Measure-Valued Flows Given Consistent Exchangeable Families

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Abstract Given a Polish space E and a consistent exchangeable family of all k-point motions in E, a universal framework on related measure-valued flow (MVF) describing how probabilities evolve under the consistent exchangeable family is given. And when $E = R^1$ and 1-point motion is a 1-dimensional diffusion, local time and Tanaka formula for MVFs are studied[1].

Keywords Measure-valued flow · strong Markov property · local time and Tanaka formula

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1. Introduction: Motivations for measure-valued flows

Given any Polish space E, let $\mathcal{B}_b(E)$ (resp. $C_b(E)$) be the set of all bounded measurable (resp. bounded continuous) functions on it, and $\|\cdot\|$ the uniform norm on $\mathcal{B}_b(E)$. Let C_E (resp. D_E) be the space of all continuous (resp. càdlàg) maps from $[0,\infty)$ to E equipped with the locally uniform (resp. Skorohod) topology. Note $M_1(E)$ (resp. $M(E)$), the set of all probabilities (resp. finite measures) on E endowed with the weak topology, can be a Polish space under a suitable compatible metric. Denote by

$$\langle \mu, f\rangle = \mu(f) = \int_E f \ d\mu$$

the integral of a measurable function f on E against a measure $\mu \in M(E)$ if it exists. Let

$$\begin{aligned} &F_{f,k}(\mu) = \langle \mu^k, f\rangle = \langle \mu^{\otimes k}, f\rangle := \int_{E^k} f \ d\mu^k, \ \mu \in M_1(E), \ f \in \mathcal{B}_b\left(E^k\right), \ k \geq 1; \\ &\mathcal{B}_p(M_1(E)) = \left\{F_{f,k} \ \middle| \ \forall k \geq 1, \ \forall f \in \mathcal{B}_b\left(E^k\right)\right\}, \\ &C_p(M_1(E)) = \left\{F_{f,k} \ \middle| \ \forall k \geq 1, \ \forall f \in C_b\left(E^k\right)\right\}. \end{aligned}$$

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Write $C_0(E)$ for the set of all continuous functions on E vanishing at infinity when E is locally compact.

For any Euclidean space R^l, let $C_b^\infty\left(R^l\right)$ be the set of all smooth functions on R^l with derivatives of any orders are bounded, and $C_b^2\left(R^l\right)$ the set of all bounded continuous functions on R^l with bounded continuous derivatives of orders one and two. Let $\mathbf{N}$ be the set of all natural numbers.

Recall some notions for Markov processes. The semigroup $\{V_t\}_{t\geq 0}$ on $\mathcal{B}_b(E)$ of an E-valued càdlàg Markov process is weakly Fellerian if

$$V_t f \in C_b(E),\ \forall f \in C_b(E),\ \forall t \in R_+ := [0,\infty);$$

$\{V_t\}_{t\geq 0}$ is said to be Fellerian on $C_b(E)$ (resp. $C_0(E)$ when E is locally compact) if it is a strongly continuous semigroup on Banach space $(C_b(E), \|\cdot\|)$ (resp. $(C_0(E), \|\cdot\|)$).

For each $k \in \mathbf{N}$, let $\mathbf{Y^k}$ be an exchangeable càdlàg Markov process on E^k with the semigroup $\left\{V_t^k\right\}_{t\geq 0}$ on $\mathcal{B}_b\left(E^k\right)$. If any n-component of $\mathbf{Y^k}$ evolves like $\mathbf{Y^n}$ for any n, k with $n \leq k$, we say the family $\left\{\mathbf{Y^k}\right\}_{k\geq 1}$ or $\left\{\left\{V_t^k\right\}_{t\geq 0}\right\}_{k\geq 1}$ is consistent. For consistent family $\left\{\mathbf{Y^k}\right\}_{k\geq 1}$, write

$$\mathbf{Y^k} = \left(\left(Y_t^1(x_1), \cdots, Y_t^k(x_k)\right)_{t\geq 0}\right)_{(x_1,\cdots,x_k)\in E^k},$$

where $\left(Y_t^1(x_1), \cdots, Y_t^k(x_k)\right)$ is the position of $\mathbf{Y^k}$ at time t starting from $(x_1, \cdots, x_k)$.

When E is locally compact, given a consistent exchangeable family $\left\{\mathbf{Y^k}\right\}_{k\geq 1}$ in E (resp. with the property that any two particles must stay together whenever they meet), assume each $\left\{V_t^k\right\}_{t\geq 0}$ is Fellerian on $C_0\left(E^k\right)$, by [25], there is a unique (in law) stochastic flow $(K_t)_{t\geq 0}$ of kernels (resp. stochastic flow $(\phi_t)_{t\geq 0}$ of measurable maps) associated to the consistent exchangeable family, and this correspondence is one-to-one. For any $\mu \in M_1(E)$, its transportation under the flow is given by $(\mu K_t)_{t\geq 0}$ (resp. $((\phi_t)_*\mu)_{t\geq 0}$), where

$$(\mu K_t)(dy) = \int_E K_t(x, dy)\ \mu(dx)\ \textit{and}\ (\phi_t)_*\mu(\cdot) = \mu \circ \phi_t^{-1}(\cdot).$$

Note in [25], the measure-valued process $(\mu K_t)_{t\geq 0}$ is constructed before constructing the flow of kernels. The mentioned càdlàg $M_1(E)$-valued Markov process is of the semigroup $\{T_t\}_{t\geq 0}$ satisfying

$$T_t F_{f,k}(\mu) = F_{V_t^k f,k}(\mu) = \left\langle \mu^k, V_t^k f\right\rangle,\ \forall F_{f,k} \in \mathcal{B}_p(M_1(E)),\ \forall t \geq 0. \tag{1.1}$$

Now the following question arises from a theoretic view point: Given any Polish space E and any consistent exchangeable càdlàg Markov process family $\left\{\mathbf{Y^k}\right\}_{k\geq 1}$ on it. How do probabilities are "transported" under the family? Namely, is there a unique càdlàg $M_1(E)$-valued Markov process $\mathbf{X} = (\mathbf{X}_t)_{t\geq 0}$ with the semigroup $\{T_t\}_{t\geq 0}$ determined by (1.1)? If yes, then study its nice properties.

Motivations for studying the above processes are described as follows.

In the deterministic situation, the flow of smooth (even measurable) maps and the (finite or infinite) measure transported by this flow are extensively studied in ergodic theory ([8], [1]). While for a stochastic flow $(\phi_t)_{t\geq 0}$ of measurable maps on E (not necessarily to be locally compact), from

$$\langle (\phi_t)_*\mu, f\rangle = \langle \mu, f\circ\phi_t\rangle,\ \forall f\in\mathcal{B}_b(E),\ \forall t\geq 0,\ \forall \mu\in M_1(E); \tag{1.2}$$

$\left(((\phi_t)_*\mu)_{t\geq 0}\right)_{\mu\in M_1(E)}$ is a dual of the flow and hence of its own interests (see [23] P135-147 for some interests of the process) and can be viewed as a measure-valued flow. Moreover, there is a probabilistic notion, decay of correlations, expressing sensitivity of the dynamics, is of importance in the characterization of complex systems ([36]); here sensitiveness means orbits forget their initial state as time increases to ∞, which may be expressed by

$$C_t^\mu(f,g) = \int_E f(z)(g\circ\phi_t)(z)\mu(dz) - \int_E f\ d\mu\ \int_E g\ d\mu$$

should converges rapidly to zero as $t\to\infty$, for any f, g in some continuous function space $\mathcal{F}$. Where $\mu\in M_1(E)$ (random or non-random) is a Sinai-Ruelle-Bowen (SRB) measure (note even for stochastic dynamical systems, SRB-measure may be deterministic, see [2]). Assume $\mu_f(dx) = f(x)\mu(dx)\in M_1(E)$. Then

$$C_t^\mu(f,g) = \int_E g(z)(\phi_t)_*\mu_f(dz) - \int_E g\ d\mu,$$

and the process $((\phi_t)_*\mu_f)_{t\geq 0}$ comes into picture.

On the other hand, if E is a Riemannian manifold with μ being the volume measure (not necessarily to be a probability), the incompressibility of $(\phi_t)_{t\geq 0}$ is defined by

$$(\phi_t)_*\mu = \mu,\ \forall t\geq 0,\ a.s..$$

Note incompressibility is important for vorticity and turbulence from a view point of physics ([18], [28]) and for ergodic theory; and stochastic flows are usually viewed as turbulence models. So it is interesting how measures or probabilities evolve under a stochastic flow.

Note 1-point motion of a stochastic flow can be a Markov process usually, and not every Markov process can correspond to a stochastic flow due to "singularities"; and representation of Markov processes by a stochastic flow is an interesting research topic. Refer to [21] Chapter I, [23], [32], [12], [16]. For some Markov process, there might be an interesting consistent exchangeable family of all k-point motions (liking those of a flow) such that one-point motion is the given Markov process, but there is no corresponding stochastic flow of measurable maps (resp. kernels).

So in order to study how probabilities are transported under flows, motivated by Examples 2.3-2.8, we need to study the $M_1(E)$-valued processes mentioned before. Notice (1.2). The $M_1(E)$-valued Markov process with the semigroup $\{T_t\}_{t\geq 0}$ satisfying (1.1) is called a measure-valued flow given consistent exchangeable family $\left\{\mathbf{Y^k}\right\}_{k\geq 1}$ $\left(\text{or } \left\{\{V_t^k\}_{t\geq 0}\right\}_{k\geq 1}\right)$.

Our long aim is to establish a theory for MVFs. The present paper is at an initial stage: a universal framework on MVFs is given by Theorem 2.1; and results on local time and Tanaka formula for MVFs (and Fleming-Viot processes) are presented by Theorem 3.1 (and Theorem 3.1′ respectively). Some interesting properties on MVFs will be studied by other papers.

2. Main result I: Framework on measure-valued flows

Theorem 2.1. Given any Polish space E and any consistent exchangeable càdlàg Markov process family $\left\{\mathbf{Y^k}\right\}_{k\geq 1}$ on it. Denote by $\left\{V_t^k\right\}_{t\geq 0}$ the semigroup of each $\mathbf{Y^k}$. Then the following results hold.

(**1**) *There is a unique family of probabilities* $\{P_\nu\}_{\nu\in M_1(E)}$ *on* $D_{M_1(E)}$ *such that the coordinate process* $\mathbf{X} = (\mathbf{X}_t)_{t\geq 0}$ *on* $D_{M_1(E)}$ *becomes a Markov process under* $\{P_\nu\}_{\nu\in M_1(E)}$ *whose semigroup* $\{T_t\}_{t\geq 0}$ *satisfies* (1.1); *and* $\mathbf{X}$ *is continuous under* P_ν *for any* $\nu \in M_1(E)$ *if so is each* $\mathbf{Y^k}$.

(**2**) *If each* $\mathbf{Y^k}$ *is strongly Markovian, then so is* $\left(\mathbf{X}, \{P_\nu\}_{\nu\in M_1(E)}\right)$.

(**3**) $\left(\mathbf{X}, \{P_\nu\}_{\nu\in M_1(E)}\right)$ *is weakly Fellerian and strongly Markovian provided each* $\mathbf{Y^k}$ *is weakly Fellerian.*

Remark 2.2. One of the most interesting objects associated to a given consistent exchangeable family is the stochastic flow of kernels (measurable maps)([25-26]). The present paper limits to how probabilities evolve under the consistent exchangeable family.

The considered measure-valued process can be viewed as a measure-valued flow given a consistent family of all k-point exchangeable processes. On one hand, measure-valued flow is of interests for stochastic flows of measurable maps or kernels; on the other hand, an interest in the measure-valued flows is that in some instances they may exist whereas the general stochastic flows of measurable maps or kernels in [25-26] do not (see Examples in section 2).

Note if E is Cosouslinian, then so are D_E and C_E ([10]). Theorem 2.1 holds for Cosouslin space, which is left as an exercise to the interested readers.

[6] studied a class of stochastic flows connected to the coalescent processes (a class of probability-valued Markov processes). In [6], an important auxiliary measure-valued process viewed as a generalized Fleming-Viot process, is a measure-valued flow.

Example 2.3. There are consistent exchangeable families $\left\{\mathbf{Y^k}\right\}_{k\geq 1}$ in Euclidean spaces such that one can not determine generally whether each $\mathbf{Y^k}$ is weakly Fellerian.

(i) Fix $d \in \mathbf{N}$. For any $k \in \mathbf{N}$, define an operator A_k as follows:

$$\begin{aligned} A_k f(z_1, \cdots, z_k) \quad = \quad & \frac{1}{2}\sum_{i=1}^{k}\sum_{p,q=1}^{d} a^{pq}(z_i)\frac{\partial^2 f}{\partial z_i^p \partial z_i^q}(z_1, \cdots, z_k) + \\ & \frac{1}{2}\sum_{1\leq i\neq j\leq k}\sum_{p,q=1}^{d} a^{pq}(z_i, z_j)\frac{\partial^2 f}{\partial z_i^p \partial z_j^q}(z_1, \cdots, z_k) + \\ & \sum_{i=1}^{k}\sum_{p=1}^{d} b^p(z_i)\frac{\partial f}{\partial z_i^p}(z_1, \cdots, z_k), \end{aligned}$$

$$\forall f \in C_b^2\left(\left(R^d\right)^k\right),\ (z_1, \cdots, z_k) \in \left(R^d\right)^k,\ z_i = \left(z_i^1, \cdots, z_i^d\right) \in R^d.$$

Here for any $1 \le p, q \le d$,

$$a^{pq}(\cdot) \in C_b\left(R^d\right), \; a^{pq}(\cdot,\cdot) \in C_b\left(\left(R^d\right)^2\right), \; b^p(\cdot) \in \mathcal{B}_b\left(R^d\right),$$
$$a^{pq}(z_1) = a^{qp}(z_1), \; a^{pq}(z_1, z_2) = a^{qp}(z_2, z_1), \; \forall (z_1, z_2) \in \left(R^d\right)^2;$$

and there is a constant $\eta > 0$ such that for any $k \ge 1$ and any $(z_1, \cdots, z_k) \in \left(R^d\right)^k$,

$$\sum_{i=1}^{k} \sum_{p,q=1}^{d} a^{pq}(z_i)\lambda_i^p \lambda_i^q + \sum_{1 \le i \ne j \le k} \sum_{p,q=1}^{d} a^{pq}(z_i, z_j)\lambda_i^p \lambda_j^q \ge \eta \sum_{i=1}^{k} \sum_{p=1}^{d} \left(\lambda_i^p\right)^2 \ge 0,$$
$$\forall \left(\lambda_i^1, \cdots, \lambda_i^d\right) \in R^d, 1 \le i \le k.$$

For each $k \ge 1$, use

$$\mathbf{Y^k} = \left(\left(Y_t^1(z_1), \cdots, Y_t^k(z_k)\right)_{t \ge 0}\right)_{(z_1, \cdots, z_k) \in (R^d)^k}$$

to denote the unique A_k diffusion process, where $\left(Y_t^1(z_1), \cdots, Y_t^k(z_k)\right)$ is the position of the process at time t starting at $(z_1, \cdots, z_k)$. Note $\left\{\mathbf{Y^k}\right\}_{k \ge 1}$ is a consistent exchangeable family. When $b \in C_b\left(R^d\right)$, each $\mathbf{Y^k}$ is weakly Fellerian. But for arbitrary $b \in \mathcal{B}_b\left(R^d\right) \setminus C_b\left(R^d\right)$, one can not determine generally whether each $\mathbf{Y^k}$ is weakly Fellerian (For 'bad' coefficients, one tends to believe $\mathbf{Y^k}$ is not weakly Fellerian).

(ii) Assume $0 \le a(\cdot) \in \mathcal{B}_b\left(R^1\right)$ and $a(\cdot)^{-1}$ is locally integrable, and let

$$Z(a) = \left\{ x \in R^1 \;\middle|\; a(x) = 0 \right\}.$$

Note

$$I(a) = \left\{ x \in R^1 \;\middle|\; \int_{-\epsilon}^{\epsilon} \frac{1}{a(x+y)}\, dy = \infty, \; \forall \epsilon > 0 \right\} = \emptyset \subseteq Z(a).$$

Then for any initial point, $\left(\frac{1}{2}a(x)\frac{\partial^2}{\partial x^2}, C_b^2\left(R^1\right)\right)$-martingale problem has a nonexploding solution, and it has a unique nonexploding solution if and only if $Z(a) = \emptyset$. See [14] and [7]. For all strong Markov processes (and Markov processes) generated by the operator $\frac{1}{2}a(x)\frac{\partial^2}{\partial x^2}$, refer to [14].

Let $\Omega = \{\omega \in C_{R^1} \mid \omega_0 = 0\}$; and μ_0 be the Wiener measure, $\omega = (\omega_t)_{t \ge 0}$ the coordinate process on Ω. Then ω is the standard 1-dimensional Brownian motion starting at 0. Let

$$T(s, x) = T(s, x)(\omega) = \int_0^s \frac{1}{a(x + \omega_u)}\, du, \forall s \in [0, \infty), \; x \in R^1.$$

Then for any fixed $x \in R^1$, due to $a(\cdot)$ is locally integrable against Lebesgue measure,

$$T(s, x) < \infty \textit{ is continuous in } s \in [0, \infty), \; \mu_0 \; - \; a.s.;$$

and due to

$$T(t, x) - T(r, x) \ge \frac{t - r}{\|a(\cdot)\|} > 0, \; T(r, x) \ge \frac{r}{\|a(\cdot)\|}, \; \forall 0 \le r < t < \infty,$$

$T(s,x)$ *strictly increases to* ∞ *as* $s\uparrow\infty$, *a.s.*. Put

$$R(t,x)=R(t,x)(\omega)=\inf\{s\geq 0\mid T(s,x)(\omega)>t\}.$$

Then for fixed $x\in R^1$, almost surely,

$$R(t,x) \textit{ is continuous and strictly increasing in } t \textit{ and } \lim_{t\to\infty}R(t,x)=\infty. \tag{2.1}$$

For any $x\in R^1$, let

$$Y_t(x)=x+\omega_{R(t,x)(\omega)},\ t\in[0,\infty).$$

Then

$$\mathbf{Y}=\Big((Y_t(x))_{t\geq 0}\Big)_{x\in R^1}$$

is a $\frac{1}{2}a(x)\frac{\partial^2}{\partial x^2}$-diffusion process. If $Z(a)\neq\emptyset$, define a different $\frac{1}{2}a(x)\frac{\partial^2}{\partial x^2}$-diffusion process $\overline{\mathbf{Y}}=\Big((\overline{Y}_t(x))_{t\geq 0}\Big)_{x\in R^1}$ as follows:

$$\overline{Y}_t(x)=Y_{t\wedge\tau(x)}(x) \textit{ with } \tau(x)=\inf\{s\geq 0\mid Y_s(x)\in Z(a)\},\ t\in[0,\infty).$$

Any two independent $\mathbf{Y}$ particles must meet at a finite time due to (2.1) and the difference process for the two independent particles is a continuous martingale whose quadratic variation process tends to $+\infty$ as time goes to $+\infty$. For any $k\geq 1$, let $\mathbf{Y^k}$ be the process obtained from k-independent copies of the process $\mathbf{Y}$ by letting any two particles stay together whenever they meet. Write

$$\mathbf{Y^k}=\Big((Y_t^1(x_1),\cdots,Y_t^k(x_k))_{t\geq 0}\Big)_{(x_1,\cdots,x_k)\in R^k}.$$

Clearly, $\{\mathbf{Y^k}\}_{k\geq 1}$ is a consistent exchangeable family. For arbitrary measurable $a(\cdot)$, one can not determine generally whether each $\mathbf{Y^k}$ is weakly Fellerian. For $\overline{\mathbf{Y}}$, if $Z(a)$ is closed, then

$$R^1\setminus Z(a)=\bigcup_{i=1}^{\infty}I_i,\ I_i=(a_i,b_i)\neq R^1,\ I_i\cap I_j=\emptyset,\ i\neq j,\ a_i,b_i\in R^1\cup\{-\infty,\infty\}.$$

For some $i\neq j$ with $[a_i,b_i]\cap[a_j,b_j]=\emptyset$, two independent $\overline{\mathbf{Y}}$ particles starting at I_i and I_j respectively can not meet; while with positive probability, any two independent $\overline{\mathbf{Y}}$ particles starting at the same I_i can meet at a finite time before they are absorbed at the boundary of I_i. Similarly to $\mathbf{Y}$, one also can get a nontrivial consistent exchangeable family for $\overline{\mathbf{Y}}$ with the property that any two particles must stay together whenever they meet. The mentioned trick can be applied to other $\frac{1}{2}a(x)\frac{\partial^2}{\partial x^2}$-diffusions.

For $b(\cdot)\in\mathcal{B}_b(R^1)$ such that $b(\cdot)^2\leq c\ a(\cdot)$ *for some constant* $c>0$, use the Girsanov argument, one can get $\frac{1}{2}a(x)\frac{\partial^2}{\partial x^2}+b(x)\frac{\partial}{\partial x}$ diffusion $\mathbf{Y^1}=\big((Y_t^1(x))_{t\geq 0}\big)_{x\in R^1}$ from $\mathbf{Y}$ and $\overline{\mathbf{Y}}$ and other $\frac{1}{2}a(x)\frac{\partial^2}{\partial x^2}$-diffusions (when $Z(a)\neq\emptyset$). Apply the trick mentioned in previous paragraph to $\frac{1}{2}a(x)\frac{\partial^2}{\partial x^2}+b(x)\frac{\partial}{\partial x}$ diffusions to get nontrivial consistent exchangeable families with the property that any two particles must stay together whenever they meet.

Example 2.4. In the situation of Example 2.3(i), assume further $(a^{pq}(x))_{1\le p,q\le d}$ is the identity matrix for any $x\in R^d$, that is, the A_1-diffusion is a d-dimensional Brownian motion with a drift. Write

$$Y_t^1(x)=\left(Y_t^{1,1}(x),\cdots,Y_t^{1,d}(x)\right)\in R^d,\ t\in[0,\infty),\ x\in R^d.$$

Then for any $x\in R^d$ and $t\in(0,\infty)$, by the Girsanov argument,

$$P\left[Y_t^1(x)=0\right]=P\left[Y_t^{1,1}(x)=0\right]=0. \tag{2.2}$$

For any $x\in R^d$, write $x=\left(x^1,\cdots,x^d\right)\in R^d$; and for each $k\in\mathbf{N}$, let

$$Z_t^i(x_i)=Y_t^i(x_i)I_{\left[x_i^1\neq 0\right]}+x_iI_{\left[x_i^1=0\right]},\ (x_1,\cdots,x_k)\in\left(R^d\right)^k,\ t\in[0,\infty),\ i\le k,$$
$$\mathbf{Z^k}=\left(\left(Z_t^1(x_1),\cdots,Z_t^k(x_k)\right)_{t\ge0}\right)_{(x_1,\cdots,x_k)\in(R^d)^k}.$$

Then by $\left\{\mathbf{Y^k}\right\}_{k\ge1}$ is a consistent exchangeable family and (2.2), we see $\left\{\mathbf{Z^k}\right\}_{k\ge1}$ is also consistent and exchangeable, and each $\mathbf{Z^k}$ is a continuous Markov process on $\left(R^d\right)^k$. But $\mathbf{Z^1}$ is not strongly Markovian and hence is not weakly Fellerian.

