)$ for all $n\geq k\geq 4$, and the rainbow numbers $rb(K_n,kK_2)$ for all $k\geq 2$ and $n\geq 3k+3$. In this paper we will determine the rainbow numbers $rb(K_{m,n},kK_2)$ for all $k\geq 1$.

Keywords: edge coloring, rainbow subgraph, rainbow number.

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1 Introduction

In this paper we consider undirected, finite and simple graphs only, and use standard notations in graph theory (see [3]). Given two graphs G and H, if G is

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edge-colored and a subgraph H of G contains no two edges of the same color, then H is called a *totally multicolored (TMC)* or *rainbow subgraph* of G and we say that G contains a TMC or rainbow H. Let $f(G,H)$ denote the maximum number of colors in an edge-coloring of the graph G with no TMC H. We now define $rb(G,H)$ as the minimum number of colors such that any edge-coloring of G with at least $rb(G,H) = f(G,H)+1$ colors contains a TMC or rainbow subgraph isomorphic to H. The number $rb(G,H)$ is called the *rainbow number of H with respect to G*. If G is the complete bipartite graph $K_{m,n}$, $rb(G,H)$ is simply called the *bipartite rainbow number of H*.

When $G = K_n$, $f(G,H)$ is called the *anti-Ramsey number* of H. Anti-Ramsey numbers were introduced by Erdős, Simonovits and Sós in the 1970s. Let P_k and C_k denote the path and the cycle with k vertices, respectively. Simonovits and Sós [8] determined $f(K_n, P_k)$ for large enough n. Erdős et al. [4] conjectured that for every fixed $k \geq 3$, $f(K_n, C_k) = n(\frac{k-2}{2} + \frac{1}{k-1}) + O(1)$, and proved it for $k = 3$ by showing that $f(K_n, C_3) = n-1$. Alon [1] showed that $f(K_n, C_4) = \lfloor \frac{4n}{3} \rfloor - 1$, and the conjecture is thus proved for $k = 4$. Recently, the conjecture is proved for all $k \geq 3$ by Montellano-Ballesteros and Neumann-Lara [5]. Axenovich, Jiang and Kündgen [2] determined $f(K_{m,n}, C_{2k})$ for all $k \geq 2$.

In 2004, Schiermeyer [7] determined the rainbow numbers $rb(K_n, K_k)$ for all $n \geq k \geq 4$, and the rainbow numbers $rb(K_n, kK_2)$ for all $k \geq 2$ and $n \geq 3k+3$, where $H = kK_2$ is a matching M of size k. The main focus of this paper is to consider the analogous problem for matchings when the host graph G is a complete bipartite graph $K_{m,n}$ (say $m \geq n$). For all positive integers $m \geq n$ and $k \geq 1$, we determine the exact values of $rb(K_{m,n}, kK_2)$.

2 Main results

Let M be a matching in a given graph G, then the subgraph of G induced by M, denoted by $\langle M \rangle_G$ or $\langle M \rangle$, is the subgraph of G whose edge set is M and whose vertex set consists of the vertices incident with some edges in M. A vertex of G is said to be *saturated* by M if it is incident with an edge of M; otherwise, it is said to be *unsaturated*. If every vertex of a vertex subset U of G is saturated, then we say that U is saturated by M. A matching with maximum cardinality is called a maximum matching.

In a given graph G, $N_G(U)$ denotes the set of vertices of G adjacent to the vertex set U. If $R, T \in V(G)$, we denote $E(R,T)$ or $E_G(R,T)$ as the set of all edges having a vertex from both R and T. Let $G(m,n)$ denote a bipartite graph with bipartition $A \cup B$, and $|A| = m$ and $|B| = n$, without loss of generality, in the following we always assume $m \geq n$.

Let $ext(m,n,H)$ denote the maximum number of edges that a bipartite graph $G(m,n)$ can have with no subgraph isomorphic to H. The bipartite graphs attaining the maximum for given m and n are called extremal graphs.