Indeed, otherwise, for any fixed $x\in R^d$ with $x^1\neq0$, let

$$\tau=\inf\left\{t>0\ \middle|\ Z_t^{1,1}(x)=0\right\},\ \textit{where}\ Z_t^1(x)=\left(Z_t^{1,1}(x),\cdots,Z_t^{1,d}(x)\right);$$

by the strong Markov property of $\mathbf{Z^1}$,

$$E\left[Z_1^{1,1}(x)\neq0,\ \tau\le1\right]=E\left[I_{[\tau\le1]}P_{1-\tau}\left(Z_\tau^1(x),\left\{y\in R^d\middle|\,y^1\neq0\right\}\right)\right]=0,$$

where $P_t(z,dy)$ is the transition probability for $\mathbf{Z^1}$ which satisfies

$$P_t\left(z,\left\{y\in R^d\middle|\,y^1\neq0\right\}\right)=0\ \textit{for}\ z=\left(z^1,\cdots,z^d\right)\in R^d\ \textit{with}\ z^1=0.$$

While due to $\left(Y_t^1(x)\right)_{t\ge0}$ is the d-dimensional Brownian motion with a drift and note (2.2),

$$E\left[Z_1^{1,1}(x)\neq0,\ \tau\le1\right]=E\left[Y_1^{1,1}(x)\neq0,\ \tau\le1\right]$$
$$=E[\tau\le1]=E\left[\exists t\le1,\ Y_t^{1,1}(x)=0\right]>0.$$

This is a contradiction!

Example 2.5. Subordination. Let $(\mu_t^\alpha)_{t>0}$ be the one-side stable convolution semigroup of probabilities on $(0,\infty)$ with order $\alpha\in(0,1]$, i.e.,

$$\mu_s^\alpha*\mu_t^\alpha=\mu_{s+t}^\alpha\ \textit{for all}\ s,t>0,\ \lim_{r\to0}\mu_r^\alpha=\delta_0\ \textit{vaguely},$$
$$\int_{(0,\infty)}\exp\{-yx\}\ \mu_t^\alpha(dx)=\exp\left\{-ty^\alpha\right\},\ y>0.$$

Where $*$ is the convolution. For any Polish space E and any E-valued càdlàg Markov process of the semigroup $\{V_t\}_{t\ge0}$ on $\mathcal{B}_b(E)$, the following Markov semigroup is called subordinated to $\{V_t\}_{t\ge0}$ by means of $(\mu_t^\alpha)_{t>0}$:

$$\widetilde{V}_tf=\int_{(0,\infty)}V_sf\ \mu_t^\alpha(ds),\ t>0,\ \widetilde{V}_0f=f,\ f\in\mathcal{B}_b(E).$$

For any $k \in \mathbf{N}$, define a Markov semigroup on $\mathcal{B}_b\left(E^k\right)$ as follows:

$$\widetilde{V}_t^k f = \int_{(0,\infty)} V_s^{\otimes k} f \ \mu_t^\alpha(ds), \ t > 0, \ \widetilde{V}_0^k f = f, \ f \in \mathcal{B}_b\left(E^k\right).$$

Then $\left\{\left\{\widetilde{V}_t^k\right\}_{t\geq 0}\right\}_{k\geq 1}$ is a consistent exchangeable family, and each Markov process associated to $\left\{\widetilde{V}_t^k\right\}_{t\geq 0}$ is càdlàg. If the Markov process $\xi = ((\xi_t(x))_{t\geq 0}, x \in E)$ is strongly Markovian, then so is each Markov process corresponding to $\left\{\widetilde{V}_t^k\right\}_{t\geq 0}$.

In fact, there is a unique (in law) R_+-valued càdlàg process with stationary independent increments such that

$$\tau_0 = 0, \ \textit{and } \tau_t \textit{ is of the law } \mu_t^\alpha \textit{ for any } t \in (0,\infty).$$

Let $(\tau_t)_{t\geq 0}$ be independent of ξ. Then $((\xi_{\tau_t}(x))_{t\geq 0}, x \in E)$ is càdlàg and of the semigroup $\left\{\widetilde{V}_t^1\right\}_{t\geq 0}$. If ξ is strongly Markovian, then by [34] P.31 Theorem 7.4.(v),

$$V_s f(\xi_r(x)) \textit{ is right continuous in } r \in R_+, \ \textit{a.s. for any fixed } s \geq 0 \textit{ and } f \in C_b(E).$$

Then for any fixed $t \geq 0$ and $f \in C_b(E)$, since

$$\widetilde{V}_t^1 f(\xi_{\tau_r}(x)) = \int_{(0,\infty)} V_s f(\xi_{\tau_r}(x)) \ \mu_t^\alpha(ds),$$

we see $\widetilde{V}_t^1 f(\xi_{\tau_r}(x))$ is right continuous in $r \in [0,\infty)$, *a.s.*; which together with [34] P.31 Theorem 7.4.(v) implies that the Markov process associated to $\left\{\widetilde{V}_t^1\right\}_{t\geq 0}$ is strongly Markovian. Similarly, any Markov process associated to $\left\{\widetilde{V}_t^k\right\}_{t\geq 0}$ is càdlàg, and further strongly Markovian if so is ξ.

Note $(\mu_t^\alpha)_{t>0}$ can be replaced by other convolution semigroups.

Example 2.6. Natural consistent exchangeable families for OU processes. It is well-known that OU-process on path spaces is fundamental in Malliavin calculus ([29]). Let

$$\mathcal{W} = \left\{ \gamma \in C\left([0,1], R^d\right) \ \middle| \ \gamma(0) = 0 \right\},$$

where $C\left([0,1], R^d\right)$ is the space of all continuous maps from [0,1] into R^d. Equip $\mathcal{W}$ with the uniform topology. Let ν be the Wiener measure on $\mathcal{W}$ and define

$$V_t\varphi(w) = \int_{\mathcal{W}} \varphi\left(e^{-t/2}w + \sqrt{1-e^{-t}}\ u\right) \ \nu(du), \ \forall \varphi \in \mathcal{B}_b(\mathcal{W}), \ \forall w \in \mathcal{W}, \ \forall t \geq 0.$$

Then $\{V_t\}_{t\geq 0}$ is the semigroup of the OU-process on $\mathcal{W}$. For any $\rho \in (0,1]$ and $k \in \mathbf{N}$, define

$$A_k f(z_1, \cdots, z_k) = \frac{1}{2} \sum_{1\leq i,j\leq k} \sum_{p=1}^{d} \{\delta_{i,j} + \rho(1-\delta_{i,j})\} \frac{\partial^2 f}{\partial z_i^p \partial z_j^p}(z_1, \cdots, z_k),$$

$$\forall f \in C_b^2\left(\left(R^d\right)^k\right), \ (z_1, \cdots, z_k) \in \left(R^d\right)^k, \ z_i = \left(z_i^1, \cdots, z_i^d\right) \in R^d,$$

where $\delta_{i,j}$ is the Kronecker delta. Then all A_k-diffusions determine a unique consistent family in R^d of exchangeable diffusions. Let $\left(w_t^1, \cdots, w_t^k\right)_{t\geq 0}$ be the A_k-diffusion and ν_k the law of $\left(\left(w_t^1\right)_{0\leq t\leq 1}, \cdots, \left(w_t^k\right)_{0\leq t\leq 1}\right)$ with

$$\left(w_0^1, \cdots, w_0^k\right) = (0, \cdots, 0) \in \left(R^d\right)^k.$$

For any $\varphi \in \mathcal{B}_b\left(\mathcal{W}^k\right)$, any $w^1, \cdots, w^k \in \mathcal{W}$, and any $t \geq 0$, define

$$\begin{aligned} & V_t^k \varphi\left(w^1, \cdots, w^k\right) \\ & \quad = \int_{\mathcal{W}^k} \varphi\left(e^{-t/2} w^1 + \sqrt{1-e^{-t}}\ u_1, \cdots, e^{-t/2} w^k + \sqrt{1-e^{-t}}\ u_k\right)\ \nu_k(du_1 \cdots du_k). \end{aligned}$$

Then it is easy to check that $\left\{V_t^k\right\}_{t\geq 0}$ is an exchangeable semigroup (resp. weakly Fellerian semigroup) on $\mathcal{B}_b\left(\mathcal{W}^k\right)$ $\left(\text{resp. } C_b\left(\mathcal{W}^k\right)\right)$, which is left to the interested readers as an exercise. Clearly, the family $\left\{\left\{V_t^k\right\}_{t\geq 0}\right\}_{k\geq 1}$ is consistent; and $\left\{\left\{V_t^k\right\}_{t\geq 0}\right\}_{k\geq 1}$ is of coalescence for $\rho = 1$; and each $\left\{V_t^k\right\}_{t\geq 0}$ is symmetric with respect to ν_k for $\rho \in (0, 1)$.

Notice time interval $[0, 1]$ can be replaced by $[0, \infty)$ and any time interval $[0, a]$ with $a \in (0, \infty)$.

Example 2.7. Brownian snakes. To begin, let $((\xi_t)_{t\geq 0}, \Pi_x)_{x\in R^d}$ be the d-dimensional Brownian motion and Π_x being the law of $\xi = (\xi_t)_{t\geq 0}$ starting at x. For any interval I of $R_+ = [0, \infty)$ and any metric space E, write $C\left(I, E\right)$ for the space of all continuous maps from I into E equipped with the locally uniform topology. Set

$$\mathcal{W} = \bigcup_{t\geq 0} C\left([0, t], R^d\right).$$

For any $w \in \mathcal{W}$, write $\zeta_w = t$ if $w \in C\left([0, t], R^d\right)$ and $\widehat{w} = w(\zeta_w)$, and call ζ_w the life time and $\widehat{w}$ the terminal point of w. The space $\mathcal{W}$ is a Polish space when endowed with the metric

$$\text{dist}\ (w, w') = |\zeta_w - \zeta_{w'}| + \sup_{t\geq 0} |w(t\wedge\zeta_w) - w'(t\wedge\zeta_{w'})|\,,\ w, w' \in \mathcal{W}.$$

For convenience, view R^d as a subset of $\mathcal{W}$ by identifying a point $x \in R^d$ with the trivial path with initial point x and lifetime 0. Let

$$\mathcal{W}_x = \{w \in \mathcal{W} \mid w(0) = x\},\ \forall x \in R^d.$$

Given $w \in \mathcal{W}$ and $a \in [0, \zeta_w],\ b \geq a$. Define the following probability measure $R_{a,b}(w, dw')$ on $\mathcal{W}$:

$$\begin{aligned} & \zeta_{w'} = b,\ w'(t) = w(t),\ \forall t \in [0, a],\ R_{a,b}(w, dw')\ -\ a.s.; \\ & (w'(a+t))_{0\leq t\leq b-a}\ \textit{under}\ R_{a,b}(w, dw')\ \textit{distributes as}\ (\xi_t)_{0\leq t\leq b-a}\ \textit{under}\ \Pi_{w(a)}. \end{aligned}$$

For any $w_0 \in \mathcal{W}$, set $\zeta_0 = \zeta_{w_0}$. Let P_{ζ_0} be the law of reflected standard Brownian motion (the modulus of standard linear Brownian motion) starting at ζ_0. Given $f \in C(R_+, R_+)$ with $f(0) = \zeta_0$, let $\Theta_{w_0}^f(d\omega)$ be the law on $\mathcal{W}^{R_+}$ of the time-inhomogeneous Markov process in $\mathcal{W}$ starting at w_0 with transition kernel between times s and $s'(> s)$ being

$$R_{m(s,s'),f(s')}(w, dw'),\ \textit{where}\ m(s, s') = \inf_{s\leq t\leq s'} f(t).$$

Then $\Theta^f_{w_0}(d\omega)$ is a probability on $C(R_+, \mathcal{W})$, P_{ζ_0} $-$ $a.s.$ f ([24] P.65 lines -5 and -4). Consider the following probability on $C(R_+, R_+) \times C(R_+, \mathcal{W})$:

$$\mathbf{P}_{w_0}(df d\omega) = P_{\zeta_0}(df)\Theta^f_{w_0}(d\omega).$$

The Brownian snake $W = (W_s)_{s\geq 0}$ $(W_0 = w_0)$ is defined under $\mathbf{P}_{w_0}(df d\omega)$ by

$$W_s(f, \omega) = \omega(s), \ \forall s \in [0, \infty).$$

Note $(W_s)_{s\geq 0}$ under $\mathbf{P}_{w_0}$ is a (time-homogeneous) diffusion process on $\mathcal{W}$, and

$$\zeta_s(f, \omega) := \zeta_{W_s} = f(s), \ \mathbf{P}_{w_0} \ - \ a.s.,$$

so that the lifetime is a reflected Brownian motion.

Let $(\phi_t)_{t\geq 0}$ ($\phi_0 =$ *the identity map*) be an arbitrary stochastic flow of homeomorphisms on R^1 such that 1-point motion $(\phi_t(x))_{t\geq 0}$ is the 1-dimensional Brownian motion and each k-point motion $(\phi_t(x_1), \cdots, \phi_t(x_k))_{t\geq 0}$ is a diffusion in R^k. For such flows, refer to [24]. Let $P_{(x_1,\cdots,x_k)}$ be the distribution of $\left((|\phi_t(x_1)|)_{t\geq 0}, \cdots, (|\phi_t(x_k)|)_{t\geq 0}\right)$ on $C(R_+, R_+)^k$ for any $(x_1, \cdots, x_k) \in R^k_+$ and any $k \in \mathbf{N}$. Define the following probability on $C(R_+, R_+)^k \times C(R_+, \mathcal{W})^k$:

$$\mathbf{P}_{\left(w^1_0,\cdots,w^k_0\right)}\left(df^1 \cdots df^k d\omega^1 \cdots d\omega^k\right) = P_{\left(\zeta_{w^1_0},\cdots,\zeta_{w^k_0}\right)}\left(df^1 \cdots df^k\right) \prod_{i=1}^k \Theta^{f^i}_{w^i_0}\left(d\omega^i\right),$$

where $w^i_0 \in \mathcal{W}, \ 1 \leq i \leq k$. On $C(R_+, R_+)^k \times C(R_+, \mathcal{W})^k$, let

$$W^i_s\left(f^1, \cdots, f^k; \omega^1, \cdots, \omega^k\right) = \omega^i(s), \ s \in [0, \infty), \ 1 \leq i \leq k.$$

Then $\mathbf{W^k} = \left(W^1_s, \cdots, W^k_s\right)_{s\geq 0}$ $\left(\left(W^1_0, \cdots, W^k_0\right) = \left(w^1_0, \cdots, w^k_0\right)\right)$ under $\mathbf{P}_{\left(w^1_0,\cdots,w^k_0\right)}$ is a k-point Markov coupling for the Brownian snake W. Similarly to [24] Chapter IV Section 4 Theorem 6, one can check $\mathbf{W^k}$ is strongly Markovian. Now we have obtained a consistent exchangeable family $\left\{\mathbf{W^k}\right\}_{k\geq 1}$ in Polish space $\mathcal{W}$.

Example 2.8. Let E be an arbitrary Polish space and $Q(x, dy)$ an arbitrary probability kernel on it. For any $n \in \mathbf{N}$, consider a conservative càdlàg regular step process

$$\mathbf{Y^n} = \left(\left(Y^1_t(x_1), \cdots, Y^n_t(x_n)\right)_{t\geq 0}\right)_{(x_1,\cdots,x_n)\in E^n}$$

whose generator is

$$\begin{aligned} A_n f(x_1, \cdots, x_n) \quad &= \quad \int_{E^n} \left[f(y_1, \cdots, y_n) - f(x_1, \cdots, x_n)\right] \prod_{i=1}^n Q(x_i, dy_i), \\ &\forall (x_1, \cdots, x_n) \in E^n, \ \forall f \in \mathcal{B}_b\left(E^n\right). \end{aligned}$$

Then $\{\mathbf{Y^n}\}^{\infty}_{n=1}$ is a consistent exchangeable family. If $Q(x, \cdot) \in M_1(E)$ is not continuous in $x \in E$, then $\mathbf{Y^1}$, further each $\mathbf{Y^n}$, is not weakly Fellerian.

3. Main result II: Local time and Tanaka formula

Local time is one of major objects in Markov process theory and Tanaka formula gives a semimartingale representation of local time. The local time and Tanaka formula for measure-valued processes have been studied extensively by many authors such as E. B. Dynkin, R. J. Adler, K. Fleischmann, I. Iscoe etc (see [30], [39] and references therein). In this section, we derive local time for measure-valued flows and Fleming-Viot (FV) processes, and establish the related Tanaka formula. To this end, for any $k \geq 1$, consider the following operator in R^k :

$$A_k f(z_1, \cdots, z_k) = \frac{1}{2}\sum_{i=1}^{k} a(z_i)\frac{\partial^2 f}{\partial z_i \partial z_i}(z_1, \cdots, z_k) + \frac{1}{2}\sum_{1\leq i\neq j\leq k} a(z_i, z_j)\frac{\partial^2 f}{\partial z_i \partial z_j}(z_1, \cdots, z_k)$$
$$+\sum_{i=1}^{k} b(z_i)\frac{\partial f}{\partial z_i}(z_1, \cdots, z_k),\ \forall f \in C_b^2\left(R^k\right),\ (z_1, \cdots, z_k) \in R^k;$$

and

$$\begin{aligned}
&a(\cdot) \in C_b\left(R^1\right),\ a(\cdot,\cdot) \in \mathcal{B}_b\left(R^2\right),\ b(\cdot) \in \mathcal{B}_b\left(R^1\right),\\
&a(z_1, z_2) = a(z_2, z_1),\ \forall (z_1, z_2) \in R^2,\\
&a(\cdot)^{-1}\ \textit{is locally integrable against the Lebesgue measure},\\
&b(\cdot)^2 \leq c\ a(\cdot)\ \textit{for some constant}\ c > 0;
\end{aligned}$$

and for any $k \geq 1$ and any $(z_1, \cdots, z_k) \in R^k$,

$$\sum_{i=1}^{k} a(z_i)(\lambda_i)^2 + \sum_{1\leq i\neq j\leq k} a(z_i, z_j)\lambda_i\lambda_j \geq 0,\ \forall\, (\lambda_1, \cdots, \lambda_k) \in R^k.$$

Assume each A_k-diffusion process $\mathbf{Y^k} = \left(\left(Y_t^1(z_1), \cdots, Y_t^k(z_k)\right)_{t\geq 0}\right)_{(z_1,\cdots,z_k)\in R^k}$ exists, and write $\{V_t^k\}_{t\geq 0}$ for its semigroup. Suppose $\left\{\mathbf{Y^k}\right\}_{k\geq 1}$ is a consistent exchangeable family. Then the measure-valued flow given the family $\left\{\mathbf{Y^k}\right\}_{k\geq 1}$, an $M_1\left(R^1\right)$ valued diffusion process $\mathbf{X} = (\mathbf{X}_t)_{t\geq 0}$ with the semigroup $\{T_t\}_{t\geq 0}$, is determined uniquely by (1.1). Recall P_μ is the law of $\mathbf{X}$ starting at $\mu \in M_1\left(R^1\right)$. If

$$\int_0^t \mathbf{X}_s\ ds \ll dz\ (\textit{the Lebesgue measure}),\ \forall t \in [0, \infty),\ P_\mu\ -\ a.s.,$$

then define the local time $\left((\mathbf{L}_t^z(\mathbf{X}))_{t\geq 0}\right)_{z\in R^1}$ of $\mathbf{X}$ as follows:

$$\mathbf{L}_t^z(\mathbf{X}) = \frac{d\left(\int_0^t \mathbf{X}_s\ ds\right)}{dz},\ \forall t \in [0, \infty),\ P_\mu\ -\ a.s..$$

Note closed subset $\{a(\cdot) = 0\}$ is of zero Lebesgue measure due to $a(\cdot)^{-1}$ is locally Lebesgue integrable. Then the following theorem is true.

Theorem 3.1. Under P_μ, $\mathbf{L}_t^z(\mathbf{X})$ is well-defined for any z and any t almost surely; and the map $(z, t) \in \{a(\cdot) \neq 0\} \times [0, \infty) \to \mathbf{L}_t^z(\mathbf{X})$ is $a.s.$ continuous; and almost surely, the map $z \in \{a(\cdot) \neq 0\} \to a(z)\mathbf{L}_t^z(\mathbf{X})$ is Hölder continuous of order α for any $\alpha \in \left(0, \frac{1}{2}\right)$ and uniformly in t on every compact interval; and given any $z \in \{a(\cdot) \neq 0\}$, $\mathbf{L}_t^z(\mathbf{X})$ is *a.s.* increasing in $t \in [0, \infty)$.

In addition, for any $\mu \in M_1\left(R^1\right)$ with $\int_{[0,\infty)} x\, \mu(dx) < \infty$ $\left(\text{resp. } \int_{(-\infty,0]} |x|\, \mu(dx) < \infty\right)$, the following Tanaka formula holds:

$$\mathbf{L}_t^z(\mathbf{X}) = \frac{2}{a(z)} \left\{ \int_{[z,\infty)} (x-z)\, \mathbf{X}_t(dx) - \int_{[z,\infty)} (x-z)\, \mu(dx) \right.$$
$$\left. - \int_0^t \left\langle \mathbf{X}_s, I_{(z,\infty)} b(\cdot) \right\rangle\, ds - \mathbf{M}_t^z(\mathbf{X}) \right\},\ \forall t \in [0,\infty),\ \forall z \in \{a(\cdot) \neq 0\},\ P_\mu\ -\ a.s.$$
$$\left(\text{resp. } \mathbf{L}_t^z(\mathbf{X}) = \frac{2}{a(z)} \left\{ \int_{(-\infty,z]} (z-x)\, \mathbf{X}_t(dx) - \int_{(-\infty,z]} (z-x)\, \mu(dx) \right. \right.$$
$$\left. \left. + \int_0^t \left\langle \mathbf{X}_s, I_{(-\infty,z]} b(\cdot) \right\rangle\, ds + \mathbf{M}_t^z(\mathbf{X}) \right\},\ \forall t \in [0,\infty),\ \forall z \in \{a(\cdot) \neq 0\},\ P_\mu\ -\ a.s. \right);$$

where $\left(\mathbf{M}_t^z(\mathbf{X})\right)_{t \geq 0}$ is a continuous L^r-martingale for any $r \in [1, \infty)$ such that

$$(z,t) \in \{a(\cdot) \neq 0\} \times [0,\infty) \to \mathbf{M}_t^z(\mathbf{X}) \textit{ is continuous a.s.},$$

and $z \in \{a(\cdot) \neq 0\} \to \mathbf{M}_t^z(\mathbf{X})$ is *a.s.* Hölder continuous of order α for any $\alpha \in \left(0, \frac{1}{2}\right)$ and uniformly in t on every compact interval; and $\mathbf{L}_t^z(\mathbf{X})$ is in $L^r\left(P_\mu\right)$ for any $t \in [0, \infty)$ and any $r \in [1, \infty)$.

Now we describe Fleming-Viot (FV) process, a model in population genetics theory introduced by [17]. By [13], FV process is one of three fundamental types superprocesses: Dawson-Watanabe (DW) superprocess, FV process, OU-superprocess. While by [22], there is a deep connection between FV process and DW superprocess (also see [15], [31]). To introduce FV process $\mathbf{X}$, let Y be a càdlàg E-valued weak Feller process with the semigroup $\{V_t\}_{t \geq 0}$; and for any $k \geq 1$, $\left\{V_t^k\right\}_{t \geq 0}$ be the semigroup of càdlàg E^k-valued k-particle look-down process over Y determined uniquely by

$$\begin{aligned} V_t^k f &= \exp\left\{-\frac{k(k-1)t}{2}\right\} V_t^{\otimes k} f + \\ &\quad \sum_{1 \leq i < j \leq k} \int_0^t \exp\left\{-\frac{k(k-1)s}{2}\right\} V_s^{\otimes k}\left(\Theta_{ij} V_{t-s}^k f\right)\, ds, \\ &\quad \forall t \geq 0,\ \forall f \in \mathcal{B}_b\left(E^k\right), \end{aligned}$$

with $\left\{V_t^{\otimes k}\right\}_{t \geq 0}$ being the semigroup of k-independent copies of Y, and

$$\Theta_{ij} f(x_1, \cdots, x_k) := f(x_1, \cdots, x_i, \cdots, x_{j-1}, x_i, x_{j+1}, \cdots, x_k),$$
$$\forall (x_1, \cdots, x_k) \in E^k,\ \forall f \in \mathcal{B}_b\left(E^k\right),\ \forall 1 \leq i < j \leq k.$$

Then FV process $\mathbf{X}$ over the mutation process Y is the unique $M_1(E)$-valued diffusion process with the semigroup $\{T_t\}_{t \geq 0}$ on $\mathcal{B}_b(M_1(E))$ determined by (1.1) with each just mentioned $\left\{V_t^k\right\}_{t \geq 0}$ (c.f. [9] and [11]).

Theorem 3.1′. For the FV-process $\mathbf{X}$ over A_1-diffusion process in this section, if P_μ still denotes the law of the process $\mathbf{X}$ starting at $\mu \in M_1\left(R^1\right)$, then the results in Theorem 3.1

hold true for the process $\mathbf{X}$.

The method for proving Theorem 3.1 is of its own interests. In Theorem 3.1, on $\{a(\cdot) = 0\}$ which is of zero Lebesgue measure, $\mathbf{L}_t^z(\mathbf{X})$ can not be defined explicitly; and so we do not consider Tanaka formula for $\mathbf{L}_t^z(\mathbf{X})$ at $z \in \{a(\cdot) = 0\}$. The condition $b(\cdot)^2 \leq c\ a(\cdot)$ *for some constant* $c > 0$ ensures that there is a Girsanov transformation theorem between $\left(\frac{1}{2}a(x)\frac{\partial^2}{\partial x^2}, C_b^2\left(R^1\right)\right)$-martingale solutions and $\left(A_1, C_b^2\left(R^1\right)\right)$-martingale solutions, and the local time L_t^z for A_1-process is continuous in $(z,t) \in R^1 \times [0,\infty)$ and further the first part of Theorem 3.1 holds. Remove the condition, Theorems 3.1 and 3.1$'$ still hold true with a few modifications (see Remark 3.1.1). For Theorem 3.1, examples are taken in Remark 3.1.2. For further remarks on Theorem 3.1 etc, refer to Remark 3.1.3.

As a contrast, recall for FV-process $\mathbf{X} = (\mathbf{X}_t)_{t\geq 0}$ over the 1-dimensional $-(-\Delta)^{\frac{\alpha}{2}}$ process ($\alpha \in (0,2]$), [22] proved that $\mathbf{X}_t$ has joint continuous density $f_t(x)$ with respect to the Lebesgue measure satisfying a stochastic partial differential equation; and so automatically, the local time for $\mathbf{X}$ exists and a Tanaka type formula holds for the local time. While our Theorem 3.1$'$ is true for any 1-dimensional diffusion described before.

3.1. Remarks on Theorems 3.1 and 3.1$'$

Remark 3.1.1. Remove the condition $b(\cdot)^2 \leq c\ a(\cdot)$ *for some constant* $c > 0$, and replace $a(\cdot) \in C_b\left(R^1\right)$ by $a(\cdot) \in \mathcal{B}_b\left(R^1\right)$; if we let

$$\Lambda = \left\{ z \in R^1 \;\middle|\; \lim_{\epsilon\downarrow 0} \frac{1}{2\epsilon}\int_{(z-\epsilon,z+\epsilon)} a(y)^{-1}\, dy = \lambda(z) \in R^1 \ exists \right\},$$

then $R^1 \setminus \Lambda$ is of zero Lebesgue measure and

$$\lambda(z) = a(z)^{-1},\ a.e.\ z \in \Lambda \ with\ respect\ to\ the\ lesbesgue\ measure,$$

and clearly, for $z \in R^1$ such that $a(z) \neq 0$ and $a(\cdot)$ is continuous at z, $\lambda(z) = a(z)^{-1}$. In this setting, since the local time L_t^z for A_1-process satisfying that $(z,t) \to L_t^z$ is continuous in t and càdlàg in z almost surely ([33] Chapter VI), the first part of Theorem 3.1 (and Theorem 3.1$'$) should be rewritten as follows:

> *Under P_μ, $\mathbf{L}_t^z(\mathbf{X})$ is well − defined for any z and any t almost surely; and $(z,t) \in \{y \in \Lambda;\ \lambda(y) \neq 0\} \times [0,\infty) \to \lambda(z)^{-1}\mathbf{L}_t^z(\mathbf{X})$ is a.s. continuous in t and càdlàg in z; and given any $z \in \{y \in \Lambda;\ \lambda(y) \neq 0\}$, $\mathbf{L}_t^z(\mathbf{X})$ is a.s. increasing in $t \in [0,\infty)$.*

While the second part of Theorem 3.1 (and Theorem 3.1$'$) is modified as follows: for any $\mu \in M_1\left(R^1\right)$ with $\int_{[0,\infty)} x\ \mu(dx) < \infty$,

$$\begin{aligned}\mathbf{L}_t^z(\mathbf{X}) = \lambda(z)\Bigg\{ & 2\int_{[z,\infty)} (x-z)\ \mathbf{X}_t(dx) - 2\int_{[z,\infty)} (x-z)\ \mu(dx) \\ & - \int_0^t \left\langle \mathbf{X}_s, I_{(z,\infty)} b(\cdot)\right\rangle\ ds - \int_0^t \left\langle \mathbf{X}_s, I_{[z,\infty)} b(\cdot)\right\rangle\ ds - \mathbf{M}_t^z(\mathbf{X})\Bigg\},\end{aligned}$$

$$\forall t \in [0,\infty),\ \forall z \in \{y \in \Lambda;\ \lambda(y) \neq 0\},\ P_\mu\ -\ a.s.;$$

where $(\mathbf{M}_t^z(\mathbf{X}))_{t\geq 0}$ *is a continuous* L^r-*martingale for any* $r \in [1,\infty)$ *such that* $(z,t) \in \{y \in \Lambda;\ \lambda(y) \neq 0\} \times [0,\infty) \to \mathbf{M}_t^z(\mathbf{X})$ *is continuous a.s., and* $z \in \{y \in \Lambda;\ \lambda(y) \neq 0\} \to \mathbf{M}_t^z(\mathbf{X})$ *is a.s. Hölder continuous of order* α *for any* $\alpha \in \left(0, \frac{1}{2}\right)$ *and uniformly in* t *on every compact interval; and* $\mathbf{L}_t^z(\mathbf{X})$ *is in* $L^r(P_\mu)$ *for any* $t \in [0,\infty)$ *and any* $r \in [1,\infty)$.

For $\mu \in M_1(R^1)$ with $\int_{(-\infty,0]} |x|\ \mu(dx) < \infty$, the Tanaka formula needs slight modification, which is left to the interested readers. Note $R^1 \setminus \{y \in \Lambda;\ \lambda(y) \neq 0\}$ is of zero Lebesgue measure; and for z in this set, either $\mathbf{L}_t^z(\mathbf{X})$ can not be defined explicitly for $z \in R^1 \setminus \Lambda$ or $\mathbf{L}_\cdot^z(\mathbf{X}) \equiv 0$ for $z \in \Lambda$ with $\lambda(z) = 0$. It is not interesting to establish Tanaka formula for $z \in R^1 \setminus \{y \in \Lambda;\ \lambda(y) \neq 0\}$.

Remark 3.1.2. (i) The existence for A_1-process. Assume $0 \leq a(\cdot) \in \mathcal{B}_b(R^1)$ and $a(\cdot)^{-1}$ is locally integrable. Recall from Example 2.3(ii),

$$Z(a) = \left\{ x \in R^1 \ \middle|\ a(x) = 0 \right\},$$

and $\mathbf{Y} = \left((Y_t(x))_{t\geq 0}\right)_{x\in R^1}$ is a $\frac{1}{2}a(x)\frac{\partial^2}{\partial x^2}$-diffusion; and $\overline{\mathbf{Y}} = \left((\overline{Y}_t(x))_{t\geq 0}\right)_{x\in R^1}$ is another $\frac{1}{2}a(x)\frac{\partial^2}{\partial x^2}$-diffusion if $Z(a) \neq \emptyset$. For all diffusions (and Markov processes) generated by the operator $\frac{1}{2}a(x)\frac{\partial^2}{\partial x^2}$, see [14].

For $b(\cdot) \in \mathcal{B}_b(R^1)$ such that $b(\cdot)^2 \leq c\ a(\cdot)$ *for some constant* $c > 0$, use the Girsanov argument, one can get $\frac{1}{2}a(x)\frac{\partial^2}{\partial x^2} + b(x)\frac{\partial}{\partial x}$ diffusion $\mathbf{Y^1} = \left((Y_t^1(x))_{t\geq 0}\right)_{x\in R^1}$ from $\mathbf{Y}$ and $\overline{\mathbf{Y}}$ and other $\frac{1}{2}a(x)\frac{\partial^2}{\partial x^2}$-diffusions (when $Z(a) \neq \emptyset$). Refer to proof of Lemma 5.1.

(ii) Consistent exchangeable families with coalescence. See Example 2.3(ii). In addition, by [5] Theorem 1.3, one can construct consistent exchangeable families with coalescence for $b(\cdot)$ therein being in $\mathcal{B}_b(R^1)$.

(iii) Consistent exchangeable families without coalescence. In the situation of Theorem 3.1 with $a(\cdot,\cdot) \in C_b(R^2)$, if there is an $\eta \in (0,\infty)$ such that

$$\sum_{i=1}^k a(z_i)(\lambda_i)^2 + \sum_{1\leq i\neq j\leq k} a(z_i,z_j)\lambda_i\lambda_j \geq \eta \sum_{i=1}^k (\lambda_i)^2 \geq 0,\ \forall (\lambda_1,\cdots,\lambda_k) \in R^k,$$

for any $k \geq 1$ and any $(z_1,\cdots,z_k) \in R^k$, then each A_k-diffusion $\mathbf{Y^k}$ exists and is unique. Clearly, $\left\{\mathbf{Y^k}\right\}_{k\geq 1}$ is a consistent exchangeable family without coalescence. For arbitrary $b \in \mathcal{B}_b(R^1)$, one can not determine generally whether each $\mathbf{Y^k}$ is weakly Fellerian. Next if $b \equiv 0$, then any two particles of each $\mathbf{Y^k}$ $(k \geq 2)$ must meet at a finite time because their difference process is a continuous martingale whose quadratic variation process tends to $+\infty$ as time goes to $+\infty$. Furthermore, for arbitrary $b \in \mathcal{B}_b(R^1)$, by the Girsanov argument, any two particles of each $\mathbf{Y^k}$ $(k \geq 2)$ must also meet at a finite time. So if let $\hat{\mathbf{Y}}^{\mathbf{k}}$ be obtained from $\mathbf{Y^k}$ by letting two particles stay together whenever they meet, then we get a nontrivial consistent exchangeable family $\left\{\hat{\mathbf{Y}}^{\mathbf{k}}\right\}_{k\geq 1}$ with coalescence.

Remark 3.1.3. For Theorem 3.1, the examples that each A_k-diffusion is the k-point motion of a common stochastic flow $(\phi_t)_{t\geq 0}$ of measurable maps (homeomorphisms) can be taken under many well-known conditions, e.g.,

$$a(x) = a(x,x),\ \forall x \in R^1;\ a(\cdot,\cdot) \in C_b^2\left(R^2\right),\ b(\cdot) \in C_b^2\left(R^1\right),\ \inf_{x\in R^1} a(x) > 0.$$

For these examples, since 1-point motion $(\phi_t(x))_{t\geq 0}$ for any point $x \in R^1$ can be given by the flow simultaneously on the same probability space, and the 1-point motion has local time and Tanaka formula, Theorem 3.1 on measure-valued flow $((\phi_t)_*\mu)_{t\geq 0}$ holds is natural. It is interesting that Theorem 3.1 (resp. Theorem 3.1′) is true when (resp. since) there is no stochastic flow of measurable maps corresponding to the (resp. look-down) family $\left\{\mathbf{Y^k}\right\}_{k\geq 1}$.

Recall 3 forms of Tanaka formulae on continuous semimartingales from [33] Chapter VI Theorem 1.2, one can see that the condition on initial measure of Theorems 3.1 and 3.1′ is optimal. Certainly, for more general $a(\cdot), a(\cdot,\cdot)$ and $b(\cdot)$, Theorem 3.1 still holds, this is left to the interested readers as an exercise. The method for proving Theorem 3.1 suggests us that when 1-point motion has local time and Tanaka formula, then the related measure-valued flow, FV process will preserve this property under a suitable condition.

Remark 3.1.4. Recall from [11], there is an infinite-particle look-down process representation for FV processes, which is exchangeable if so is its initial distribution. Similarly to Theorem 3.1, one can verify Theorem 3.1′.

Remark 3.1.5. Fractional diffusion. Let $(F, \rho(\cdot,\cdot))$ be a complete metric space and μ a Borel measure on it. $(F, \rho(\cdot,\cdot), \mu)$ is called a *fractional metric space* (FMS) if **(1)** $(F, \rho(\cdot,\cdot))$ has the *midpoint property* in the sense that

$$for\ each\ x, y \in F\ there\ is\ a\ z \in F\ satisfying\ \rho(x,z) = \rho(z,y) = \frac{1}{2}\rho(x,y);$$

and **(2)** there are $d_f > 0$, and constants c_1, c_2 such that

$$c_1 r^{d_f} \leq \mu((B(x,r)) \leq c_2 r^{d_f},\ \forall x \in F,\ 0 < r \leq r_0 := \sup_{x,y\in F} \rho(x,y).$$

Here $B(x,r) = \{y \in F \mid \rho(x,y) < r\}$. If G is the Sierpinski gasket, $d_G(\cdot,\cdot)$ is the geodesic metric on G, and μ_G is the natural fractional measure on it, then $(G, d_G(\cdot,\cdot), \mu)$ is a FMS with $d_f = \log 3/\log 2$ and $r_0 = 1$. Any FMS $(F, \rho(\cdot,\cdot), \mu)$ satisfies

$$F\ is\ locally\ compact,\ dim_H(F) = dim_P(F) = d_f \geq 1.$$

Recall any p.c.f.s.s. set is a compact FMS ([3] Definition 5.13), where p.c.f.s.s. means *post critically finite self-similar*. A Markov process $Y = ((Y_t)_{t\geq 0}, P^x)_{x\in F}$ is a *fractional diffusion* (FD) on FMS $(F, \rho(\cdot,\cdot), \mu)$ if Y is a μ-symmetric conservative Feller diffusion on F, and Y has a symmetric transition density $p(t,x,y) = p(t,y,x),\ t > 0,\ x, y \in F$, which satisfies the Chapman-Kolmogorov equation and each $p(t,x,y)$ is continuous in $(x,y) \in F^2$, and there are constants $\alpha, \beta, \gamma, c_1, \cdots, c_4$ satisfying that for any $(x,y) \in F^2$ and any $0 < t \leq t_0 := r_0^\beta$,

$$c_1 t^{-\alpha} \exp\left\{-c_2 \rho(x,y)^{\beta\gamma} t^{-\gamma}\right\} \leq p(t,x,y) \leq c_3 t^{-\alpha} \exp\left\{-c_4 \rho(x,y)^{\beta\gamma} t^{-\gamma}\right\}.$$

We say above FD is a $FD'(d_f, \alpha, \beta, \gamma)$. Particularly,

$$Y\ is\ a\ FD(d_f, \beta)\ if\ it\ is\ a\ FD'(d_f, d_f/\beta, \beta, 1/(\beta - 1))\ with\ \beta > 1.$$

For these notions, see [3]. From [4] and [3], the Brownian motion on the Sierpinski gasket is a $FD(\log 3/\log 2, \log 5/\log 2)$.

When $d_f < d_w := \beta$, or equivalently $d_s := \frac{2d_f}{d_w} < 2$, by [3] Theorem 3.32, $FD(d_f, \beta)$ Y on F have jointly continuous local time $(L_t^x, x \in F, t \geq 0)$ which satisfying

$$\int_0^t f(Y_s)\ ds = \int_F f(x)\ L_t^x\ \mu(dx),\ \forall f \in \mathcal{B}_b(F)$$
$$(see\ [4]\ Theorem\ 1.11\ for\ local\ time\ of\ the\ Brownian\ motion\ on\ the\ Sierpinski\ gasket).$$

For diffusions on p.c.f.s.s. sets, the just mentioned result is still true (c.f. [3] Theorem 7.21).

Note [9] Lemma 3.4.2.2 can be extended to *fractional metric space* (FDS) $(F, \rho(\cdot,\cdot), \mu)$. Similarly to Theorem 3.1, one can show

For the FV process $\mathbf{X}$ *over the FD* Y *which is a* $FD(d_f, \beta)$ *with* $1 \leq d_f < \beta$ *on FMS* $(F, \rho(\cdot,\cdot), \mu)$ *or a diffusion on p.c.f.s.s. sets* F*, it has jointly measurable local time* $\mathbf{L}_t^z(\mathbf{X})$ *in the sense that*

$$\mathbf{L}_t^z(\mathbf{X}) = \lim_{\epsilon \downarrow 0} \frac{1}{\mu(B(z,\epsilon))} \int_0^t \mathbf{X}_s(B(z,\epsilon))\ ds,\ \mu\ -\ a.e.\ z,\ \forall t \in [0,\infty),\ a.s.;$$

where $B(z,\epsilon)$ *is the open ball centered at* z *of radius* ϵ*; and for fixed* $z \in F$, $t \to \mathbf{L}_t^z(\mathbf{X})$ *is a.s. continuous, and* $\mathbf{L}_t^z(\mathbf{X})$ *is* L^1*-integrable for any* $(z,t) \in F \times [0,\infty)$.

Note by [4] Corollary 1.8 and [3] Theorem 2.28, the Brownian motion on the Sierpinski gasket is not semimartingales and hence does not have Tanaka formulae. In addition, for fractional diffusions on fractional metric spaces F, by [3] Lemma 3.25 and Theorem 3.41, their paths fail to be Hölder continuous of order $\gamma \in \left(0, \frac{1}{2}\right)$; so if F is a subspace of an Euclidean space, then fractional diffusions are not semimartingales and there is no Tanaka formula for the diffusions. So we do not consider Tanaka formulae for the FV process $\mathbf{X}$ over the FD Y.

4. Proof of Theorem 2.1

Proof of Theorem 2.1(1)

To begin the proof, we need the following preliminaries. For any Polish space S, there is a homeomorphic map from S into a G_δ-subset of some compact metrizable space $\mathcal{T}$. Without loss of generality, suppose S is a G_δ-subset of $\mathcal{T}$. Hence the closure $\overline{S}$ of S in $\mathcal{T}$ is compact. Let

$$C_{restrict}(S) = \left\{ f|_S \ \middle|\ f \in C_b\left(\overline{S}\right) \right\},$$

where $f|_S$ is the restriction of f to S. Then $C_{restrict}(S)$ is separable in the uniform norm and weak convergence determining, and

$$\sigma\left(f,\ f \in C_{restrict}(S)\right)\ equals\ to\ the\ Borel\ \sigma-field\ on\ S.$$

Write $C_{sym}(S^n)$ for the set of all continuous symmetric functions from S^n to R^1 for any $n \in \mathbf{N}$.

Now we are in the position to give an extended de Finetti's theorem.

Lemma 4.1. Let S be a Polish space.

(1) Assume $P \in M_1(M_1(S))$. Then there exists a probability Q on $S^{\mathbf{N}}$ under which the coordinate process $Z = (Z_n)_{n\geq 1}$ satisfies that $\{Z_n\}_{n\geq 1}$ is exchangeable and $(Z_1, \cdots, Z_n)$ has distribution

$$P^{(n)}(dx_1, \cdots, dx_n) := \int_{M_1(S)} \nu(dx_1) \cdots \nu(dx_n) P(d\nu), \ \forall n \geq 1.$$

(2) Let $\{Z_n\}_{n\geq 1}$ be a sequence of S-valued exchangeable random variables on probability space $\left(S^{\mathbf{N}}, Q\right)$ and $W_n(\omega) = \frac{1}{n}\sum_{i=1}^{n} \delta_{Z_i(\omega)}$. Then $\{W_n(\omega)\}_{n\geq 1}$ converges to some $W(\omega) \in M_1(S)$ in the $\mathcal{H}$-topology on $M_1(S)$, $Q - a.\ s.\ \omega$. Where $\mathcal{H} \subseteq \mathcal{B}_b(S)$ is separable in the uniform norm, and $\sigma(\mathcal{H})$ equals to the Borel σ-field on S; $\{\nu_n\}_{n\geq 1} \subset M_1(S)$ converges to $\nu \in M_1(S)$ in $\mathcal{H}$-topology if and only if

$$\lim_{n\to\infty} \langle \nu_n, f\rangle = \langle \nu, f\rangle, \ \forall f \in \mathcal{H}.$$

Let $\mathcal{G}_n = \sigma\left\{ f_n(Z_1, \cdots, Z_n) \mid f_n \in C_{sym}(S^n) \right\}$, and

$$\mathcal{Y}_n = \sigma\{\mathcal{G}_n, Z_{n+1}, Z_{n+2}, \cdots\}, \ \mathcal{Y}_\infty = \bigcap_{n=1}^{\infty} \mathcal{Y}_n.$$

Then W is $\mathcal{Y}_\infty$-measurable, and $\{Z_n\}_n$ is an i.i.d. random variable sequence with marginal distribution W conditioned on $\mathcal{Y}_\infty$.

Proof. **(1)** By Kolmogorov's extension theorem, similarly to [9] Theorem 11.2.1 (a), Lemma 4.1(1) can be proven.

(2) Notice (Z_i, Y) and (Z_1, Y), $1 \leq i \leq n$, have the identical law when Y is $\mathcal{Y}_n$-measurable. So $Z_1, \cdots, Z_n$ are identically distributed given $\mathcal{Y}_n$. Thus for any $\varphi \in \mathcal{B}_b(S)$, following from that $\langle W_n, \varphi\rangle$ is $\mathcal{Y}_n$-measurable, we have

$$Q\left(\varphi(Z_1)\middle|\, \mathcal{Y}_n\right) = Q\left(\frac{1}{n}\sum_{i=1}^{n} \varphi(Z_i)\middle|\, \mathcal{Y}_n\right) = \langle W_n, \varphi\rangle, \ Q - a.\ s.$$

However, $\mathcal{Y}_{n+1} \subset \mathcal{Y}_n$ and $Q(\varphi(Z_1)|\, \mathcal{Y}_n)$ is a reverse martingale, by the reverse martingale convergence theorem, we get as $n \to \infty$,

$$\langle W_n(\omega), \varphi\rangle = \frac{1}{n}\sum_{i=1}^{n} \varphi(Z_i(\omega)) \to Q\left(\varphi(Z_1)|\, \mathcal{Y}_\infty\right)(\omega), \ Q - a.\ s.\ \omega. \tag{4.1}$$

Note $S^{\mathbf{N}}$ is a Polish space. Let $W(\omega)$ be the regular conditional distribution of Z_1 given $\mathcal{Y}_\infty$. Then as $n \to \infty$,

$$\langle W_n(\omega), \varphi\rangle = \frac{1}{n}\sum_{i=1}^{n} \varphi(Z_i(\omega)) \to \langle W(\omega), \varphi\rangle, \ Q - a.\ s.\ \omega$$

for any fixed $\varphi \in \mathcal{B}_b(S)$. Therefore from separability of $\mathcal{H}$, we obtain that as $n \to \infty$,

$$\langle W_n(\omega), \varphi \rangle \to \langle W(\omega), \varphi \rangle, \ \forall \varphi \in \mathcal{H}, \ Q \ - \ a.\ s.\ \omega,$$

which says $\{W_n(\omega)\}_{n \geq 1}$ converges to $W(\omega)$ in the $\mathcal{H}$-topology, $Q \ - \ a.\ s.\ \omega$.