We now determine the value $ext(m,n,H)$ for $H = kK_2$. The first lemma is due to Ore and can be found in [6].

Lemma 2.1 *Let $G(m,n)$ be a bipartite graph with bipartition $A \cup B$, and M a maximum matching in G. Then the size of M is $m - d$, where*

$$d = \max\{|S| - |N_G(S)| : S \subseteq A\}.$$

Theorem 2.2

$$ext(m,n,kK_2) = m(k-1) \text{ for all } 1 \leq k \leq n,$$

that is, for any given bipartite graph $G(m,n)$, if $|E(G(m,n))| > m(k-1)$, then $kK_2 \subset G$. Moreover, $K_{m,(k-1)}$ is the unique such extremal graph.

Proof. Suppose that G contains no kK_2. Let M be a maximum matching of G and the size of M is $k - i$, where $i \geq 1$. By Lemma 2.1, there exists a subset $S \subset A$ such that $|S| - |N_G(S)| = m - k + i$. Thus

$$|E(G)| \leq |S||N_G(S)| + n(m - |S|) = (|N_G(S)| + m - k + i)|N_G(S)| + n(k - i - |N_G(S)|).$$

Since $0 \leq |N_G(S)| \leq k - i \leq k - 1$, we obtain

$$|E(G)| \leq \max\{m(k-1), n(k-1)\} \leq m(k-1),$$

where the equality is possible only if $i = 1$ and $G \cong K_{m,k-1}$. So, $K_{m,k-1}$ is the unique such extremal graph. ∎

For $k = 1$, it is clear that $rb(K_{m,n}, K_2)$=1. Now we determine the value of $rb(K_{m,n}, 2K_2)$ (for $k = 2$).

Theorem 2.3

$$rb(K_{2,2}, 2K_2) = 3,$$

and

$$rb(K_{m,n}, 2K_2) = 2 \text{ for all } m \geq 3 \text{ and } n \geq 2.$$

Proof. It is obvious that $rb(K_{2,2}, 2K_2) \leq 3$. Let $\{a_1, a_2\} \cup \{b_1, b_2\}$ be the two parts of $K_{2,2}$. If $K_{2,2}$ is edge-colored with 2 colors such that $c(a_1b_1) = c(a_2b_2) = 1$ and $c(a_1b_2) = c(a_2b_1) = 2$, then $K_{2,2}$ contains no TMC $2K_2$. So, $rb(K_{2,2}, 2K_2) = 3$.

For $m \geq 3$ and $n \geq 2$, let the edges of $G = K_{m,n}$ be colored with at least 2 colors. We suppose that the two parts of $K_{m,n}$ are A and B with $|A| = m$ and $|B| = n$. Suppose that $K_{m,n}$ contains no TMC $2K_2$. Let $e_1 = a_1b_1$, $a_1 \in A$, $b_1 \in B$, be an edge with $c(e_1) = 1$, and $R = V(K_{m,n}) - \{a_1, b_1\}$. Then $c(e) = 1$ for all edges $e \in E(G[R])$. Moreover, $c(e) = 1$ for all edges $e \in E(b_1, R)$, since $m \geq 3$. Thus $c(e) = 1$ for all edges $e \in E(a_1, R)$. But then $K_{m,n}$ is monochromatic, a contradiction. So, $rb(K_{m,n}, 2K_2) = 2$ for all $m \geq 3$ and $n \geq 2$. ∎

The next proposition provides a lower and upper bound for $rb(K_{m,n}, kK_2)$, and the proof of which is the same as that of [7].

Proposition 2.4 $ext(m, n, (k-1)K_2) + 2 \leq rb(K_{m,n}, kK_2) \leq ext(m, n, kK_2) + 1$.