The rest of this part can be proven similarly to [9] Theorem 11.2.1 (c). □

Lemma 4.2. Given $\mu \in M_1(E)$, let $\{z_i\}_{i \geq 1}$ be a sequence of independent random variables with identical distribution μ. Write

$$\mathbf{X}_t^k = \frac{1}{k} \sum_{i=1}^{k} \delta_{Y_t^i(z_i)}, \ \mathbf{X}^k = \left(\mathbf{X}_t^k\right)_{t \geq 0}, \ \forall k \geq 1.$$

Let P_μ^k be the distribution of $\mathbf{X}^k = \left(\mathbf{X}_t^k\right)_{t \geq 0}$ on $D_{M_1(E)}$. Then P_μ^k converges weakly to a probability $P_\mu \in M_1\left(D_{M_1(E)}\right)$ in finite-dimensional distributions; and in addition, if all $\mathbf{Y^n}$ are continuous, then so is the coordinate process $\mathbf{X}$ on $D_{M_1(E)}$ under P_μ.

Proof. Since D_E is a Polish space under a suitable metric, there is a function family $\mathcal{H}_1 \subset C_b(D_E)$ on D_E which is separable with respect to the uniform norm and weak convergence determining, and $\sigma(\mathcal{H}_1)$ equals to the Borel σ-field on D_E.

To prove P_μ^k converges weakly to a probability $P_\mu \in M_1\left(D_{M_1(E)}\right)$ in finite-dimensional distributions, it suffices to check that for any $0 \leq t_1 < \cdots < t_n, \ n < \infty$,

$$(\Pi_{t_1,\cdots,t_n})_* P_\mu^k \text{ converges weakly to } (\Pi_{t_1,\cdots,t_n})_* P_\mu \text{ on } M_1(E)^k,$$

where $\Pi_{t_1,\cdots,t_n} : \ \omega \in D_{M_1(E)} \to (\omega_{t_1}, \cdots, \omega_{t_n}) \in M_1(E)^n$.

For any fixed $0 \leq t_1 < \cdots < t_n$, define the following function family on D_E :

$$\mathcal{H}_2(t_1, \cdots, t_n) = \{f(\omega_{t_1}, \cdots, \omega_{t_n}), \ \omega = (\omega_t)_{t \geq 0} \in D_E \ | \ f \in C_b(E^n)\}.$$

Then $\mathcal{H}_2(t_1, \cdots, t_n)$ is separable in the uniform norm. Let

$$\mathcal{H} = \mathcal{H}(t_1, \cdots, t_n) = \mathcal{H}_1 \cup \mathcal{H}_2(t_1, \cdots, t_n).$$

Then $(\mathcal{H}, \|\cdot\|)$ is separable and $\sigma(\mathcal{H})$ equals to the Borel σ-field on D_E. Define

$$\begin{aligned} \pi \quad : \quad & M_1(D_E) \to D_{M_1(E)}, \\ & \omega \ \to \pi(\omega) = ((\pi_t)_* \omega)_{t \geq 0}, \end{aligned}$$

where π_t is the projection from D_E to E at time t. Then π is measurable.

From Lemma 4.1, we have that

on some probability space supporting $Y_i = \left(Y_t^i(z_i)\right)_{t \geq 0}, \ i \geq 1, \ \frac{1}{k}\sum_{i=1}^{k} \delta_{Y_i}$ *converges in* $\mathcal{H}-$*topology to a random probability measure* Z *on* D_E *as* $k \to \infty$;

where $\left(Y_t^1(z_1), \cdots, Y_t^k(z_k)\right)_{t \geq 0}$ distributes as $\mathbf{Y^k}$ with initial distribution μ^k for $k \geq 1$. Particularly,

$$\text{almost surely, } (\pi_{t_i})_* \left(\frac{1}{k}\sum_{j=1}^{k} \delta_{Y_j}\right) \text{ converges weakly to } (\pi_{t_i})_* Z, \ \forall 1 \leq i \leq n.$$

Let Q_μ be the law of Z on $M_1(D_E)$, and P_μ the image measure of Q_μ under map π. Then $\int_E f(x)(\pi_t)_* Z(dx) = \int_{D_E} f(\gamma(t))\ Z(d\gamma)$ is càdlàg in $t \in R_+ = [0,\infty)$, for any $f \in C_b(E)$; and $((\pi_t)_* Z)_{t\geq 0}$ is càdlàg in $t \in R_+$ on $M_1(E)$. So the first part of the lemma holds.

The rest of the lemma follows from a similar deduction. □

Lemma 4.3. There is a unique family of probabilities $\{P_\nu\}_{\nu\in M_1(E)}$ on $D_{M_1(E)}$ such that the coordinate process $\mathbf{X} = (\mathbf{X}_t)_{t\geq 0}$ on $D_{M_1(E)}$ becomes a Markov process of the semigroup $\{T_t\}_{t\geq 0}$ determined by (1.1) under $\{P_\nu\}_{\nu\in M_1(E)}$. Moreover, if all $\mathbf{Y^k}$ are continuous, then each P_ν is in $M_1(C_{M_1(E)})$.

Proof. Step 1. For any $f \in \mathcal{B}_b(E^n)$ with $n \in \mathbf{N}$, there is a constant $C(n,f)$ depending on n and f such that uniformly in $\mu \in M_1(E)$ and $t \in [0,\infty)$,

$$\left| E\left[\left\langle \left(\mathbf{X}_t^k\right)^{\otimes n}, f \right\rangle \right] - \left\langle \mu^{\otimes n}, V_t^n f \right\rangle \right| \leq \frac{C(n,f)}{k},\ \textit{for large enough } k.$$

Where $\mathbf{X}_t^k$ is specified in Lemma 4.2.

In fact, with the consistency property, for $k \geq n$, we have

$$\begin{aligned} E\left[\left\langle \left(\mathbf{X}_t^k\right)^{\otimes n}, f \right\rangle \right] &= \frac{1}{k^n} \sum_{j_1,\cdots,j_n=1}^{k} E\left[f\left(Y_t^{j_1}(z_{j_1}), \cdots, Y_t^{j_n}(z_{j_n}) \right) \right] \\ &= \frac{1}{k^n} \sum_{(j_1,\cdots,j_n)\in I_n^k} E\left[f\left(Y_t^{j_1}(z_{j_1}), \cdots, Y_t^{j_n}(z_{j_n}) \right) \right] + \\ &\quad \frac{1}{k^n} \sum_{1\leq j_1\neq\cdots\neq j_n\leq k} \left\langle \mu^{\otimes n}, V_t^n f \right\rangle, \end{aligned}$$

where $I_n^k = \{(j_1,\cdots,j_n) \mid 1 \leq j_1,\cdots,j_n \leq k,\ j_i = j_l \text{ for some } i \neq l\}$. Note

$$\frac{1}{k^n} \sum_{1\leq j_1\neq\cdots\neq j_n\leq k} 1 = \frac{k(k-1)\cdots(k-n+1)}{k^n}.$$

We obtain that uniformly in $\mu \in M_1(E)$ and $t \in [0,\infty)$, for large enough k,

$$\left| \left(\frac{1}{k^n} \sum_{1\leq j_1\neq\cdots\neq j_n\leq k} \left\langle \mu^{\otimes n}, V_t^n f \right\rangle \right) - \left\langle \mu^{\otimes n}, V_t^n f \right\rangle \right| \leq \frac{C_1(n,f)}{k},$$

$$\left| \frac{1}{k^n} \sum_{(j_1,\cdots,j_n)\in I_n^k} E\left[f\left(Y_t^{j_1}(z_{j_1}), \cdots, Y_t^{j_n}(z_{j_n}) \right) \right] \right| \leq \frac{1}{k^n} \sum_{(j_1,\cdots,j_n)\in I_n^k} \|f\| \leq \frac{C_2(n,f)}{k}.$$

Here $C_1(n,f)$ and $C_2(n,f)$ are two constants depending on n and f. The claim holds.

Step 2. For any $F = F_{f,n} \in \mathcal{B}_p(M_1(E))$ and any $\mu \in M_1(E)$, let

$$T_t F_{f,n}(\mu) = \langle \mu^n, V_t^n f \rangle = \left\langle \mu^{\otimes n}, V_t^n f \right\rangle,\ \forall t \geq 0. \tag{4.2}$$

Note (4.2) does not depend on the expression of $F = F_{f,n}$, $\{T_t\}_{t\geq 0}$ is well defined,

$$\begin{aligned} T_{t+s}F_{f,n}(\mu) &= \langle \mu^n, V_{t+s}^n f\rangle = \langle \mu^n, V_t^n[V_s^n f]\rangle = T_t[F_{V_s^n f,n}](\mu) \\ &= T_t[T_s F_{f,n}](\mu) = T_t T_s F_{f,n}(\mu),\ \forall s,t\geq 0. \end{aligned}$$

Given any $t \in [0,\infty)$. By Lemma 4.2, $P_t^k(\mu, d\nu)$, the law of $\mathbf{X}_t^k$ under P_μ^k, converges weakly to a probability kernel $P_t(\mu, d\nu)$, the law of $\mathbf{X}_t$ under P_μ, as $k\to\infty$. Therefore, by Step 1,

$$T_t F(\mu) = \int_{M_1(E)} F(\nu)P_t(\mu, d\nu),\ \forall F\in C_p(M_1(E)).$$

Since for any $f\in\mathcal{B}_b(E^n)$ with $n\in\mathbf{N}$, there is a sequence $\{f_m\}_{m=1}^\infty \subseteq C_b(E^n)$ such that

$$\sup_{m\geq 1}\|f_m\| < \infty,\ \lim_{m\to\infty} f_m(x) = f(x),\ \forall x\in E^n,$$

we have

$$T_t F(\mu) = \int_{M_1(E)} F(\nu)P_t(\mu, d\nu),\ \forall F = F_{f,n}\in\mathcal{B}_p(M_1(E)).$$

Step 3. Extend $T_t F$ to any $F\in\mathcal{B}_b(M_1(E))$ by letting

$$T_t F(\mu) = \int_{M_1(E)} F(\nu)P_t(\mu, d\nu),\ \forall F\in\mathcal{B}_b(M_1(E)).$$

Then $\{T_t\}_{t\geq 0}$ is a Markov semigroup on $\mathcal{B}_b(M_1(E))$.

In fact, since $C_p(M_1(E))$, further $\mathcal{B}_p(M_1(E))$, separates points in $M_1(M_1(E))$ and

$$T_{t+s}F_{f,n} = T_t T_s F_{f,n},\ \forall F_{f,n}\in\mathcal{B}_p(M_1(E)),$$

we obtain

$$\begin{aligned} &\int_{M_1(E)}\int_{M_1(E)} F_{f,n}(w)P_t(\mu, d\nu)P_s(\nu, dw) = \int_{M_1(E)} F_{f,n}(w)P_{t+s}(\mu, dw), \\ &\int_{[\nu\in M_1(E)]} P_t(\mu, d\nu)P_s(\nu, dw) = P_{t+s}(\mu, dw), \end{aligned}$$

which shows $\{T_t\}_{t\geq 0}$ is a Markov semigroup on $\mathcal{B}_b(M_1(E))$.

Step 4. Let $\omega = (\omega_t)_{t\geq 0}$ be the unique $M_1(E)$-valued Markov process associated to the semigroup $\{T_t\}_{t\geq 0}$ on $\mathcal{B}_b(M_1(E))$. Then $\omega = (\omega_t)_{t\geq 0}$ has a càdlàg (resp. continuous) realization on $D_{M_1(E)}$ $\big($resp. $C_{M_1(E)}$ if all $\mathbf{Y^k}$ are continuous$\big)$.

In fact, with Lemma 4.2 in mind, it suffices to show $\left(\mathbf{X}^k, P_\mu^k\right)$ converges weakly to $\omega = (\omega_t)_{t\geq 0}$ in finite-dimensional distributions, that is to check

$$\begin{aligned} &\forall 0\leq s_1 < \cdots < s_r < \infty,\ \text{and}\ \forall f_i\in C_b(E^{n_i}),\ 1\leq n_i<\infty,\ 1\leq i\leq r, \\ &\lim_{k\to\infty} P_\mu^k\left[\prod_{i=1}^r \left\langle \left(\mathbf{X}_{s_i}^k\right)^{n_i}, f_i\right\rangle\right] = E\left[\prod_{i=1}^r \langle(\omega_{s_i})^{n_i}, f_i\rangle \,\middle|\, \omega_0 = \mu\right]. \end{aligned}$$

Note

$$\left\langle \left(\mathbf{X}_{s_i}^k\right)^{n_i}, f_i \right\rangle = \frac{1}{k^{n_i}} \sum_{1\le j_1,\cdots,j_{n_i}\le k} f_i\left(Y_{s_i}^{j_1}\left(z_{j_1}\right),\cdots,Y_{s_i}^{j_{n_i}}\left(z_{j_{n_i}}\right)\right),$$

$$\prod_{i=1}^r \left\langle \left(\mathbf{X}_{s_i}^k\right)^{n_i}, f_i \right\rangle = \frac{1}{k^n} \sum_{\substack{1\le j_1^u,\cdots,j_{n_u}^u\le k \\ 1\le u\le r}} \prod_{i=1}^r f_i\left(Y_{s_i}^{j_1^i}\left(z_{j_1^i}\right),\cdots,Y_{s_i}^{j_{n_i}^i}\left(z_{j_{n_i}^i}\right)\right)$$

$$= \frac{1}{k^n}\left[\Sigma_1+\Sigma_2\right]\prod_{i=1}^r f_i\left(Y_{s_i}^{j_1^i}\left(z_{j_1^i}\right),\cdots,Y_{s_i}^{j_{n_i}^i}\left(z_{j_{n_i}^i}\right)\right).$$

Where $n = n_1+\cdots+n_r$, and in Σ_1, the sum is taken over all $1\le j_1^u \ne \cdots \ne j_{n_u}^u \le k,\ 1\le u\le r$ such that

$$\{j_1^u,\cdots,j_{n_u}^u\}\cap\{j_1^v,\cdots,j_{n_v}^v\}=\emptyset,\ \forall 1\le u\ne v\le r;$$

and in Σ_2, the sum is taken over other indexes. For large enough k, we have that

$$\left|\frac{1}{k^n}\Sigma_2\prod_{i=1}^r f_i\left(Y_{s_i}^{j_1^i}\left(z_{j_1^i}\right),\cdots,Y_{s_i}^{j_n^i}\left(z_{j_{n_i}^i}\right)\right)\right|$$
$$\le \frac{C^r}{k^n}\Sigma_2 1 \le \frac{C^r}{k^n}(k^n-\Sigma_1 1) = \frac{C^r[k^n-k(k-1)\cdots(k-n+1)]}{k^n} \le \frac{C^r C(n)}{k},$$

where $C = \max_{1\le i\le r}\|f_i\|$ and $C(n)$ is a constant depending on n. Thus, with the exchangeability and consistency of $\left(Y_t^1(z_1),\cdots,Y_t^k(z_k)\right)_{t\ge 0}$, we obtain

$$\lim_{k\to\infty} E\left[\frac{1}{k^n}\Sigma_1\prod_{i=1}^r f_i\left(Y_{s_i}^{j_1^i}\left(z_{j_1^i}\right),\cdots,Y_{s_i}^{j_{n_i}^i}\left(z_{j_{n_i}^i}\right)\right)\right]$$
$$= \lim_{k\to\infty}\frac{1}{k^n}\Sigma_1 E\left[\prod_{i=1}^r f_i\left(Y_{s_i}^{j_1^i}\left(z_{j_1^i}\right),\cdots,Y_{s_i}^{j_{n_i}^i}\left(z_{j_{n_i}^i}\right)\right)\right]$$
$$= E\left[\prod_{i=1}^r f_i\left(Y_{s_i}^{m_{i-1}+1}(z_{m_{i-1}+1}),\cdots,Y_{s_i}^{m_i}(z_{m_i})\right)\right]$$
$$= \Big\langle \mu^{n_1+\cdots+n_r}, V_{s_1}^{n_1+\cdots+n_r}\Big[f_1\otimes V_{s_2-s_1}^{n_2+\cdots+n_r}\Big[f_2\otimes V_{s_3-s_2}^{n_3+\cdots+n_r}$$
$$\Big[\cdots\Big[f_{r-1}\otimes V_{s_r-s_{r-1}}^{n_r} f_r\Big]\cdots\Big]\Big]\Big]\Big\rangle,$$

where $m_0=0,\ m_i=n_1+\cdots+n_i,\ 1\le i\le r$; and the Markov property has been used, and for any pair $h_i\in\mathcal{B}_b\left(E^{l_i}\right),\ 1\le i\le 2,\ h_1\otimes h_2$ is a function on $E^{l_1+l_2}$ such that

$$h_1\otimes h_2(x_1,\cdots,x_{l_1+l_2}) = h_1(x_1,\cdots,x_{l_1})h_2(x_{l_1+1},\cdots,x_{l_1+l_2}).$$

On the other hand, by the Markov property,

$$E\left[\prod_{i=1}^r\langle(\omega_{s_i})^{n_i},f_i\rangle \;\middle|\; \omega_0=\mu\right]$$
$$= E\left[\left[\prod_{i=1}^{r-1}\langle(\omega_{s_i})^{n_i},f_i\rangle\right]\left\langle(\omega_{s_{r-1}})^{n_r}, V_{s_r-s_{r-1}}^{n_r}f_r\right\rangle \;\middle|\; \omega_0=\mu\right]$$

$$= E\left[\left[\prod_{i=1}^{r-2}\langle(\omega_{s_i})^{n_i}, f_i\rangle\right]\left\langle(\omega_{s_{r-1}})^{n_{r-1}+n_r}, f_{r-1}\otimes V^{n_r}_{s_r-s_{r-1}}f_r\right\rangle \,\middle|\, \omega_0=\mu\right]$$
$$= \cdots$$
$$= \left\langle \mu^{n_1+\cdots+n_r}, V^{n_1+\cdots+n_r}_{s_1}\left[f_1\otimes V^{n_2+\cdots+n_r}_{s_2-s_1}\left[f_2\otimes V^{n_3+\cdots+n_r}_{s_3-s_2}\right.\right.\right.$$
$$\left.\left.\left.\left[\cdots\left[f_{r-1}\otimes V^{n_r}_{s_r-s_{r-1}}f_r\right]\cdots\right]\right]\right]\right\rangle.$$

We are done. □

Proof of Theorem 2.1(2)

Note E is a Polish space, and there is a homeomorphic map φ from E into a G_δ-subset (intersection of countable many open subsets) of some compact metrizable space $\mathcal{S}$ of metric $d(\cdot,\cdot)$. Without loss of generality, suppose E is a G_δ-subset of $\mathcal{S}$ and φ the identity map on E. Hence the closure $\overline{E}$ of E in $(\mathcal{S}, d(\cdot,\cdot))$ is compact. Note $d(\cdot,\cdot)$ is a compatible metric on E and $(E, d(\cdot,\cdot))$ is not necessarily a Polish space in general. Recall $\mathbf{N}$ is the set of all natural numbers and define a metric $d_{\mathbf{N}}(\cdot,\cdot)$ on product space $\overline{E}^{\mathbf{N}}$:

$$d_{\mathbf{N}}\left(\{x_i\}_{i=1}^\infty, \{y_i\}_{i=1}^\infty\right) = \sum_{i=1}^\infty 2^{-n} d(x_i, y_i),\ \forall \{x_i\}_{i=1}^\infty, \{y_i\}_{i=1}^\infty \in \overline{E}^{\mathbf{N}}.$$

Since $\max\limits_{x,y\in\mathcal{S}} d(x,y) < \infty$, $d_{\mathbf{N}}(\cdot,\cdot)$ is a finite metric on $\overline{E}^{\mathbf{N}}$. For each $n\in\mathbf{N}$, introduce the following metric $d_n(\cdot,\cdot)$ on $\overline{E}^n$:

$$d_n((x_1,\cdots,x_n),(y_1,\cdots,y_n)) = \sum_{i=1}^n d(x_i,y_i),\ \forall x_i, y_i\in\overline{E},\ \forall 1\le i\le n.$$

Lemma 4.4. Let $C_u(E^n)$ be the family of all bounded uniformly continuous functions on $(E^n, d_n(\cdot,\cdot))$ for any $n\in\mathbf{N}$. View each $f\in C_u(E^n)$ as a bounded uniformly continuous function on $\left(E^{\mathbf{N}}, d_{\mathbf{N}}(\cdot,\cdot)\right)$ by letting

$$f(\{x_i\}_{i=1}^\infty) = f(x_1,\cdots,x_n),\ \forall\{x_i\}_{i=1}^\infty\in E^{\mathbf{N}}.$$

Then $\bigcup\limits_{n\ge1} C_u(E^n)$ is separable and dense in $C_u\left(E^{\mathbf{N}}\right)$, the set of all bounded uniformly continuous functions on $\left(E^{\mathbf{N}}, d_{\mathbf{N}}(\cdot,\cdot)\right)$, with respect to the uniform norm.

Proof. Notice $\left(\overline{E}^{\mathbf{N}}, d_{\mathbf{N}}(\cdot,\cdot)\right)$ is compact, and for each $f\in C_u(E^n)$ $\left(\text{resp. } C_u\left(E^{\mathbf{N}}\right)\right)$, there is a unique continuous function $\overline{f}\in C_b\left(\overline{E}^n\right)$ $\left(\text{resp. } C_b\left(\overline{E}^{\mathbf{N}}\right)\right)$ whose restriction to E^n $\left(\text{resp. } E^{\mathbf{N}}\right)$ is f and

$$f\in C_u(E^n)\ \left(\text{resp. } C_u\left(E^{\mathbf{N}}\right)\right)\to \overline{f}\in C_b\left(\overline{E}^n\right)\ \left(\text{resp. } C_b\left(\overline{E}^{\mathbf{N}}\right)\right)$$

is one-to-one and onto. Then through an obvious way, we can view any $f\in C_u(E^n)$ as a continuous function on $\overline{E}^{\mathbf{N}}$. With that $\bigcup\limits_{n\ge1} C_u(E^n)$ is an algebra and separates points in

$\overline{E}^{\mathbf{N}}$, and the Stone-Weierstrass theorem, we obtain $\bigcup_{n\geq 1} C_u(E^n)$ is dense in $C_b\left(\overline{E}^{\mathbf{N}}\right)$ in the uniform norm. Note $\left(C_b\left(\overline{E}^{\mathbf{N}}\right), \|\cdot\|\right)$ is separable, we get the lemma at once. □

Note n-point motion process

$$\mathbf{Y^n} = (\mathbf{Y^n}(x_1, \cdots, x_n))_{(x_1,\cdots,x_n)\in E^n} := \left(\left(Y_t^1(x_1), \cdots, Y_t^n(x_n)\right)_{t\geq 0}\right)_{(x_1,\cdots,x_n)\in E^n}$$

is a càdlàg E^n-valued strong Markov process. Let $Q_{x_1,\cdots,x_n}$ be the law of $\mathbf{Y^n}(x_1,\cdots,x_n)$ on D_{E^n} (resp. C_{E^n} when $\mathbf{Y^n}$ is continuous). For any $\{x_i\}_{i=1}^{\infty} \in E^{\mathbf{N}}$, by Kolmogorov's extension theorem, there is a unique probability $Q_{\{x_i\}_{i=1}^{\infty}}$ on $D_{E^{\mathbf{N}}}$ (resp. $C_{E^{\mathbf{N}}}$ when each $\mathbf{Y^n}$ is continuous) under which the coordinate process

$$\mathbf{w^N} = \left(\mathbf{w_t^N}\right)_{t\geq 0} = \left(\{w_t^i\}_{i=1}^{\infty}\right)_{t\geq 0}$$

on $D_{E^{\mathbf{N}}}$ (resp. $C_{E^{\mathbf{N}}}$) starts at $\{x_i\}_{i=1}^{\infty}$ and

$$\mathbf{w^n} = (\mathbf{w_t^n})_{t\geq 0} = \left(w_t^1, \cdots, w_t^n\right)_{t\geq 0}$$

distributes as $Q_{x_1,\cdots,x_n}$ for any $n \in \mathbf{N}$.

Lemma 4.5. The family $\left(Q_{\{x_i\}_{i=1}^{\infty}}\right)_{\{x_i\}_{i=1}^{\infty}\in E^{\mathbf{N}}}$ is strongly Markovian on $D_{E^{\mathbf{N}}}$ (resp. $C_{E^{\mathbf{N}}}$ when each $\mathbf{Y^n}$ is continuous).

Proof. By the Markov property of each $\mathbf{Y^n}$ and Lemma 4.4, we see $\left(Q_{\{x_i\}_{i=1}^{\infty}}\right)_{\{x_i\}_{i=1}^{\infty}\in E^{\mathbf{N}}}$ is Markovian on $D_{E^{\mathbf{N}}}$ (resp. $C_{E^{\mathbf{N}}}$ when each $\mathbf{Y^n}$ is continuous). To prove Lemma 4.5, by Lemma 4.4 and [34] P.31 Theorem 7.4.(v), it suffices to verify

$$\begin{array}{l} V_s^n f(Y_t^1(x_1), \cdots, Y_t^n(x_n)) \textit{ is right continuous in } t, \textit{ a.s.} - Q_{x_1,\cdots,x_n} \\ \textit{for any fixed } s \geq 0,\ n \in \mathbf{N},\ f \in C_u(E^n) \subset C_u\left(E^{\mathbf{N}}\right). \end{array}$$

However, by the strong Markov property of each $\mathbf{Y^n}$ and [34] P.31 Theorem 7.4.(v) again, this is true. □

Note D_E (resp. C_E) can be a Polish space because so is E, and there is a countable weak convergence determining function family $\mathcal{H}_1 \subset C_b(D_E)$ (resp. $C_b(C_E)$) whose bounded pointwise convergence limit closure is $\mathcal{B}_b(D_E)$ (resp. $\mathcal{B}_b(C_E)$). Let $\mathcal{U}$ be an arbitrary dense countable subset of $R_+ = [0, \infty)$. For any $n \in \mathbf{N}$ and any $t_i \in \mathcal{U},\ 1 \leq i \leq n$ with $0 \leq t_1 < \cdots < t_n$, and any $f \in C_u(E^n)$, define a function $F^f_{t_1,\cdots,t_n} \in \mathcal{B}_b(D_E)$ (resp. $C_b(C_E)$) by letting

$$F^f_{t_1,\cdots,t_n}(\gamma) = f(\gamma_{t_1}, \cdots, \gamma_{t_n}),\ \forall \gamma \in D_E \text{ (resp. } C_E).$$

Let

$$\begin{aligned} \mathcal{H}_2 &= \bigcup_{n\in\mathbf{N}} \bigcup_{\substack{t_i\in\mathcal{U}, 1\leq i\leq n \\ t_1<\cdots<t_n}} \left\{ F^f_{t_1,\cdots,t_n} \;\middle|\; f \in C_u(E^n) \right\} \subset \mathcal{B}_b(D_E) \text{ (resp. } C_b(D_E)), \\ \mathcal{H} &= \mathcal{H}_1 \cup \mathcal{H}_2. \end{aligned}$$

Then $\mathcal{H}$ is separable in the uniform norm. Endow $M_1(D_E)$ (resp. $M_1(C_E)$) with the $\mathcal{H}$-topology as follows:

$$\begin{aligned}&A\ sequence\ \{\nu_n\}_{n=1}^{\infty} \subset M_1(D_E)\ \text{(resp. } M_1(C_E))\ converges\ to\ \nu \in M_1(D_E)\\ &\text{(resp. } M_1(C_E))\ in\ \mathcal{H}-topology\ if\ and\ only\ if\\ &\qquad \lim_{n\to\infty}\langle \nu_n, f\rangle = \langle \nu, f\rangle,\ \forall f \in \mathcal{H}.\end{aligned}$$

Write $C_{sym}(D_E^n)$ (resp. $C_{sym}(C_E^n)$) for the set of all symmetric bounded continuous functions on $D_E \times \cdots \times D_E$ (resp. $C_E \times \cdots \times C_E$) ($n$ folds), for any $n \in \mathbf{N}$. On $D_{E^{\mathbf{N}}}$ (resp. $C_{E^{\mathbf{N}}}$) define the following σ-algebras:

$$\begin{aligned}&\mathcal{Y}_n = \sigma\left\{f(w^1,\cdots,w^n),\ f \in C_{sym}(D_E^n);\ w^{n+1}, w^{n+2}, \cdots\right\},\ n \in \mathbf{N}\\ &\big(\text{resp. } \mathcal{Y}_n = \sigma\left\{f(w^1,\cdots,w^n),\ f \in C_{sym}(C_E^n);\ w^{n+1}, w^{n+2}, \cdots\right\},\ n \in \mathbf{N}\big);\\ &\mathcal{Y}_\infty = \bigcap_{n=1}^{\infty} \mathcal{Y}_n;\end{aligned}$$

where $\mathbf{w}^{\mathbf{N}} = \left(\mathbf{w}_{\mathbf{t}}^{\mathbf{N}}\right)_{t\geq 0} = \left(\{w_t^i\}_{i=1}^{\infty}\right)_{t\geq 0}$ is the coordinate process on $D_{E^{\mathbf{N}}}$ (resp. $C_{E^{\mathbf{N}}}$), and

$$w^i = \left(w_t^i\right)_{t\geq 0} \in D_E \text{ (resp. } C_E),\ \forall i \in \mathbf{N}.$$

Note

$$\mathcal{I}:\ \{w^i\}_{i=1}^{\infty} \in D_E^{\mathbf{N}}\ \left(\text{resp. } C_E^{\mathbf{N}}\right) \to \left(\{w_t^i\}_{i=1}^{\infty}\right)_{t\geq 0} \in D_{E^{\mathbf{N}}}\ \left(\text{resp. } C_{E^{\mathbf{N}}}\right)$$

is a topological isomorphism, identify $D_E^{\mathbf{N}}$ $\left(\text{resp. } C_E^{\mathbf{N}}\right)$ with $D_{E^{\mathbf{N}}}$ $\left(\text{resp. } C_{E^{\mathbf{N}}}\right)$ by the $\mathcal{I}$.

Lemma 4.6. Note $Q_{\{x_i\}_{i=1}^{\infty}}$ is measurable in $\{x_i\}_{i=1}^{\infty} \in E^{\mathbf{N}}$. Given any $\mu \in M_1(E)$, let $Q_{\mu^{\mathbf{N}}}$ be the probability on $D_{E^{\mathbf{N}}}$ (resp. $C_{E^{\mathbf{N}}}$ when each $\mathbf{Y^n}$ is continuous) defined by

$$Q_{\mu^{\mathbf{N}}} = \int_{E^{\mathbf{N}}} Q_{\{x_i\}_{i=1}^{\infty}}\ \mu^{\mathbf{N}}(dx_1 dx_2 \cdots).$$

For the coordinate process

$$\mathbf{w}^{\mathbf{N}} = \left(\mathbf{w}_{\mathbf{t}}^{\mathbf{N}}\right)_{t\geq 0} = \left(\{w_t^i\}_{i=1}^{\infty}\right)_{t\geq 0}$$

on $D_{E^{\mathbf{N}}}$ (resp. $C_{E^{\mathbf{N}}}$), write

$$w^i = (w_t^i)_{t\geq 0} \in D_E \text{ (resp. } C_E),\ \forall i \in \mathbf{N}.$$

Then $Q_{\mu^{\mathbf{N}}}$ *a.s.*, as $n \to \infty$, $Z^{(n)} := \frac{1}{n}\sum_{i=1}^{n} \delta_{w^i}$ converges to some random probability Z on D_E (resp. C_E when each $\mathbf{Y^k}$ is continuous) in $\mathcal{H}$-topology; Z is $\mathcal{Y}_\infty$-measurable not depending on the choice of $\mathcal{H}$; and $\{w^i\}_{i=1}^{\infty}$ is an *i.i.d* random variable sequence with marginal distribution Z conditioned on $\mathcal{Y}_\infty$. Moreover, for any measurable function ϕ on D_E (resp. C_E when each $\mathbf{Y^k}$ is continuous) such that

$$Q_{\mu^{\mathbf{N}}}\left[\left|\phi(w^1)\right|^r\right] < \infty\ for\ some\ r \in [1, \infty),$$

the following results hold:

$$\begin{aligned}&\left\langle Z^{(n)}, \phi\right\rangle \to \left\langle Z\left(\mathbf{w}^{\mathbf{N}}\right)(d\gamma), \phi\right\rangle,\ a.s.\ \mathbf{w}^{\mathbf{N}}\ and\ in\ L^r\left(Q_{\mu^{\mathbf{N}}}\right)\ as\ n \to \infty,\\ &Q_{\mu^{\mathbf{N}}}\left[\left\langle Z\left(\mathbf{w}^{\mathbf{N}}\right)(d\gamma), |\phi|^r\right\rangle\right] = \lim_{n\to\infty} Q_{\mu^{\mathbf{N}}}\left[\left\langle Z^{(n)}, |\phi|^r\right\rangle\right] = Q_{\mu^{\mathbf{N}}}\left[\left|\phi(w^1)\right|^r\right],\\ &Q_{\mu^{\mathbf{N}}}\left[\left\langle Z\left(\mathbf{w}^{\mathbf{N}}\right)(d\gamma), \phi\right\rangle\right] = Q_{\mu^{\mathbf{N}}}\left[\phi(w^1)\right],\ Q_{\mu^{\mathbf{N}}}\left[\left\langle Z\left(\mathbf{w}^{\mathbf{N}}\right)(d\gamma), |\phi|\right\rangle\right] = Q_{\mu^{\mathbf{N}}}\left[\left|\phi(w^1)\right|\right].\end{aligned}$$

Proof. The proof of the lemma is similar to Lemma 4.1(2). The readers may skip it. Note (w^i, Y) and (w^1, Y), $1 \le i \le n$, have the same law for $\mathcal{Y}_n$-measurable Y. So $w^1, \cdots, w^n$ are identically distributed given $\mathcal{Y}_n$. Thus $\forall \varphi \in \mathcal{H}$, following from that $\int_{D_E} \varphi(\gamma)\ Z^{(n)}(d\gamma)$ $\Big($resp. $\int_{C_E} \varphi(\gamma)\ Z^{(n)}(d\gamma)$ *if each* $\mathbf{Y^k}$ *is continuous*$\Big)$ is $\mathcal{Y}_n$-measurable,

$$Q_{\mu^{\mathbf{N}}}\left[\varphi(w^1)\ \middle|\ \mathcal{Y}_n\right] = Q_{\mu^{\mathbf{N}}}\left[\frac{1}{n}\sum_{i=1}^{n}\varphi(w^i)\ \middle|\mathcal{Y}_n\right] = \left\langle Z^{(n)}(d\gamma), \varphi\right\rangle,\ Q_{\mu^{\mathbf{N}}}\ -\ a.s.. \tag{4.3}$$

However, $\mathcal{Y}_{n+1} \subset \mathcal{Y}_n$ and $Q_{\mu^{\mathbf{N}}}\left[\varphi(w^1)\middle|\,\mathcal{Y}_n\right]$ is a reverse martingale, by the reverse martingale convergence theorem, we get that as $n \to \infty$,

$$\left\langle Z^{(n)}(d\gamma), \varphi\right\rangle = \frac{1}{n}\sum_{i=1}^{n}\varphi(w^i) \to Q_{\mu^{\mathbf{N}}}\left[\varphi(w^1)\ \middle|\mathcal{Y}_\infty\right],\ a.s.\ \mathbf{w^N}\ and\ in\ L^1\left(Q_{\mu^{\mathbf{N}}}\right). \tag{4.4}$$

Let $Z\left(\mathbf{w^N}\right)$ be the regular conditional distribution of w^1 given $\mathcal{Y}_\infty$ under $Q_{\mu^{\mathbf{N}}}$. Then $Q_{\mu^{\mathbf{N}}}$ $-$ $a.s.$ $\mathbf{w^N}$, as $n \to \infty$,

$$\int_{D_E}\varphi(\gamma)\ Z^{(n)}\left(\mathbf{w^N}\right)(d\gamma) = \frac{1}{n}\sum_{i=1}^{n}\varphi(w^i) \to \int_{D_E}\varphi(\gamma)\ Z\left(\mathbf{w^N}\right)(d\gamma)$$
$$\left(\text{resp. } \int_{C_E}\varphi(\gamma)\ Z^{(n)}\left(\mathbf{w^N}\right)(d\gamma) = \frac{1}{n}\sum_{i=1}^{n}\varphi(w^i) \to \int_{C_E}\varphi(\gamma)\ Z\left(\mathbf{w^N}\right)(d\gamma)\right),$$

for any fixed $\varphi \in \mathcal{H}$. Thus from separability of $(\mathcal{H}, \|\cdot\|)$, we get that as $n \to \infty$,

$$\left\langle Z^{(n)}\left(\mathbf{w^N}\right)(d\gamma), \varphi(\cdot)\right\rangle \to \left\langle Z\left(\mathbf{w^N}\right)(d\gamma), \varphi(\cdot)\right\rangle,\ \forall\varphi \in \mathcal{H},\ Q_{\mu^{\mathbf{N}}}\ -\ a.s.\ \mathbf{w^N},$$

which says $\left\{Z^{(n)}\left(\mathbf{w^N}\right)\right\}_{n=1}^{\infty}$ converges to $Z\left(\mathbf{w^N}\right)$ in the $\mathcal{H}$-topology, $Q_{\mu^{\mathbf{N}}}$ $-$ $a.s.$ $\mathbf{w^N}$. Clearly, Z does not depend on the choice of $\mathcal{H}$. Similarly to [9] Theorem 11.2.1(c), one can prove that $\{w^i\}_{i=1}^{\infty}$ is an $i.i.d$ random variable sequence with marginal distribution Z conditioned on $\mathcal{Y}_\infty$.

For any measurable function ϕ on D_E (resp. C_E) satisfying

$$Q_{\mu^{\mathbf{N}}}\left[\left|\phi(w^1)\right|^r\right] < \infty\ for\ some\ r \in [1, \infty),$$

similarly to (4.3)-(4.4), we have that as $n \to \infty$,

$$\left\langle Z^{(n)}, \phi\right\rangle = Q_{\mu^{\mathbf{N}}}\left[\phi(w^1)\middle|\,\mathcal{Y}_n\right] \to \left\langle Z\left(\mathbf{w^N}\right), \phi\right\rangle = Q_{\mu^{\mathbf{N}}}\left[\phi(w^1)\middle|\,\mathcal{Y}_\infty\right],\ a.s.\ \mathbf{w^N};$$

and note $\left\{Q_{\mu^{\mathbf{N}}}\left[\phi(w^1)\middle|\,\mathcal{Y}_n\right]\right\}_{n=1}^{\infty}$ is uniformly L^r-integrable due to $Q_{\mu^{\mathbf{N}}}\left[\left|\phi(w^1)\right|^r\right] < \infty$, we see that

$$\left\langle Z^{(n)}, \phi\right\rangle \to \left\langle Z\left(\mathbf{w^N}\right), \phi\right\rangle\ in\ L^r\left(Q_{\mu^{\mathbf{N}}}\right)\ as\ n \to \infty,$$
$$Q_{\mu^{\mathbf{N}}}\left[\left\langle Z\left(\mathbf{w^N}\right)(d\gamma), \phi\right\rangle\right] = \lim_{n\to\infty} Q_{\mu^{\mathbf{N}}}\left[\left\langle Z^{(n)}, \phi\right\rangle\right] = Q_{\mu^{\mathbf{N}}}\left[\phi(w^1)\right].$$

So replace ϕ by $|\phi|^r$ or $|\phi|$, we obtain

$$Q_{\mu^{\mathbf{N}}}\left[\left\langle Z\left(\mathbf{w}^{\mathbf{N}}\right)(d\gamma), |\phi|^r\right\rangle\right] = \lim_{n\to\infty} Q_{\mu^{\mathbf{N}}}\left[\left\langle Z^{(n)}, |\phi|^r\right\rangle\right] = Q_{\mu^{\mathbf{N}}}\left[\left|\phi(w^1)\right|^r\right],$$
$$Q_{\mu^{\mathbf{N}}}\left[\left\langle Z\left(\mathbf{w}^{\mathbf{N}}\right)(d\gamma), |\phi|\right\rangle\right] = Q_{\mu^{\mathbf{N}}}\left[\left|\phi(w^1)\right|\right].$$

□

Lemma 4.7. Define the following map Ξ from $D_{E^{\mathbf{N}}}$ (resp. $C_{E^{\mathbf{N}}}$) into $M_1(D_E)$ (resp. $M_1(C_E)$):

$$\Xi\left(\left(\{w_t^i\}_{i=1}^{\infty}\right)_{t\geq 0}\right) = \begin{cases} \textit{the limit of } \left\{\frac{1}{n}\sum\limits_{i=1}^{n}\delta_{w_\cdot^i}\right\}_{n=1}^{\infty} \textit{ in } M_1(D_E) \text{ (resp. } M_1(C_E)) \\ \quad\quad \textit{if it exists,} \\ Z_0 \in M_1(D_E) \text{ (resp. } M_1(C_E)), \textit{ otherwise,} \end{cases}$$

with $Z_0 \in M_1(D_E)$ (resp. $M_1(C_E)$) being fixed. Then Ξ is measurable.

In addition, define the following map Θ from $E^{\mathbf{N}}$ into $M_1(E)$:

$$\Theta\left(\{x_i\}_{i=1}^{\infty}\right) = \begin{cases} \textit{the limit of } \left\{\frac{1}{n}\sum\limits_{i=1}^{n}\delta_{x_i}\right\}_{n=1}^{\infty} \textit{ in } M_1(E) \textit{ if it exists,} \\ \mu_0 \in M_1(E), \textit{ otherwise,} \end{cases}$$

where $\mu_0 \in M_1(E)$ is fixed. Then Θ is measurable.

Proof. Note D_E (resp. C_E) can be a Polish space, and there is a homeomorphic map ϕ from D_E (resp. C_E) into a G_δ-subset (intersection of countable many open subsets) of some compact metrizable space $\mathcal{T}$ of metric $r(\cdot,\cdot)$. Without loss of generality, suppose D_E (resp. C_E) is a G_δ-subset of $\mathcal{T}$. Then $\mathbf{T}$, the closure of D_E (resp. C_E) in $\mathcal{T}$, is compact. Let $C_u(D_E)$ (resp. $C_u(C_E)$) be the set of all bounded uniformly continuous functions on $(D_E, r(\cdot,\cdot))$ (resp. $(C_E, r(\cdot,\cdot))$). Notice for each $f \in C_u(D_E)$ (resp. $C_u(C_E)$), there is a unique continuous function $\overline{f} \in C_b(\mathbf{T})$ whose restriction to D_E (resp. C_E) is f and

$$f \in C_u(D_E) \text{ (resp. } C_u(C_E)) \to \overline{f} \in C_b(\mathbf{T})$$

is one-to-one and onto. Then $(C_u(D_E), \|\cdot\|)$ (resp. $(C_u(C_E), \|\cdot\|)$) is separable. Write $\{g_n\}_{n=1}^{\infty}$ for a dense subset of $(C_u(D_E), \|\cdot\|)$ (resp. $(C_u(C_E), \|\cdot\|)$). Define a map $\Xi_{M_1(\mathbf{T})}$ from $D_{E^{\mathbf{N}}}$ (resp. $C_{E^{\mathbf{N}}}$) into $M_1(\mathbf{T})$ as follows:

$$\Xi_{M_1(\mathbf{T})}\left(\left(\{w_t^i\}_{i=1}^{\infty}\right)_{t\geq 0}\right) = \begin{cases} \textit{the limit of } \left\{\frac{1}{n}\sum\limits_{i=1}^{n}\delta_{w_\cdot^i}\right\}_{n=1}^{\infty} \textit{ in } M_1(\mathbf{T}) \textit{ if it exists,} \\ Z_0, \textit{ otherwise,} \end{cases}$$

For convenience, write $\mathbf{I} = D_{E^{\mathbf{N}}}$ (resp. $C_{E^{\mathbf{N}}}$). Then

$$\begin{aligned} \mathbf{I}(+) &:= \left\{\left(\left\{w_t^i\right\}_{i=1}^{\infty}\right)_{t\geq 0} \in \mathbf{I} \;\middle|\; \textit{the limit of } \left\{\frac{1}{k}\sum_{i=1}^{k}\delta_{w_\cdot^i}\right\}_{k=1}^{\infty} \textit{ in } M_1(\mathbf{T}) \textit{ exists}\right\} \\ &= \left\{\left(\left\{w_t^i\right\}_{i=1}^{\infty}\right)_{t\geq 0} \in \mathbf{I} \;\middle|\; \lim_{k\to\infty}\frac{1}{k}\sum_{i=1}^{k} g_m\left(w_\cdot^i\right) \textit{ exists for any } m \in \mathbf{N}\right\} \\ &= \bigcap_{m\in\mathbf{N}}\left\{\left(\left\{w_t^i\right\}_{i=1}^{\infty}\right)_{t\geq 0} \in \mathbf{I} \;\middle|\; \limsup_{k\to\infty}\frac{1}{k}\sum_{i=1}^{k} g_m\left(w_\cdot^i\right) = \liminf_{k\to\infty}\frac{1}{k}\sum_{i=1}^{k} g_m\left(w_\cdot^i\right)\right\}, \end{aligned}$$

which implies $\mathbf{I}(+)$ is a measurable subset of $\mathbf{I}$. Clearly, the map

$$\left(\{w_t^i\}_{i=1}^{\infty}\right)_{t\geq 0} \in \mathbf{I}(+) \to \lim_{k\to\infty} \frac{1}{k}\sum_{i=1}^{k} g\left(w_\cdot^i\right)$$

is measurable, for any $g \in C_u(D_E)$ (resp. $C_u(C_E)$), which shows $\Xi_{M_1(\mathbf{T})}$ is measurable. Note $M_1(\mathbf{I})$ is a measurable subset of $M_1(\mathbf{T})$. So Ξ is measurable.