Proof. The upper bound is obvious. For the lower bound, an extremal coloring of $K_{m,n}$ can be obtained from an extremal graph $K_{m,k-2}$ for $ext(m, n, (k-1)K_2)$ by coloring the edges of $K_{m,k-2}$ differently and the edges of $\overline{K_{m,k-2}}$ by one extra color. So, the coloring does not contain a TMC kK_2. ∎

In the sequel, for a set of vertices X of a graph G we use $G - X$ to denote the subgraph of G obtained by deleting from G all the vertices in X and all the edges incident with them; whereas for a set of edges F of G we use $G - F$ to denote the subgraph of G obtained by deleting from G only all the edges in F. If $F = \{f\}$ has a single edge, we simply use $G - f$ to denote $G - \{f\}$. We will now show that the lower bound can be achieved for all $m \geq n \geq k \geq 3$, and thus obtain the exact value of $rb(K_{m,n}, kK_2)$ for all $k \geq 3$.

Theorem 2.5 $rb(K_{m,n}, kK_2) = ext(m, n, (k-1)K_2) + 2 = m(k-2) + 2$ *for all* $m \geq n \geq k \geq 3$.

Proof. For $m \geq n \geq k \geq 3$, let the edges of $K_{m,n}$ be colored with $m(k-2)+2$ colors. Suppose that $K_{m,n}$ contains no TMC kK_2. Since $m(k-2) + 2 = ext(m, n, (k-1)K_2) + 2$, there is a TMC $(k-1)K_2$ in the coloring of $K_{m,n}$. Now let $G \subset K_{m,n}$ be a TMC spanning subgraph of size $m(k-2) + 2$ containing a $(k-1)K_2$, and so $|E(G)| = m(k-2) + 2$. We suppose that the two parts of the bipartite graph G

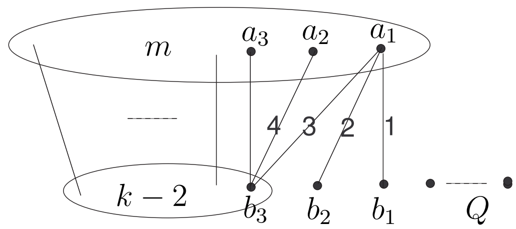

Figure 1: The special case of graph G in Claim 1.

are A and B with $|A| = m$ and $|B| = n$. First, we prove the following two assertions.

Claim 1: If one component of G consists of a $K_{m,k-2}$ and two adjacent pendant edges and the other components are isolated vertices (see Figure 1), then $K_{m,n}$ contains a TMC kK_2.

Denote SG_1 as the special case of graph G with the property in Claim 1, and Q as the set of isolated vertices of G. The proof of the claim is given by distinguishing the following two cases:

Case I. $m = n = k = 3$.

Without loss of generality, we suppose that $A = \{p_1, p_2, p_3\}$, $B = \{q_1, q_2, q_3\}$, $c(p_1q_1) = 1$, $c(p_2q_1) = 2$, $c(p_3q_1) = 3$, $c(p_3q_2) = 4$ and $c(p_3q_3) = 5$. Suppose $K_{m,n}$ contains no $3K_2$. Hence $c(p_2q_2) \in \{1, 5\}$ and $c(p_1q_3) \in \{2, 4\} \cap \{3, c(p_2q_2)\} = \emptyset$, a contradiction.

Case II. $m \geq 4$.

Without loss of generality, we suppose that $c(a_1b_1) = 1$, $c(a_1b_2) = 2$, $c(a_1b_3) = 3$ and $c(a_2b_3) = 4$ (see Figure 1).

We will show that $c(a_2b_2) = 1$. If $c(a_2b_2) = 2$, 3 or 4, it is obvious that there is a TMC kK_2 in $K_{m,n}$. Otherwise, $c(a_2b_2) = q$ $(5 \leq q \leq m(k-2)+2)$. In $G_1 = G - \{Q \cup a_1 \cup a_2 \cup b_1 \cup b_2\}$, the number of edges whose colors are not q is at least $(m-2)(k-2)-1$. Since $m \geq 4$, we have $(m-2)(k-2)-1 \geq ext(m-2, k-2, (k-2)K_2)+1 = (m-2)(k-3)+1$. Thus we can obtain a TMC $H = (k-2)K_2$ in G_1, and hence there is a TMC $kK_2 = H \cup \{a_1b_1, a_2b_2\}$ in $K_{m,n}$. So $c(a_2b_2)$ must be 1.