The measurability of Θ can be proven similarly. □

On $D_{E^{\mathbf{N}}}$ (resp. $C_{E^{\mathbf{N}}}$ when each $\mathbf{Y^n}$ is continuous), by Lemmas 4.6-4.7,

$$Z\left(\mathbf{w^N}\right) = \Xi\left(\mathbf{w^N}\right),\ Q_{\mu^{\mathbf{N}}}\ -\ a.s..$$

Here and hereafter, we take

$$Z\left(\mathbf{w^N}\right) = \Xi\left(\mathbf{w^N}\right),\ \forall \mathbf{w^N} \in D_{E^{\mathbf{N}}}\ \text{(resp. } C_{E^{\mathbf{N}}} \text{ when each } \mathbf{Y^n} \text{ is continuous)}.$$

For any $t \in R_+$, let $\pi_t:\ \gamma \in D_E$ (resp. C_E) $\to \gamma(t) \in E$ be the canonical projection at time t. Note π_t is measurable (resp. continuous). For any $t \in R_+$, define

$$\mathbf{X}_t := \mathbf{X}_t\left(\mathbf{w^N}\right) = (\pi_t)_* Z\left(\mathbf{w^N}\right),\ \textit{the image measure of } Z\left(\mathbf{w^N}\right) \textit{ under } \pi_t.$$

Then $\int_E f(x)\mathbf{X}_t\left(\mathbf{w^N}\right)(dx) = \int f(\gamma(t))Z\left(\mathbf{w^N}\right)(d\gamma)$ is càdlàg (resp. continuous) in $t \in R_+$, for any $f \in C_b(E)$. So $\mathbf{X} = (\mathbf{X}_t)_{t\geq 0}$ is càdlàg (resp. continuous) on $M_1(E)$; and under $Q_{\mu^{\mathbf{N}}}$, $\mathbf{X} = (\mathbf{X})_{t\geq 0}$ distributes as P_μ.

Lemma 4.8. Let $\omega = (\omega_t)_{t\geq 0}$ be the coordinate process on $D_{M_1(E)}$ $\left(\text{resp. } C_{M_1(E)}\right)$ and

$$\mathcal{F}_t = \bigcap_{s>t} \sigma(\omega_u, u \leq s),\ \forall t \geq 0;\ \mathcal{F} = \sigma(\mathcal{F}_t, t \geq 0).$$

Given arbitrary bounded $(\mathcal{F}_t)$-stopping time $\tau := \tau(\omega)$ on $D_{M_1(E)}$ $\left(\text{resp. } C_{M_1(E)}\right)$. Then on $(D_{E^{\mathbf{N}}}, Q_{\mu^{\mathbf{N}}})$ $\left(\text{resp. } (C_{E^{\mathbf{N}}}, Q_{\mu^{\mathbf{N}}}) \text{ when each } \mathbf{Y^n} \text{ is continuous}\right)$, as $n \to \infty$,

$$\mathbf{X}^n_{\tau(\mathbf{X}(\mathbf{w^N}))}\left(\mathbf{w^N}\right) \textit{ converges weakly to } \mathbf{X}_{\tau(\mathbf{X}(\mathbf{w^N}))}\left(\mathbf{w^N}\right),\ Q_{\mu^{\mathbf{N}}}\ -\ a.s.\ \mathbf{w^N};$$

and conditioned on $\mathcal{Y}_\infty$, $\left\{w^k_{\tau(\mathbf{X}(\mathbf{w^N}))}\right\}_{k=1}^{\infty}$ is an *i.i.d* random variable sequence distributed as $\mathbf{X}_{\tau(\mathbf{X}(\mathbf{w^N}))}\left(\mathbf{w^N}\right)$.

Proof. Note if conditioned on $\mathcal{Y}_\infty$, $\left\{w^k_{\tau(\mathbf{X}(\mathbf{w^N}))}\right\}_{k=1}^{\infty}$ is an *i.i.d* random variable sequence distributed as $\mathbf{X}_{\tau(\mathbf{X}(\mathbf{w^N}))}\left(\mathbf{w^N}\right)$; then by the law of large numbers, conditioned on $\mathcal{Y}_\infty$, $\mathbf{X}^n_{\tau(\mathbf{X}(\mathbf{w^N}))}\left(\mathbf{w^N}\right)$ converges weakly to $\mathbf{X}_{\tau(\mathbf{X}(\mathbf{w^N}))}\left(\mathbf{w^N}\right)$, which implies

$$\mathbf{X}^n_{\tau(\mathbf{X}(\mathbf{w^N}))}\left(\mathbf{w^N}\right) \textit{ converges weakly to } \mathbf{X}_{\tau(\mathbf{X}(\mathbf{w^N}))}\left(\mathbf{w^N}\right),\ Q_{\mu^{\mathbf{N}}}\ -\ a.s.\ \mathbf{w^N}.$$

So it suffices to prove that for any $n \in \mathbf{N}$, and any $f_k \in C_b(E),\ 1 \leq k \leq n$,

$$Q_{\mu^{\mathbf{N}}}\left[\prod_{k=1}^{n} f_k\left(w^k_{\tau(\mathbf{X}(\mathbf{w^N}))}\right)\ \middle|\ \mathcal{Y}_\infty\right] = \int_{E^n}\prod_{k=1}^{n} f_k(x_k)\ \left(\mathbf{X}_{\tau(\mathbf{X}(\mathbf{w^N}))}\left(\mathbf{w^N}\right)\right)^n (dx_1\cdots dx_n). \tag{4.5}$$

In fact, since $\mathbf{X}\left(\mathbf{w}^{\mathbf{N}}\right)$ and $\tau\left(\mathbf{X}\left(\mathbf{w}^{\mathbf{N}}\right)\right)$ are $\mathcal{Y}_\infty$-measurable,

$$\int_{E^n}\prod_{k=1}^n f_k(x_k)\ \left(\mathbf{X}_{\tau(\mathbf{X}(\mathbf{w}^{\mathbf{N}}))}\left(\mathbf{w}^{\mathbf{N}}\right)\right)^n(dx_1\cdots dx_n)$$

is $\mathcal{Y}_\infty$-measurable. For any $l\in\mathbf{N}$, define

$$\tau_l\left(\mathbf{w}^{\mathbf{N}}\right)=\frac{i}{2^l},\ if\ \tau\left(\mathbf{X}\left(\mathbf{w}^{\mathbf{N}}\right)\right)\in\left[\frac{i-1}{2^l},\frac{i}{2^l}\right)\ for\ some\ i\in\mathbf{N}.$$

By Lemma 4.6, $Q_{\mu^{\mathbf{N}}}\ -\ a.s.\ \mathbf{w}^{\mathbf{N}}$,

$$\mathbf{X}_t\left(\mathbf{w}^{\mathbf{N}}\right)=\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^n\delta_{w_t^i}\ (in\ the\ weak\ topology)=\Theta\left(\left\{w_t^i\right\}_{i=1}^\infty\right),\ \forall t\in\mathcal{U}.$$

Note $\mathcal{U}$ is a dense countable subset of R_+ and $\mathbf{X}=(\mathbf{X}_t)_{t\geq0}$ is càdlàg in $t\in R_+$. Then each $\tau_l\left(\mathbf{w}^{\mathbf{N}}\right)$ is a bounded stopping time of the process $\mathbf{w}^{\mathbf{N}}=\left(\{w_t^i\}_{i=1}^\infty\right)_{t\geq0}$. For any $i\in\mathbf{N}$, write

$$G_{l,i}=\left\{\tau\left(\mathbf{X}\left(\mathbf{w}^{\mathbf{N}}\right)\right)\in\left[\frac{i-1}{2^l},\frac{i}{2^l}\right)\right\},\ \forall l,i\in\mathbf{N}.$$

Given any $F\in\mathcal{Y}_\infty$, note $G_{l,i}\bigcap F\in\mathcal{Y}_\infty$. By Lemma 4.6,

$$\begin{aligned}
&Q_{\mu^{\mathbf{N}}}\left[\left(\int_{E^n}\prod_{k=1}^n f_k(x_k)\ \left(\mathbf{X}_{\tau_l(\mathbf{w}^{\mathbf{N}})}\left(\mathbf{w}^{\mathbf{N}}\right)\right)^n(dx_1\cdots dx_n)\right)I_{G_{l,i}\bigcap F}\right]\\
&=Q_{\mu^{\mathbf{N}}}\left[\left(\int_{D_E^n}\prod_{k=1}^n f_k\circ\pi_{\frac{i}{2^l}}(\widetilde{w}^k)\ \left(Z\left(\mathbf{w}^{\mathbf{N}}\right)\right)^n(d\widetilde{w}^1\cdots d\widetilde{w}^n)\right)I_{G_{l,i}\bigcap F}\right]\\
&=Q_{\mu^{\mathbf{N}}}\left[\left(Q_{\mu^{\mathbf{N}}}\left[\prod_{k=1}^n f_k\circ\pi_{\frac{i}{2^l}}(w^k)\ \middle|\ \mathcal{Y}_\infty\right]\right)I_{G_{l,i}\bigcap F}\right]\\
&=Q_{\mu^{\mathbf{N}}}\left[\left(\prod_{k=1}^n f_k\circ\pi_{\tau_l(\mathbf{w}^{\mathbf{N}})}(w^k)\right)I_{G_{l,i}\bigcap F}\right].
\end{aligned}$$

Therefore,

$$\begin{aligned}
&Q_{\mu^{\mathbf{N}}}\left[\left(\int_{E^n}\prod_{k=1}^n f_k(x_k)\ \left(\mathbf{X}_{\tau_l(\mathbf{w}^{\mathbf{N}})}\left(\mathbf{w}^{\mathbf{N}}\right)\right)^n(dx_1\cdots dx_n)\right)I_F\right]\\
&=Q_{\mu^{\mathbf{N}}}\left[\left(\prod_{k=1}^n f_k\circ\pi_{\tau_l(\mathbf{w}^{\mathbf{N}})}(w^k)\right)I_F\right].
\end{aligned}$$

Let $l\uparrow\infty$, then $\tau_l\left(\mathbf{w}^{\mathbf{N}}\right)\downarrow\tau\left(\mathbf{X}\left(\mathbf{w}^{\mathbf{N}}\right)\right)$, $Q_{\mu^{\mathbf{N}}}\ -\ a.s.\ \mathbf{w}^{\mathbf{N}}$. Note process $\mathbf{X}$ is right continuous and each $f_k\in C_b(E)$, by the bounded convergence theorem,

$$\begin{aligned}
&Q_{\mu^{\mathbf{N}}}\left[\left(\int_{E^n}\prod_{k=1}^n f_k(x_k)\ \left(\mathbf{X}_{\tau(\mathbf{X}(\mathbf{w}^{\mathbf{N}}))}\left(\mathbf{w}^{\mathbf{N}}\right)\right)^n(dx_1\cdots dx_n)\right)I_F\right]\\
&=Q_{\mu^{\mathbf{N}}}\left[\left(\prod_{k=1}^n f_k\circ\pi_{\tau(\mathbf{X}(\mathbf{w}^{\mathbf{N}}))}(w^k)\right)I_F\right]=Q_{\mu^{\mathbf{N}}}\left[\left(\prod_{k=1}^n f_k\left(w^k_{\tau(\mathbf{X}(\mathbf{w}^{\mathbf{N}}))}\right)\right)I_F\right],
\end{aligned}$$

which implies (4.5). □

Lemma 4.9. Given any dense countable subset $\mathcal{U}$ of R_+, and any τ specified in Lemma 4.8, and any $k, l \in \mathbf{N}$ and any constants

$$0 < s_1 < s_2 < \cdots < s_k < \infty \textit{ and } 0 < u_i < \infty, \ 1 \le i \le l; \\ (\textit{need not to assume each } s_i \in \mathcal{U})$$

then for any $f_j \in C_b(M_1(E)),\ 1 \le j \le k$, and any $g_i \in C_b(M_1(E)),\ 1 \le i \le l$,

$$\begin{aligned} & P_\mu \left[\left(\prod_{j=1}^{k} f_j \left(\omega_{\tau(\omega) \wedge s_j} \right) \right) \prod_{i=1}^{l} g_i \left(\omega_{\tau(\omega)+u_i} \right) \right] \\ & = P_\mu \left[\left(\prod_{j=1}^{k} f_j \left(\omega_{\tau(\omega) \wedge s_j} \right) \right) P_{\omega_{\tau(\omega)}} \left[\prod_{i=1}^{l} g_i(\omega_{u_i}) \right] \right], \end{aligned}$$

where $\omega = (\omega_t)_{t \ge 0}$ is specified in Lemma 4.8, which implies Theorem 2.1(2).

Proof. By Lemmas 4.6-4.8, we have

$$\begin{aligned} & \Theta\left(\mathbf{w}_{\mathbf{t}}^{\mathbf{N}}\right) = \mathbf{X}_t\left(\mathbf{w}^{\mathbf{N}}\right), \ \forall t \in \mathcal{U}, \ Q_{\nu^{\mathbf{N}}} \ - \ a.s.; \\ & \Theta\left(\mathbf{w}^{\mathbf{N}}_{\tau(\mathbf{X}(\mathbf{w}^{\mathbf{N}}))+\mathbf{u_i}}\right) = \mathbf{X}_{\tau(\mathbf{X}(\mathbf{w}^{\mathbf{N}}))+u_i}\left(\mathbf{w}^{\mathbf{N}}\right), \ 1 \le i \le l, \ Q_{\nu^{\mathbf{N}}} \ - \ a.s.; \\ & \Theta\left(\mathbf{w}^{\mathbf{N}}_{\tau(\mathbf{X}(\mathbf{w}^{\mathbf{N}}))\wedge \mathbf{s_j}}\right) = \mathbf{X}_{\tau(\mathbf{X}(\mathbf{w}^{\mathbf{N}}))\wedge s_j}\left(\mathbf{w}^{\mathbf{N}}\right), \ 1 \le j \le k, \ Q_{\nu^{\mathbf{N}}} \ - \ a.s.; \end{aligned}$$

for arbitrary $\nu \in M_1(E)$. Therefore,

$$\begin{aligned} & P_\mu \left[\left(\prod_{j=1}^{k} f_j \left(\omega_{\tau(\omega) \wedge s_j} \right) \right) \prod_{i=1}^{l} g_i \left(\omega_{\tau(\omega)+u_i} \right) \right] \\ & = Q_{\mu^{\mathbf{N}}} \left[\left(\prod_{j=1}^{k} f_j \left(\mathbf{X}_{\tau(\mathbf{X}) \wedge s_j} \right) \right) \prod_{i=1}^{l} g_i \left(\mathbf{X}_{\tau(\mathbf{X})+u_i} \right) \right] \\ & = Q_{\mu^{\mathbf{N}}} \left[\left(\prod_{j=1}^{k} f_j \left(\Theta \left(\mathbf{w}^{\mathbf{N}}_{\tau(\mathbf{X}(\mathbf{w}^{\mathbf{N}}))\wedge \mathbf{s_j}} \right) \right) \right) \prod_{i=1}^{l} g_i \left(\Theta \left(\mathbf{w}^{\mathbf{N}}_{\tau(\mathbf{X}(\mathbf{w}^{\mathbf{N}}))+\mathbf{u_i}} \right) \right) \right] \\ & = Q_{\mu^{\mathbf{N}}} \left[\left(\prod_{j=1}^{k} f_j \left(\Theta \left(\mathbf{w}^{\mathbf{N}}_{\tau(\mathbf{X}(\mathbf{w}^{\mathbf{N}}))\wedge \mathbf{s_j}} \right) \right) \right) Q_{\left\{ w^i_{\tau(\mathbf{X}(\mathbf{w}^{\mathbf{N}}))} \right\}_{i=1}^{\infty}} \left[\prod_{i=1}^{l} g_i \left(\Theta \left(\mathbf{w}^{\mathbf{N}}_{\mathbf{u_i}} \right) \right) \right] \right] \\ & \left(\textit{note } \tau\left(\mathbf{X}\left(\mathbf{w}^{\mathbf{N}}\right)\right) \textit{ is a stopping time for process } \mathbf{w}^{\mathbf{N}} = \left(\mathbf{w}^{\mathbf{N}}_{\mathbf{t}}\right)_{t \ge 0} = \left(\{w^i_t\}_{i=1}^{\infty}\right)_{t \ge 0} \right. \\ & \left. \textit{and } \left\{ Q_{\{x_i\}_{i=1}^{\infty}} \right\}_{\{x_i\}_{i=1}^{\infty} \in E^{\mathbf{N}}} \textit{ is a strong Markov family} \right) \\ & = Q_{\mu^{\mathbf{N}}} \left[\left(\prod_{j=1}^{k} f_j \left(\Theta \left(\mathbf{w}^{\mathbf{N}}_{\tau(\mathbf{X}(\mathbf{w}^{\mathbf{N}}))\wedge \mathbf{s_j}} \right) \right) \right) Q_{\mu^{\mathbf{N}}} \left[Q_{\left\{ w^i_{\tau(\mathbf{X}(\mathbf{w}^{\mathbf{N}}))} \right\}_{i=1}^{\infty}} \left[\prod_{i=1}^{l} g_i \left(\Theta \left(\mathbf{w}^{\mathbf{N}}_{\mathbf{u_i}} \right) \right) \right] \middle| \ \mathcal{Y}_\infty \right] \right] \\ & \left(\textit{since any } \Theta\left(\mathbf{w}^{\mathbf{N}}_{\tau(\mathbf{X}(\mathbf{w}^{\mathbf{N}}))\wedge \mathbf{s_j}} \right), \ \mathbf{X}_{\tau(\mathbf{X}(\mathbf{w}^{\mathbf{N}}))\wedge s_j}\left(\mathbf{w}^{\mathbf{N}}\right) \textit{ are } \mathcal{Y}_\infty - \textit{measurable} \right) \end{aligned}$$

$$= Q_{\mu^{\mathbf{N}}}\left[\left(\prod_{j=1}^{k} f_j\left(\Theta\left(\mathbf{w}^{\mathbf{N}}_{\tau(\mathbf{X}(\mathbf{w}^{\mathbf{N}}))\wedge \mathbf{s_j}}\right)\right)\right) Q_{\left(\mathbf{X}_{\tau(\mathbf{X})}\right)^{\mathbf{N}}}\left[\prod_{i=1}^{l} g_i\left(\Theta\left(\mathbf{w}^{\mathbf{N}}_{\mathbf{u_i}}\right)\right)\right]\right]$$

$\Big($*conditioned on* $\mathcal{Y}_\infty$, $\left\{w^k_{\tau(\mathbf{X}(\mathbf{w}^{\mathbf{N}}))}\right\}_{k=1}^{\infty}$ *is an i.i.d random variable sequence distributed as* $\mathbf{X}_{\tau(\mathbf{X})}\Big)$

$$= Q_{\mu^{\mathbf{N}}}\left[\left(\prod_{j=1}^{k} f_j\left(\mathbf{X}_{\tau(\mathbf{X})\wedge s_j}\right)\right) Q_{(\mathbf{X}_{\tau(\mathbf{X})})^{\mathbf{N}}}\left[\prod_{i=1}^{l} g_i\left(\mathbf{X}_{u_i}\right)\right]\right]$$

$$= P_\mu\left[\left(\prod_{j=1}^{k} f_j\left(\omega_{\tau(\omega)\wedge s_j}\right)\right) P_{\omega_{\tau(\omega)}}\left[\prod_{i=1}^{l} g_i(\omega_{u_i})\right]\right]$$

$$\left(\forall \nu\in M_1(E),\ P_\nu[\varphi(\omega)] = Q_{\nu^{\mathbf{N}}}\left[\varphi\left(\mathbf{X}\left(\mathbf{w}^{\mathbf{N}}\right)\right)\right],\ \forall \varphi\in\mathcal{B}_b\left(D_{M_1(E)}\right)\right).$$

By a standard argument, Theorem 2.1(2) holds. □

Proof of Theorem 2.1(3). See [27]. □

5. Proof of Theorem 3.1

Note A_k-diffusion process is denoted by

$$\mathbf{Y^k} = \left(\left(Y^1_t(x_1),\cdots,Y^k_t(x_k)\right)_{t\geq 0}\right)_{(x_1,\cdots,x_k)\in R^k}$$

and $Q_{x_1,\cdots,x_k}$ is the law of $\mathbf{Y^k}$ on C_{R^k} starting at $(x_1,\cdots,x_k)\in R^k$; and $Q_{\{x_i\}_{i=1}^\infty}$ is the unique probability on $C_{R^{\mathbf{N}}}$ under which the coordinate process

$$\mathbf{w^N} = \left(\mathbf{w^N_t}\right)_{t\geq 0} = \left(\left\{w^i_t\right\}_{i=1}^\infty\right)_{t\geq 0}$$

on $C_{R^{\mathbf{N}}}$ starts at $\{x_i\}_{i=1}^\infty$ and

$$\mathbf{w^k} = \left(\mathbf{w^k_t}\right)_{t\geq 0} = \left(w^1_t,\cdots,w^k_t\right)_{t\geq 0}$$

is of the law $Q_{x_1,\cdots,x_k}$ for any $k\in\mathbf{N}$. For any $\mu\in M_1\left(R^1\right)$, let

$$Q_{\mu^k} = \int_{R^k} Q_{x_1,\cdots,x_k}\ \mu^k(dx_1\cdots dx_k),\ k\in\mathbf{N};\ Q_{\mu^{\mathbf{N}}} = \int_{R^{\mathbf{N}}} Q_{\{x_i\}_{i=1}^\infty}\ \mu^{\mathbf{N}}(dx_1dx_2\cdots).$$

Then under $Q_{\mu^{\mathbf{N}}}$, $\mathbf{w^N}$ is infinite exchangeable and of initial distribution $\mu^{\mathbf{N}}$, and each $\mathbf{w^k}$ distributes as Q_{μ^k}.

Recall from Section 4, $\mathbf{X}_t\left(\mathbf{w^N}\right) = (\pi_t)_* Z\left(\mathbf{w^N}\right),\ t\in[0,\infty)$; and

$$Z\left(\mathbf{w^N}\right) = \Xi\left(\mathbf{w^N}\right),\ \mathbf{X} = \left(\mathbf{X}_t\left(\mathbf{w^N}\right)\right)_{t\geq 0} = \left((\pi_t)_*\Xi\left(\mathbf{w^N}\right)\right)_{t\geq 0},\ Q_{\mu^{\mathbf{N}}}\ -\ a.s.\ \mathbf{w^N};$$

and under $Q_{\mu^{\mathbf{N}}}$, diffusion process $\mathbf{X}$ is just the measure-valued flow given $\left\{\mathbf{Y^k}\right\}_{k\geq 1}$ starting at μ.

Notation. For any stochastic process $\xi = (\xi_t)_{t\geq 0}$ defined on $\left(C_{R^{\mathbf{N}}}, Q_{\mu^{\mathbf{N}}}\right)$, $(\mathcal{F}_t(\xi))_{t\geq 0}$ is the natural filtration associated to ξ :

$$\mathcal{F}_t(\xi) = \bigcap_{s>t} \sigma(\xi_u, u \leq s), \ \forall t \in [0,\infty).$$

Lemma 5.1. For A_1-diffusion $\mathbf{Y^1} = (Y_t^1)_{t\geq 0}$, its local time $L_t^z(\mathbf{Y^1})$ satisfies that almost surely, $z \in R^1 \to \mathbf{L}_t^z(\mathbf{Y^1})$ is Hölder continuous of order α for any $\alpha \in \left(0, \frac{1}{2}\right)$ and uniformly in t on every compact interval, and $(z,t) \in R^1 \times [0,\infty) \to \mathbf{L}_t^z(\mathbf{Y^1})$ is continuous.