We shall show that $c(a_3b_1) \in \{2, 4\}$. If $c(a_3b_1) = 1$ or 3, we can obtain a TMC $H' = (k-3)K_2$ in $G_2 = G - \{Q \cup a_1 \cup a_2 \cup a_3 \cup b_1 \cup b_2 \cup b_3\}$, and hence there is a TMC $kK_2 = H' \cup \{a_1b_2, a_2b_3, a_3b_1\}$ in $K_{m,n}$. Otherwise, $c(a_3b_1) = q$ $(5 \leq q \leq m(k-2)+2)$. In $G_3 = G - \{Q \cup a_1 \cup a_3 \cup b_1 \cup b_2\}$, the number of edges whose colors are not q

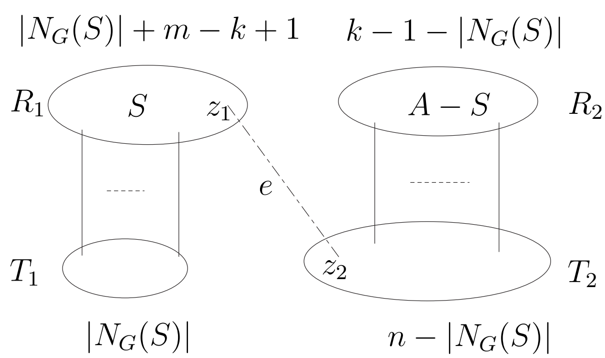

Figure 2: T_1 and R_2 are saturated by M.

is at least $(m-2)(k-2)-1$. Similar to the above arguments, we can obtain a TMC $H''=(k-2)K_2$ in G_3, and hence there is a TMC $kK_2=H''\cup\{a_1b_2,a_3b_1\}$ in $K_{m,n}$. Thus, $c(a_3b_1)\in\{2,4\}$.

Now we can obtain a TMC $H'''=(k-3)K_2$ in $G_2=G-\{Q\cup a_1\cup a_2\cup a_3\cup b_1\cup b_2\cup b_3\}$, and hence there is a TMC $kK_2=H'''\cup\{a_1b_3,a_2b_2,a_3b_1\}$ in $K_{m,n}$.

Claim 2: If one component of G consists of a $K_{m-1,m-1}$ and a pendant edge (say pv with $d_G(v)=1$) and the other component is an isolated vertex (say u), then $K_{m,n}$ contains a TMC mK_2.

Denote SG_2 as the special case of graph G with the property in Claim 2. The proof of the claim is given as follows:

Now $|E(G)|=(m-1)^2+1=m(k-2)+2$, and $m=n=k$. Without loss of generality, we suppose $c(uv)=1$. Then in $G_3=G-u-v$, the number of edges whose colors are not 1 is at least $(m-1)(m-1)-1$. Since $m\geq 3$, we have $(m-1)(m-1)-1\geq ext(m-1,m-1,(m-1)K_2)+1=(m-1)(m-2)+1$. Thus we can obtain a TMC $H=(m-1)K_2$ in G_3, and hence there is a TMC $mK_2=H\cup uv$ in $K_{m,n}$.

From Lemma 2.1 we have that there exists a subset S of A such that $|S|-|N_G(S)|=m-k+1$ and $0\leq|N_G(S)|\leq k-1$. We define in G $R_1=S$ and $R_2=A-S$, $T_1=N_G(S)$ and $T_2=B-N_G(S)$. Let M be a maximum matching of G, then T_1 and R_2 are saturated by M. There exists an edge $e=z_1z_2\in E_{K_{m,n}}(R_1,T_2)$, $z_1\in R_1$, $z_2\in T_2$, $z_1\notin\langle M\rangle, z_2\notin\langle M\rangle$. Without loss of generality, we suppose $c(e)=1$, see Figure 2. So, there exists an edge $e_1\in M$ such that $c(e_1)=1$. Now we distinguish three cases to finish the proof of the theorem.