Proof. Since $b^2(\cdot) \leq c\ a(\cdot)$, for any $\left(A_1, C_b^2\left(R^1\right)\right)$-martingale solution $Q_\mu \in M_1\left(C_{R^1}\right)$ with any initial distribution $\mu \in M_1\left(R^1\right)$, if let

$$m_t(\gamma) = \exp\left\{-\int_0^t a(\gamma_s)^{-1} b(\gamma_s)\ d\widetilde{\gamma}_s - \frac{1}{2}\int_0^t b(\gamma_s)^2 a(\gamma_s)^{-1}\ ds\right\},\ t \in [0,\infty),\ \textit{under } Q_\mu,$$

where $\gamma = (\gamma_t)_{t\geq 0}$ is the coordinate process on C_{R^1}, $\widetilde{\gamma}_t = \gamma_t - \gamma_0 - \int_0^t b(\gamma_s)\ ds$, and $a(x)^{-1}b(x) = 0$ provided $a(x) = 0$; then $(m_t(\gamma))_{t\geq 0}$ is a martingale by the Novikov criterion; and there is a unique $\widetilde{Q}_\mu \in M_1\left(C_{R^1}\right)$ such that for any $t \in [0,\infty)$, when restricted to $F_t(\gamma) = \bigcap_{s>t} \sigma(\gamma_u, u \leq s)$,

$$\left.\widetilde{Q}_\mu\right|_{F_t(\gamma)} = m_t(\gamma)\ Q_\mu|_{F_t(\gamma)},$$

and $\widetilde{Q}_\mu$ is a $\left(\frac{1}{2}a(x)\frac{\partial^2}{\partial x^2}, C_b^2\left(R^1\right)\right)$-martingale solution of initial distribution μ. Here $Q_\mu|_{F_t(\gamma)}$ and $\left.\widetilde{Q}_\mu\right|_{F_t(\gamma)}$ are the restrictions of Q_μ and $\widetilde{Q}_\mu$ to $F_t(\gamma)$ respectively.

Conversely, for any $\left(\frac{1}{2}a(x)\frac{\partial^2}{\partial x^2}, C_b^2\left(R^1\right)\right)$-martingale solution $\widetilde{Q}_\mu \in M_1\left(C_{R^1}\right)$ with any initial distribution $\mu \in M_1\left(R^1\right)$, let

$$\widetilde{m}_t(\gamma) = \exp\left\{\int_0^t a(\gamma_s)^{-1} b(\gamma_s)\ d\overline{\gamma}_s - \frac{1}{2}\int_0^t b(\gamma_s)^2 a(\gamma_s)^{-1}\ ds\right\},\ t \in [0,\infty),\ \textit{under } \widetilde{Q}_\mu,$$

where $\overline{\gamma}_t = \gamma_t - \gamma_0$, then $(\widetilde{m}_t(\gamma))_{t\geq 0}$ is a martingale; and there is a unique $Q_\mu \in M_1\left(C_{R^1}\right)$ such that for any $t \in [0,\infty)$, when restricted to $F_t(\gamma)$,

$$Q_\mu|_{F_t(\gamma)} = \widetilde{m}_t(\gamma)\ \left.\widetilde{Q}_\mu\right|_{F_t(\gamma)},$$

and Q_μ is an $\left(A_1, C_b^2\left(R^1\right)\right)$-martingale solution of initial distribution μ.

Recall for a continuous semimartingale $X = (X_t)_{t\geq 0}$, by [19] Chapter 19 Corollary 19.6, outside a fixed null-probability set,

$$\textit{local time } L_t^x(X) = \lim_{\epsilon\downarrow 0} \frac{1}{\epsilon}\int_0^t I_{[x,x+\epsilon)}\left(X_s\right)\ d\langle X\rangle_s,\ \forall t \geq 0,\ \forall x \in R^1.$$

However, for any $\frac{1}{2}a(x)\frac{\partial^2}{\partial x^2}$-diffusion process $\widetilde{\mathbf{Y}}^{\mathbf{1}}$, almost surely, $z \to L_t^z\left(\widetilde{\mathbf{Y}}^{\mathbf{1}}\right)$ is Hölder continuous of order α for every $\alpha \in \left(0, \frac{1}{2}\right)$ and uniformly in t on every compact interval ([33] Chapter VI Corollary 1.8) and $(z,t) \to L_t^z\left(\widetilde{\mathbf{Y}}^{\mathbf{1}}\right)$ is continuous in $(z,t) \in R^1 \times [0,\infty)$ ([33] Chapter VI Theorem 1.7).

Note $d\left\langle \widetilde{Y}^1_\cdot \right\rangle_s = a\left(\widetilde{Y}^1_s\right)\ ds$ and $d\langle Y^1_\cdot\rangle_s = a(Y^1_s)\ ds$, and the Girsanov argument for the two processes. We see the lemma holds. □

Lemma 5.2. Given any $\mu \in M_1\left(R^1\right)$. For any $t \in [0,\infty)$,

$$\sup_{z\in R^1}\left\{\int_{C_{R^1}} \left(L^z_t(w^1)\right)^r\ Q_\mu(dw^1)\right\} < \infty,\ \forall r \in (0,\infty); \tag{5.1}$$

and for any $T \in (0,\infty)$ and any $r \in [2,\infty)$,

$$Q_\mu\left[\sup_{t\le T}\left|L^{z_1}_t(w^1) - L^{z_2}_t(w^1)\right|^r\right] \le c(T,r)\left\{|z_1-z_2|^r + |z_1-z_2|^{\frac{r}{2}}\right\},\ \forall (z_1,z_2)\in R^2, \tag{5.2}$$

with $c(T,r)$ being a constant depending on T and r.

Proof. Step 1. Prove (5.1). Note

$$w^1_t = w^1_0 + \int_0^t b(w^1_s)\ ds + M_t(w^1),\ t\ge 0,\ Q_\mu\ -\ a.s.\ w^1 \in C_{R^1};$$

where $(M_t(w^1))_{t\ge 0}$ is a continuous martingale with

$$\langle M_\cdot(w^1)\rangle_t = \int_0^t a(w^1_s)\ ds,\ t\ge 0.$$

By the Tanaka formula for continuous semimartingales,

$$\begin{aligned} L^z_t(w^1) \quad &= \quad 2\left[(w^1_t - z)^+ - (w^1_0 - z)^+ - \int_0^t I_{(z,\infty)}(w^1_s)\ dM_s(w^1) - \right. \\ &\left. \int_0^t I_{(z,\infty)}(w^1_s) b(w^1_s)\ ds\right],\ \forall t\in[0,\infty),\ Q_\mu\ -\ a.s.\ w^1. \end{aligned} \tag{5.3}$$

Moreover, recall from [33] Chapter VI, the following occupation time formula holds:

$$\int_0^t \phi\left(w^1_s\right) a\left(w^1_s\right)\ ds = \int_{-\infty}^{\infty}\phi(z) L^z_t\left(w^1\right)\ dz,$$
$$\forall t\in[0,\infty),\ \forall 0<\phi\in\mathcal{B}_b\left(R^1\right),\ Q_\mu\ -\ a.s.;$$

by the monotone convergence theorem,

$$\int_0^t \phi\left(w^1_s\right) a\left(w^1_s\right)\ ds = \int_{-\infty}^{\infty}\phi(z) L^z_t\left(w^1\right)\ dz,$$
$$\forall t\in[0,\infty),\ \forall 0\le\phi\in\mathcal{B}_b\left(R^1\right),\ Q_\mu\ -\ a.s..$$

By the Burkholder-Davis-Gundy inequality, for some constant c_r depending on r,

$$\begin{aligned} Q_\mu\left[\left|M_t(w^1)\right|^r\right] &\le Q_\mu\left[\sup_{s\le t}\left|M_s(w^1)\right|^r\right] \le c_r Q_\mu\left[\langle M_\cdot(w^1)\rangle_t^{\frac{r}{2}}\right] \\ &= c_r Q_\mu\left[\left(\int_0^t a(w^1_s)\ ds\right)^{\frac{r}{2}}\right] \le c_r\|a(\cdot)\|^{\frac{r}{2}} t^{\frac{r}{2}}, \end{aligned}$$

$$Q_\mu\left[\left|\int_0^t I_{(z,\infty)}(w_s^1)\ dM_s(w^1)\right|^r\right] \le Q_\mu\left[\sup_{s\le t}\left|\int_0^s I_{(z,\infty)}(w_u^1)\ dM_u(w^1)\right|^r\right]$$
$$\le c_r Q_\mu\left[\left(\int_0^t I_{(z,\infty)}(w_s^1)\ a(w_s^1)\ ds\right)^{\frac{r}{2}}\right] \le c_r\|a(\cdot)\|^{\frac{r}{2}} t^{\frac{r}{2}}.$$

Combining with (5.3) and

$$\left|(w_t^1-z)^+ - (w_0^1-z)^+\right| \le \left|w_t^1 - w_0^1\right| \le \|b(\cdot)\|t + \left|M_t(w^1)\right|,$$
$$\left|\int_0^t I_{(z,\infty)}(w_s^1) b(w_s^1)\ ds\right| \le \|b(\cdot)\|t;$$

we see (5.1) is true.

Step 2. The case $b(\cdot)\equiv 0$. In the case, use $\widetilde{Q}_\mu$ to denote Q_μ, we have

$$\widetilde{Q}_\mu\left[\sup_{t\le T}\left|L_t^{z_1}(w^1) - L_t^{z_2}(w^1)\right|^r\right] \le c_1(T,r)\left\{|z_1-z_2|^r + |z_1-z_2|^{\frac{r}{2}}\right\},\ \forall (z_1,z_2)\in R^2,$$

for some constant $c_1(T,r)$ depending on T and $r\in[2,\infty)$.

Note for any $t\in[0,\infty)$,

$$\left|(w_t^1-z_1)^+ - (w_0^1-z_1)^+ - (w_t^1-z_2)^+ + (w_0^1-z_2)^+\right|$$
$$\le \left|(w_t^1-z_1)^+ - (w_t^1-z_2)^+\right| + \left|(w_0^1-z_1)^+ - (w_0^1-z_2)^+\right|$$
$$\le 2|z_1-z_2|,\ \forall (z_1,z_2)\in R^2.$$

By (5.3), it suffices to check that for some constant $c_2(T,r)$ depending on T and r,

$$\widetilde{Q}_\mu\left[\sup_{t\le T}\left|\int_0^t \left(I_{(z_1,\infty)}(w_s^1) - I_{(z_2,\infty)}(w_s^1)\right)\ dM_s(w^1)\right|^r\right]$$
$$\le c_2(T,r)|z_1-z_2|^{\frac{r}{2}},\ \forall (z_1,z_2)\in R^2.$$

In fact, by the Burkholder-Davis-Gundy inequality and the occupation time formula,

$$\widetilde{Q}_\mu\left[\sup_{t\le T}\left|\int_0^t \left(I_{(z_1,\infty)}(w_s^1) - I_{(z_2,\infty)}(w_s^1)\right)\ dM_s(w^1)\right|^r\right]$$
$$\le c_r\widetilde{Q}_\mu\left[\left(\int_0^T I_{(z_1\wedge z_2, z_1\vee z_2]}(w_s^1) a(w_s^1)\ ds\right)^{\frac{r}{2}}\right] = c_r\widetilde{Q}_\mu\left[\left(\int_{z_1\wedge z_2}^{z_1\vee z_2} L_T^z(w^1)\ dz\right)^{\frac{r}{2}}\right]$$
$$= c_r|z_1-z_2|^{\frac{r}{2}}\widetilde{Q}_\mu\left[\left(\frac{1}{|z_1-z_2|}\int_{z_1\wedge z_2}^{z_1\vee z_2} L_T^z(w^1)\ dz\right)^{\frac{r}{2}}\right]$$
$$\le c_r|z_1-z_2|^{\frac{r}{2}}\widetilde{Q}_\mu\left[\frac{1}{|z_1-z_2|}\int_{z_1\wedge z_2}^{z_1\vee z_2}\left(L_T^z(w^1)\right)^{\frac{r}{2}}\ dz\right]$$
$$\le c_r|z_1-z_2|^{\frac{r}{2}}\sup_{z\in R^1}\left\{\widetilde{Q}_\mu\left[\left(L_T^z(w^1)\right)^{\frac{r}{2}}\right]\right\},\ \forall z_1\ne z_2,\ \textit{we are done.}$$

Step 3. Prove (5.2). Note the proof of Lemma 5.1,

$$
\begin{aligned}
&\widetilde{Q}_\mu\left[\left(\widetilde{m}_T(w^1)\right)^{\frac{1+\epsilon}{\epsilon}}\right]\\
&=\widetilde{Q}_\mu\left[\exp\left\{\frac{1+\epsilon}{\epsilon}\int_0^T a(w_s^1)^{-1}b(w_s^1)\ d\overline{w}_s^1-\frac{1}{2}\left(\frac{1+\epsilon}{\epsilon}\right)^2\int_0^T b(w_s^1)^2a(w_s^1)^{-1}\ ds+\right.\right.\\
&\qquad\qquad\left.\left.\frac{1+\epsilon}{2\epsilon^2}\int_0^T b(w_s^1)^2a(w_s^1)^{-1}\ ds\right\}\right]\\
&\leq\exp\left\{\frac{(1+\epsilon)cT}{2\epsilon^2}\right\}\\
&\quad\widetilde{Q}_\mu\left[\exp\left\{\frac{1+\epsilon}{\epsilon}\int_0^T a(w_s^1)^{-1}b(w_s^1)\ d\overline{w}_s^1-\frac{1}{2}\left(\frac{1+\epsilon}{\epsilon}\right)^2\int_0^T b(w_s^1)^2a(w_s^1)^{-1}\ ds\right\}\right]\\
&=\exp\left\{\frac{(1+\epsilon)cT}{2\epsilon^2}\right\},\ \ for\ any\ \epsilon\in(0,\infty).
\end{aligned}
$$

Then by Step 2,

$$
\begin{aligned}
&Q_\mu\left[\sup_{t\leq T}\left|L_t^{z_1}(w^1)-L_t^{z_2}(w^1)\right|^r\right]=\widetilde{Q}_\mu\left[\widetilde{m}_T(w^1)\ \sup_{t\leq T}\left|L_t^{z_1}(w^1)-L_t^{z_2}(w^1)\right|^r\right]\\
&\leq\left\{\widetilde{Q}_\mu\left[\left(\widetilde{m}_T(w^1)\right)^{\frac{1+\epsilon}{\epsilon}}\right]\right\}^{\frac{\epsilon}{1+\epsilon}}\left\{\widetilde{Q}_\mu\left[\sup_{t\leq T}\left|L_t^{z_1}(w^1)-L_t^{z_2}(w^1)\right|^{r(1+\epsilon)}\right]\right\}^{\frac{1}{1+\epsilon}}\\
&\leq\exp\left\{\frac{cT}{2\epsilon}\right\}\left\{c_1(T,(1+\epsilon)r)\right\}^{\frac{1}{1+\epsilon}}\left\{|z_1-z_2|^{r(1+\epsilon)}+|z_1-z_2|^{\frac{r(1+\epsilon)}{2}}\right\}^{\frac{1}{1+\epsilon}}\\
&\leq\exp\left\{\frac{cT}{2\epsilon}\right\}\left\{c_1(T,(1+\epsilon)r)\right\}^{\frac{1}{1+\epsilon}}\left\{|z_1-z_2|^{r}+|z_1-z_2|^{\frac{r}{2}}\right\},
\end{aligned}
$$

take $\epsilon=1$ to complete the proof. □

Lemma 5.3. For any $\mu\in M_1\left(R^1\right)$ with $\int_{[0,\infty)}x\ \mu(dx)<\infty$,

$$
Q_{\mu^{\mathbf{N}}}\left[\sup_{s\leq t}\int_{R^1}(y-z)^+\ \mathbf{X}_s(dy)\right]\leq Q_{\mu^{\mathbf{N}}}\left[\int_{C_{R^1}}\sup_{s\leq t}(\gamma_s-z)^+\ Z\left(\mathbf{w^N}\right)(d\gamma)\right]<\infty,
$$

for any $t\in[0,\infty)$ *and any* $z\in R^1$;

and for any $z\in R^1$,

$$
\int_{R^1}(y-z)^+\ \mathbf{X}_t(dy)\ is\ continuous\ in\ t,\ Q_{\mu^{\mathbf{N}}}\ -\ a.s.;
$$

and

$$
\begin{aligned}
&\sup_{s\leq t}\left|\int_{R^1}(y-z_1)^+\ \mathbf{X}_s(dy)-\int_{R^1}(y-z_2)^+\ \mathbf{X}_s(dy)\right|\\
&\leq|z_1-z_2|,\ \forall t\in[0,\infty),\ \forall(z_1,z_2)\in R^2,\ Q_{\mu^{\mathbf{N}}}\ -\ a.s..
\end{aligned}
$$

So $\int_{R^1}(y-z)^+\ \mathbf{X}_t(dy)$ is continuous in $(z,t)\in R^1\times[0,\infty),\ Q_{\mu^{\mathbf{N}}}\ -\ a.s..$

Proof. Note

$$
\begin{aligned}
& Q_{\mu^{\mathbf{N}}}\left[\sup_{s\le t}\left(w_s^1 - z\right)^+\right] \\
& \le Q_{\mu^{\mathbf{N}}}\left[\sup_{s\le t}\left(w_s^1 - w_0^1 - \int_0^s b\left(w_r^1\right)\ dr\right) + \|b\|t + w_0^1 I_{\{w_0^1\ge z\}} + z\right] \\
& \le c_1 Q_{\mu^{\mathbf{N}}}\left[\sqrt{\int_0^t a\left(w_s^1\right)\ ds}\right] + \|b\|t + \int_{[z,\infty)} x\ \mu(dx) + z < \infty \\
& \quad (where\ we\ have\ used\ Burkholder - Davis - Gundy\ inequality),
\end{aligned}
$$

for some constant $c_1 > 0$. By the second part of Lemma 4.6,

$$
Q_{\mu^{\mathbf{N}}}\left[\int_{C_{R^1}} \sup_{s\le t}(\gamma_s - z)^+\ Z\left(\mathbf{w}^{\mathbf{N}}\right)(d\gamma)\right] = Q_{\mu^{\mathbf{N}}}\left[\sup_{s\le t}\left(w_s^1 - z\right)^+\right] < \infty.
$$

Thus,

$$
\begin{aligned}
& Q_{\mu^{\mathbf{N}}}\left[\sup_{s\le t}\int_{R^1}(y-z)^+\ \mathbf{X}_s(dy)\right] = Q_{\mu^{\mathbf{N}}}\left[\sup_{s\le t}\int_{C_{R^1}}(\gamma_s - z)^+\ Z\left(\mathbf{w}^{\mathbf{N}}\right)(d\gamma)\right] \\
& \le Q_{\mu^{\mathbf{N}}}\left[\int_{C_{R^1}} \sup_{s\le t}(\gamma_s - z)^+\ Z\left(\mathbf{w}^{\mathbf{N}}\right)(d\gamma)\right] < \infty.
\end{aligned}
$$

Note

$$
\int_{R^1}(y-z)^+\ \mathbf{X}_t(dy) = \int_{C_{R^1}}(\gamma_t - z)^+\ Z\left(\mathbf{w}^{\mathbf{N}}\right)(d\gamma),\ \forall t\in[0,\infty),
$$

we see that for any $z \in R^1$,

$$
\int_{R^1}(y-z)^+\ \mathbf{X}_t(dy)\ is\ continuous\ in\ t,\ Q_{\mu^{\mathbf{N}}}\ -\ a.s..
$$

While,

$$
\begin{aligned}
& \sup_{s\le t}\left|\int_{R^1}(y-z_1)^+\ \mathbf{X}_s(dy) - \int_{R^1}(y-z_2)^+\ \mathbf{X}_s(dy)\right| \\
& \le \sup_{s\le t}\int_{R^1}\left|(y-z_1)^+ - (y-z_2)^+\right|\ \mathbf{X}_s(dy) \\
& \le |z_1 - z_2|,\ \forall t\in[0,\infty),\ \forall(z_1,z_2)\in R^2,\ Q_{\mu^{\mathbf{N}}}\ -\ a.s..
\end{aligned}
$$

Now the lemma holds. □

Lemma 5.4. Under $Q_{\mu^{\mathbf{N}}}$, almost surely,

$$
\begin{aligned}
& \int_0^t \mathbf{X}_s\ ds \ll dz\ (the\ Lebesgue\ measure),\ \forall t\in[0,\infty), \\
& \mathbf{L}_t^z(\mathbf{X}) := \frac{d\left(\int_0^t \mathbf{X}_s\ ds\right)}{dz}\ is\ well-defined\ for\ any\ z\in R^1\ and\ any\ t\in[0,\infty);
\end{aligned}
$$

and $\forall z \in \{a(\cdot) \neq 0\}$, $\mathbf{L}_t^z(\mathbf{X}) \in L^r\left(Q_{\mu^{\mathbf{N}}}\right)$, $\forall t \in [0,\infty)$, $\forall r \in [1,\infty)$. Moreover, under $Q_{\mu^{\mathbf{N}}}$, $\left((\mathbf{L}_t^z(\mathbf{X}))_{t\geq 0}\right)_{z\in\{a(\cdot)\neq 0\}}$ has a modified version $\left(\left(\widetilde{\mathbf{L}}_t^z(\mathbf{X})\right)_{t\geq 0}\right)_{z\in\{a(\cdot)\neq 0\}}$ such that

(**i**) *the map* $(z,t) \in \{a(\cdot)\neq 0\} \times [0,\infty) \to \widetilde{\mathbf{L}}_t^z(\mathbf{X})$ *is a.s. continuous*;

(**ii**) *almost surely, the map* $z \in \{a(\cdot) \neq 0\} \to a(z)\mathbf{L}_t^z(\mathbf{X})$ *is Hölder continuous of order* α *for any* $\alpha \in \left(0, \frac{1}{2}\right)$ *and uniformly in* t *on every compact interval*;

(**iii**) *for any fixed* $z \in \{a(\cdot)\neq 0\}$, $\widetilde{\mathbf{L}}_t^z(\mathbf{X})$ *is a.s. increasing in* $t \in [0,\infty)$, *and* $\widetilde{\mathbf{L}}_t^z(\mathbf{X}) = \int_{C_{R^1}} L_t^z(\gamma) a(z)^{-1}\, Z\left(\mathbf{w}^{\mathbf{N}}\right)(d\gamma)$, $\forall t \in [0,\infty)$, *a.s.*;

(**iv**) *given any* $z \in \{a(\cdot)\neq 0\}$, $\widetilde{\mathbf{L}}_t^z(\mathbf{X})$ *is* $\mathcal{F}_t(\mathbf{X})-$*measurable for any* $t \in [0,\infty)$.

Proof. Step 1. Representation of $\int_0^t \mathbf{X}_s((z-\epsilon, z+\epsilon))\, ds$ *for* $\epsilon > 0$ *and* $z \in R^1$.