Case 1. $|N_G(S)|=k-1$.

In this case, $R_1 = A$, there is no $(k-1)K_2$ in $G' = G - (T_2 \cup z_1) - e_1$. By Theorem 2.2, $|E(G')| \leq (m-1)(k-2)$. Thus,

$$|E(G)| = 1 + |E(G')| + |E_G(z_1, T_1)| \leq 1 + (m-1)(k-2) + (k-1) \leq m(k-2) + 2.$$

If $m > k$, since $|E(G)| = m(k-2)+2$, then $G' = K_{m-1,k-2}$ and $|E_G(z_1, T_1)| = k-1$. It is easy to check that $(G - e_1 + e) \cong SG_1$, and by the proof of **Claim 1** we can obtain a TMC kK_2 in $K_{m,n}$. If $m = k$, again since $|E(G)| = m(k-2)+2$, it is easy to check that $(G - e_1 + e) \cong SG_1$ or $G \cong SG_2$, and by **Claim 1** and **Claim 2** we can obtain a TMC kK_2 in $K_{m,n}$.

Case 2. $|N_G(S)| = 0$.

In this case, $G' = G - (R_1 \cup z_2) - e_1$ and there is no $(k-1)K_2$ in G'. Similarly,

$$|E(G)| = 1 + |E(G')| + |E_G(z_2, R_2)| \leq 1 + (n-1)(k-2) + (k-1) \leq n(k-2) + 2.$$

If $m > n$, this contradicts that G has $m(k-2)+2$ edges; if $m = n$, by Case 1 we can obtain a TMC kK_2 in $K_{m,n}$.

Case 3. $1 \leq |N_G(S)| \leq k-2$.

Subcase 3.1. $e_1 \in E_G(R_2, T_2)$.

In this case, there is no TMC $(k-1-|N_G(S)|)K_2$ in $G' = G[R_2 \cup T_2] - z_2 - e_1$. Thus,

$$\begin{aligned}
|E(G)| &= 1 + |E(G')| + |E_G(z_2, R_2)| + |E_G(T_1, A)| \\
&\leq 1 + (k-2-|N_G(S)|)(n - |N_G(S)| - 1) \\
&\quad +(k-1-|N_G(S)|) + m|N_G(S)| \\
&\leq \max\{3 + n(k-2) + (m-n) - (k-2), m(k-2)+2\} \\
&\leq m(k-2)+2.
\end{aligned}$$

Since $|E(G)| = m(k-2)+2$, it is easy to check that G' is an empty graph and $G \cong SG_1$, and hence there is a TMC kK_2 in $K_{m,n}$.

Subcase 3.2. $e_1 \in E_G(R_1, T_1)$.

In this case, $G' = G[R_1 \cup T_1] - z_1 - e_1$ and there is no TMC $|N_G(S)|K_2$ in G'. Similarly,

$$\begin{aligned}
|E(G)| &= 1 + |E(G')| + |E_G(z_1, T_1)| + |E_G(R_2, B)| \\
&\leq 1 + (|N_G(S)|-1)(|N_G(S)| + m - k) + |N_G(S)| + n(k-1-|N_G(S)|) \\
&\leq \max\{3 + m(k-2) + (n-m) - (k-2), n(k-2)+2\} \\
&\leq m(k-2)+2.
\end{aligned}$$

If $m > n$, then $|E(G)| < m(k-2)+2$, a contradiction. Otherwise, $|E(G)| = m(k-2)+2$ only if $|N_G(S)| = 1$ and G' is an empty graph and $G \cong SG_1$, hence there is a TMC kK_2 in $K_{m,n}$. The proof is now complete. ∎

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