Write $B(z,\epsilon) = (z-\epsilon, z+\epsilon)$. Since under $Q_{\mu^{\mathbf{N}}}$, each $w^i = \left(w_s^i\right)_{s\geq 0}$ is the A_1-diffusion process with the initial distribution μ, w^i is a continuous semimartingale with

$$d\left\langle w_\cdot^i\right\rangle_s = a\left(w_s^i\right)\, ds \textit{ and variation part } \int_0^\cdot b\left(w_s^i\right)\, ds;$$

then by the occupation time formula,

$$\int_0^t \Phi\left(w_s^i\right) a\left(w_s^i\right)\, ds = \int_{-\infty}^{\infty} \Phi(z) L_t^z\left(w^i\right)\, dz,$$
$$\forall t \in [0,\infty),\ \forall 0 < \Phi \in \mathcal{B}_b\left(R^1\right),\ Q_{\mu^{\mathbf{N}}}\ -\ a.s.,$$

with $L_t^z(\cdot)$ being the local time functional in z up to time t for the A_1-diffusion process. By the monotone convergence theorem,

$$\int_0^t \Phi\left(w_s^i\right) a\left(w_s^i\right)\, ds = \int_{-\infty}^{\infty} \Phi(z) L_t^z\left(w^i\right)\, dz,$$
$$\forall t \in [0,\infty),\ \forall 0 \leq \Phi \in \mathcal{B}_b\left(R^1\right),\ Q_{\mu^{\mathbf{N}}}\ -\ a.s.,$$

which implies

$$\int_0^t \Phi\left(w_s^i\right) a\left(w_s^i\right)\, ds = \int_{-\infty}^{\infty} \Phi(z) L_t^z\left(w^i\right)\, dz,\ \forall t \in [0,\infty),$$
$$\textit{for any nonnegative measurable function } \Phi \textit{ on } R^1,\ Q_{\mu^{\mathbf{N}}}\ -\ a.s.;$$

$$\int_0^t I_{B(z,\epsilon)}\left(w_s^i\right)\, ds = \int_{-\infty}^{\infty} I_{B(z,\epsilon)}(y) a(y)^{-1} L_t^y\left(w^i\right)\, dy,$$
$$\forall t \in [0,\infty),\ \forall z \in R^1,\ \forall \epsilon > 0,\ Q_{\mu^{\mathbf{N}}}\ -\ a.s..$$

Therefore,

$$\frac{1}{k}\sum_{i=1}^k \int_0^t I_{B(z,\epsilon)}\left(w_s^i\right)\, ds = \frac{1}{k}\sum_{i=1}^k \int_{B(z,\epsilon)} L_t^y\left(w^i\right) a(y)^{-1}\, dy,$$
$$\forall t \in [0,\infty),\ \forall z \in R^1,\ \forall \epsilon > 0,\ Q_{\mu^{\mathbf{N}}}\ -\ a.s.;$$

and by the second part of Lemma 4.6, we have that $Q_{\mu^{\mathbf{N}}} - a.s.$, as $k \to \infty$,

$$\begin{aligned}
&\frac{1}{k}\sum_{i=1}^{k}\int_{B(z,\epsilon)} L_t^y\left(w^i\right)a(y)^{-1}\ dy \to \int_{C_{R^1}}\int_{B(z,\epsilon)} L_t^y(\gamma)a(y)^{-1}\ dyZ\left(\mathbf{w}^{\mathbf{N}}\right)(d\gamma) \\
&\qquad = \int_{B(z,\epsilon)}\int_{C_{R^1}} L_t^y(\gamma)a(y)^{-1}\ Z\left(\mathbf{w}^{\mathbf{N}}\right)(d\gamma)dy, \\
&Q_{\mu^{\mathbf{N}}}\left[\int_{B(z,\epsilon)}\int_{C_{R^1}} L_t^y(\gamma)a(y)^{-1}\ Z\left(\mathbf{w}^{\mathbf{N}}\right)(d\gamma)dy\right] \\
&\qquad = \lim_{k\to\infty} Q_{\mu^{\mathbf{N}}}\left[\frac{1}{k}\sum_{i=1}^{k}\int_0^t I_{B(z,\epsilon)}\left(w_s^i\right)\ ds\right] \\
&\qquad = \lim_{k\to\infty} Q_{\mu^{\mathbf{N}}}\left[\frac{1}{k}\sum_{i=1}^{k}\int_{-\infty}^{\infty} I_{B(z,\epsilon)}(y)a(y)^{-1}L_t^y\left(w^i\right)\ dy\right] \\
&\qquad = \int_0^t \left\langle \mu, V_s^1 I_{B(z,\epsilon)}\right\rangle\ ds = \int_{B(z,\epsilon)} a(y)^{-1}Q_\mu\left[L_t^y\left(w^1\right)\right]\ dy; \\
&\frac{1}{k}\sum_{i=1}^{k}\int_0^t I_{B(z,\epsilon)}\left(w_s^i\right)\ ds \to \int_{C_{R^1}}\int_0^t I_{B(z,\epsilon)}(\gamma_s)\ dsZ\left(\mathbf{w}^{\mathbf{N}}\right)(d\gamma) \\
&\qquad = \int_0^t\int_{C_{R^1}} I_{B(z,\epsilon)}(\gamma_s)\ Z\left(\mathbf{w}^{\mathbf{N}}\right)(d\gamma)\ ds = \int_0^t \mathbf{X}_s(B(z,\epsilon))\ ds.
\end{aligned}$$

Hence,

$$\int_0^t \mathbf{X}_s(B(z,\epsilon))\ ds = \int_{B(z,\epsilon)}\int_{C_{R^1}} L_t^y(\gamma)a(y)^{-1}Z\left(\mathbf{w}^{\mathbf{N}}\right)(d\gamma)dy,\ Q_{\mu^{\mathbf{N}}}\ -\ a.s..$$

Step 2. For any $t \in [0,\infty)$ and any $r \in [1,\infty)$,

$$\begin{aligned}
&\sup_{z\in R^1} Q_{\mu^{\mathbf{N}}}\left[\left(\int_{C_{R^1}} L_t^z(\gamma)\ Z\left(\mathbf{w}^{\mathbf{N}}\right)(d\gamma)\right)^r\right] \\
&\qquad \le \sup_{z\in R^1} Q_{\mu^{\mathbf{N}}}\left[\int_{C_{R^1}} \left(L_t^z(\gamma)\right)^r Z\left(\mathbf{w}^{\mathbf{N}}\right)(d\gamma)\right] < \infty.
\end{aligned}$$

With Lemma 5.2 in mind, from the second part of Lemma 4.6, we obtain

$$\begin{aligned}
&\frac{1}{k}\sum_{i=1}^{k}\left(L_t^z(w^i)\right)^r \to \int_{C_{R^1}} \left(L_t^z(\gamma)\right)^r Z\left(\mathbf{w}^{\mathbf{N}}\right)(d\gamma),\ Q_{\mu^N}\ -\ a.s., \\
&Q_{\mu^{\mathbf{N}}}\left[\int_{C_{R^1}} \left(L_t^z(\gamma)\right)^r Z\left(\mathbf{w}^{\mathbf{N}}\right)(d\gamma)\right] = \lim_{k\to\infty} Q_{\mu^{\mathbf{N}}}\left[\frac{1}{k}\sum_{i=1}^{k}\left(L_t^z(w^i)\right)^r\right] = Q_\mu\left[\left(L_t^z(w^1)\right)^r\right]; \\
&\sup_{z\in R^1} Q_{\mu^{\mathbf{N}}}\left[\int_{C_{R^1}} \left(L_t^z(\gamma)\right)^r Z\left(\mathbf{w}^{\mathbf{N}}\right)(d\gamma)\right] = \sup_{z\in R^1} Q_\mu\left[\left(L_t^z(w^1)\right)^r\right] < \infty.
\end{aligned}$$

Step 3. Note $L^z_t(\cdot)$ is increasing in $t \in [0,\infty)$. By Lemma 5.1 and Step 2, for fixed $z \in R^1$,

$$\int_{C_{R^1}} L^z_t(\gamma)\ Z(\mathbf{w}^{\mathbf{N}})(d\gamma) \textit{ is continuous in } t \in [0,\infty),\ Q_{\mu^{\mathbf{N}}}\ -\ a.s..$$

Step 4. Under $Q_{\mu^{\mathbf{N}}}$, almost surely, the map $z \in R^1 \to \int_{C_{R^1}} L^z_t(\gamma)\ Z(\mathbf{w}^{\mathbf{N}})(d\gamma)$ is Hölder continuous of order α for any $\alpha \in \left(0, \frac{1}{2}\right)$ and uniformly in t on every compact interval, which together with Step 3 imply almost surely the map $(z,t) \in R^1 \times [0,\infty) \to \int_{C_{R^1}} L^z_t(\gamma)\ Z(\mathbf{w}^{\mathbf{N}})(d\gamma)$ is continuous.

Indeed, for any $T \in (0,\infty)$ and $r \in [2,\infty)$, by (5.2) and the second part of Lemma 4.6,

$$\begin{aligned}
&Q_{\mu^{\mathbf{N}}}\left[\int_{C_{R^1}} \sup_{t\le T} |L^{z_1}_t(\gamma) - L^{z_2}_t(\gamma)|^r\, Z\left(\mathbf{w}^{\mathbf{N}}\right)(d\gamma)\right]\\
&= \lim_{k\to\infty} Q_{\mu^{\mathbf{N}}}\left[\frac{1}{k}\sum_{i=1}^k \sup_{t\le T}\left|L^{z_1}_t(w^i) - L^{z_2}_t(w^i)\right|^r\right] = Q_\mu\left[\sup_{t\le T}\left|L^{z_1}_t(w^1) - L^{z_2}_t(w^1)\right|^r\right]\\
&\le c(T,r)\left(|z_1-z_2|^r + |z_1-z_2|^{\frac{r}{2}}\right),\ \forall (z_1,z_2) \in R^2.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&Q_{\mu^{\mathbf{N}}}\left[\sup_{t\le T}\left|\int_{C_{R^1}} L^{z_1}_t(\gamma) Z\left(\mathbf{w}^{\mathbf{N}}\right)(d\gamma) - \int_{C_{R^1}} L^{z_2}_t(\gamma) Z\left(\mathbf{w}^{\mathbf{N}}\right)(d\gamma)\right|^r\right]\\
&\le Q_{\mu^{\mathbf{N}}}\left[\int_{C_{R^1}} \sup_{t\le T} |L^{z_1}_t(\gamma) - L^{z_2}_t(\gamma)|^r\, Z\left(\mathbf{w}^{\mathbf{N}}\right)(d\gamma)\right]\\
&\le c(T,r)\left(|z_1-z_2|^r + |z_1-z_2|^{\frac{r}{2}}\right),\ \forall (z_1,z_2) \in R^2.
\end{aligned}$$

Note Step 3. By the Kolmogorov's continuity criterion on Banach spaces, the result of Step 4 holds.

Step 5. Given any sequence $\{\epsilon_n\}_{n\ge1} \subset (0,\infty)$ converging decreasingly to 0. For any $z \in \{a(\cdot) \neq 0\}$, note $\int_0^t \mathbf{X}_s(B(z,\epsilon_n))\ ds$ is continuous in t and $L^y_t(\gamma)$ is continuous and increasing in t, with Steps 1, 3 and 4,

$$\begin{aligned}
&\int_{B(z,\epsilon_n)}\int_{C_{R^1}} L^y_t(\gamma) a(y)^{-1} Z\left(\mathbf{w}^{\mathbf{N}}\right)(d\gamma) dy < \infty,\ \forall t \in [0,\infty),\ Q_{\mu^{\mathbf{N}}}\ -\ a.s.,\\
&\int_{B(z,\epsilon_n)}\int_{C_{R^1}} L^y_t(\gamma) a(y)^{-1} Z\left(\mathbf{w}^{\mathbf{N}}\right)(d\gamma) dy \textit{ is continuous in } t,\ Q_{\mu^{\mathbf{N}}}\ -\ a.s.;\\
&\int_0^t \mathbf{X}_s(B(z,\epsilon_n))\ ds = \int_{B(z,\epsilon_n)}\int_{C_{R^1}} L^y_t(\gamma) a(y)^{-1} Z\left(\mathbf{w}^{\mathbf{N}}\right)(d\gamma) dy,\\
&\qquad \forall t \in [0,\infty),\ \forall n \in \mathbf{N},\ Q_{\mu^{\mathbf{N}}}\ -\ a.s.;\\
&\lim_{n\to\infty}\frac{1}{2\epsilon_n}\int_0^t \mathbf{X}_s(B(z,\epsilon_n))\ ds = \lim_{n\to\infty}\frac{1}{2\epsilon_n}\int_{B(z,\epsilon_n)}\int_{C_{R^1}} L^y_t(\gamma) a(y)^{-1} Z\left(\mathbf{w}^{\mathbf{N}}\right)(d\gamma)\ dy\\
&= \int_{C_{R^1}} L^z_t(\gamma) a(z)^{-1} Z\left(\mathbf{w}^{\mathbf{N}}\right)(d\gamma),\ \forall t \ge 0,\ Q_{\mu^{\mathbf{N}}}\ -\ a.s.. \qquad (5.4)
\end{aligned}$$

From the proof of Step 2,

$$Q_{\mu^{\mathbf{N}}}\left[\int_{C_{R^1}} L_t^z(\gamma)a(z)^{-1}Z\left(\mathbf{w}^{\mathbf{N}}\right)(d\gamma)\right]=a(z)^{-1}Q_\mu\left[L_t^z\left(w^1\right)\right],\ z\in\{a(\cdot)\neq 0\}.$$

Recall $\{a(\cdot)=0\}$ is of zero Lebesgue measure. Then for any $\phi\in C_b\left(R^1\right)$,

$$\begin{aligned}
&Q_{\mu^{\mathbf{N}}}\left[\int_0^t\langle\mathbf{X}_s,\phi\rangle\ ds\right]=\int_0^t\left\langle\mu,V_s^1\phi\right\rangle\ ds=Q_\mu\left[\int_0^t\phi\left(w_s^1\right)\ ds\right]\\
&=Q_\mu\left[\int_{-\infty}^{\infty}\phi(y)L_t^y\left(w^1\right)a(y)^{-1}\ dy\right]=\int_{-\infty}^{\infty}\phi(y)a(y)^{-1}Q_\mu\left[L_t^y\left(w^1\right)\right]\ dy\\
&=\int_{-\infty}^{\infty}\phi(y)Q_{\mu^{\mathbf{N}}}\left[\int_{C_{R^1}}L_t^y(\gamma)a(y)^{-1}Z\left(\mathbf{w}^{\mathbf{N}}\right)(d\gamma)\right]\ dy.
\end{aligned}$$

So by [9] Lemma 3.4.2.2, $Q_{\mu^{\mathbf{N}}}\ -\ a.s.$, $\int_0^t\mathbf{X}_s\ ds$ is absolutely continuous with respect to the Lebesgue measure dz. Combining with $\int_0^t\mathbf{X}_s\ ds$ is increasing in $t\in[0,\infty)$, we have that $Q_{\mu^{\mathbf{N}}}$ almost surely,

$$\begin{aligned}
&\int_0^t\mathbf{X}_s\ ds\ll dz,\ \forall t\in[0,\infty);\\
&\mathbf{L}_t^z(\mathbf{X})=\frac{d\left(\int_0^t\mathbf{X}_s\ ds\right)}{dz}\ \ is\ well-defined\ for\ any\ z\in R^1\ and\ t\in[0,\infty).
\end{aligned}$$

Define $\left(\left(\widetilde{\mathbf{L}}_t^z(\mathbf{X})\right)_{t\geq 0}\right)_{z\in R^1}$ as follows:

$$\widetilde{\mathbf{L}}_t^z(\mathbf{X})=\begin{cases}\lim\limits_{n\to\infty}\frac{1}{2\epsilon_n}\int_0^t\mathbf{X}_s((z-\epsilon_n,z+\epsilon_n))\ ds\ if\ the\ limit\ exists,\\ 0,\ \ otherwise.\end{cases}$$

Note $\int_0^t\mathbf{X}_s\ ds\ll dz,\ \forall t\in[0,\infty),\ Q_{\mu^{\mathbf{N}}}\ -\ a.s.$. We see that $Q_{\mu^{\mathbf{N}}}\ -\ a.s.$,

$$\begin{aligned}
&\widetilde{\mathbf{L}}_t^z(\mathbf{X})=\mathbf{L}_t^z(\mathbf{X}),\ dz\ -\ a.e.\ z\in R^1,\ \forall t\in[0,\infty),\\
&\lim_{n\to\infty}\frac{1}{2\epsilon_n}\int_0^t\mathbf{X}_s((z-\epsilon_n,z+\epsilon_n))\ ds\ exists,\ dz\ -\ a.e.\ z\in R^1,\ \forall t\in[0,\infty).
\end{aligned}$$

By (5.4), for any $z\in\{a(\cdot)\neq 0\}$,

$$\widetilde{\mathbf{L}}_t^z\left(\mathbf{X}\right)=\int_{C_{R^1}}L_t^z(\gamma)a(z)^{-1}Z\left(\mathbf{w}^{\mathbf{N}}\right)(d\gamma),\ \forall t\geq 0,\ Q_{\mu^{\mathbf{N}}}\ -\ a.s.,\tag{5.5}$$

which implies that $\widetilde{\mathbf{L}}_t^z\left(\mathbf{X}\right)$ is $a.s.$ increasing in $t\in[0,\infty)$.

Clearly, for any $z\in R^1$, $\widetilde{\mathbf{L}}_t^z(\mathbf{X})$ is $\mathcal{F}_t(\mathbf{X})$-measurable for any $t\in[0,\infty)$. So far, combining with Step 4 and (5.5), we have obtained that $\left(\left(\widetilde{\mathbf{L}}_t^z(\mathbf{X})\right)_{t\geq 0}\right)_{z\in\{a(\cdot)\neq 0\}}$ is a version of $\left((\mathbf{L}_t^z(\mathbf{X}))_{t\geq 0}\right)_{z\in\{a(\cdot)\neq 0\}}$ under $Q_{\mu^{\mathbf{N}}}$ specified in Lemma 5.4. The lemma holds. □

Proof of Theorem 3.1. Step 1. Recall from the proof of Lemma 5.2,

$$\gamma_t = \gamma_0 + \int_0^t b(\gamma_s)\ ds + M_t(\gamma),\ t \geq 0,\ Q_\mu\ -\ a.s.\ \gamma \in C_{R^1},$$

where $(M_t(\gamma))_{t\geq 0}$ is a continuous martingale of quadratic variation process $\left(\int_0^t a(\gamma_s)\ ds\right)_{t\geq 0}$; and

$$\begin{aligned} L_t^z(\gamma) &= 2\left[(\gamma_t - z)^+ - (\gamma_0 - z)^+ - \int_0^t I_{(z,\infty)}(\gamma_s)\ dM_s(\gamma) - \right. \\ &\left. \int_0^t I_{(z,\infty)}(\gamma_s) b(\gamma_s)\ ds\right],\ \forall t \in [0,\infty),\ Q_\mu\ -\ a.s.\ \gamma. \end{aligned}$$

While by [20] Chapter 3 Lemma 7.5,

$$(z,t) \in R^1 \times [0,\infty) \to \int_0^t I_{(z,\infty)}(\gamma_s)\ dM_s(\gamma)\ \textit{is continuous},\ Q_\mu\ -\ a.s.\ \gamma \in C_{R^1}.$$

Hence, by Lemma 5.1, $Q_\mu\ -\ a.s.$, each term in above Tanaka formula is continuous in $(z,t) \in R^1 \times [0,\infty)$; and further

$$\begin{aligned} L_t^z(\gamma) &= 2\left[(\gamma_t - z)^+ - (\gamma_0 - z)^+ - \int_0^t I_{(z,\infty)}(\gamma_s)\ dM_s(\gamma) - \right. \\ &\left. \int_0^t I_{(z,\infty)}(\gamma_s) b(\gamma_s)\ ds\right],\ \forall t \in [0,\infty),\ \forall z \in R^1,\ Q_\mu\ -\ a.s.\ \gamma. \end{aligned}$$

Now without loss of generality, assume for every $\gamma \in C_{R^1}$,

$$\begin{aligned} L_t^z(\gamma) = 2&\left[(\gamma_t - z)^+ - (\gamma_0 - z)^+ - \int_0^t I_{(z,\infty)}(\gamma_s)\ dM_s(\gamma) - \right. \\ &\left. \int_0^t I_{(z,\infty)}(\gamma_s) b(\gamma_s)\ ds\right],\ \forall t \in [0,\infty),\ \forall z \in R^1, \\ \textit{and}\ &\int_0^t I_{(z,\infty)}(\gamma_s)\ dM_s(\gamma)\ \textit{is continuous in}\ (z,t) \in R^1 \times [0,\infty). \end{aligned}$$

By (5.5) and Lemma 5.3, for any $z \in \{a(\cdot) \neq 0\}$ and $\mu \in M_1\left(R^1\right)$ with $\int_{[0,\infty)} x\ \mu(dx) < \infty$,

$$\begin{aligned} \widetilde{\mathbf{L}}_t^z(\mathbf{X}) = &\frac{2}{a(z)} \int_{C_{R^1}} (\gamma_t - z)^+\ Z\left(\mathbf{w}^{\mathbf{N}}\right)(d\gamma) - \frac{2}{a(z)} \int_{C_{R^1}} (\gamma_0 - z)^+\ Z\left(\mathbf{w}^{\mathbf{N}}\right)(d\gamma) - \\ &\frac{2}{a(z)} \int_0^t \int_{C_{R^1}} I_{(z,\infty)}(\gamma_s) b(\gamma_s)\ Z\left(\mathbf{w}^{\mathbf{N}}\right)(d\gamma) ds - \\ &\frac{2}{a(z)} \int_{C_{R^1}} \int_0^t I_{(z,\infty)}(\gamma_s) dM_s(\gamma) Z\left(\mathbf{w}^{\mathbf{N}}\right)(d\gamma) \\ = &\frac{2}{a(z)} \left\{ \int_{[z,\infty)} (x - z)\ \mathbf{X}_t(dx) - \int_{[z,\infty)} (x - z)\ \mu(dx) - \int_0^t \left\langle \mathbf{X}_s, I_{(z,\infty)} b(\cdot)\right\rangle\ ds \right. \\ &\left. - \int_{C_{R^1}} \int_0^t I_{(z,\infty)}(\gamma_s) dM_s(\gamma) Z\left(\mathbf{w}^{\mathbf{N}}\right)(d\gamma) \right\},\ \ \forall t \in [0,\infty),\ Q_{\mu^{\mathbf{N}}}\ -\ a.s.. \end{aligned}$$

Step 2. For any $z \in \{a(\cdot) \neq 0\}$, let

$$\begin{aligned}\mathbf{M}_t^z(\mathbf{X}) &= \int_{[z,\infty)} (x-z)\ \mathbf{X}_t(dx) - \int_{[z,\infty)} (x-z)\ \mu(dx) - \\ &\int_0^t \left\langle \mathbf{X}_s, I_{(z,\infty)} b(\cdot) \right\rangle\ ds - \frac{1}{2} a(z) \widetilde{\mathbf{L}}_t^z(\mathbf{X}),\ \forall t \in [0,\infty).\end{aligned}$$

Then $(\mathbf{M}_t^z(\mathbf{X}))_{t\geq 0}$ is a $(\mathcal{F}_t(\mathbf{X}))_t$-adapted L^r-martingale for any $r \in [1,\infty)$; and almost surely, $(z,t) \in \{a(\cdot) \neq 0\} \times [0,\infty) \to \mathbf{M}_t^z(\mathbf{X})$ is continuous, and $z \in \{a(\cdot) \neq 0\} \to \mathbf{M}_t^z(\mathbf{X})$ is Hölder continuous of order α for any $\alpha \in \left(0, \frac{1}{2}\right)$ and uniformly in t on every compact interval.

Let

$$\left(\overline{\mathbf{M}}_t^z(\mathbf{X})\right)_{t\geq 0} := \left(\int_{C_{R^1}} \int_0^t I_{(z,\infty)}(\gamma_s) dM_s(\gamma) Z\left(\mathbf{w}^{\mathbf{N}}\right)(d\gamma) \right)_{t\geq 0},\ \forall z \in R^1.$$

For any $z \in R^1$, by the second part of Lemma 4.6, we have that for any $t \geq 0$, as $k \to \infty$,

$$M_t^z(k) := \frac{1}{k} \sum_{i=1}^k \int_0^t I_{(z,\infty)}\left(w_s^i\right)\ dM_s\left(w^i\right) \to \overline{\mathbf{M}}_t^z(\mathbf{X})\ \textit{in}\ L^1\left(Q_{\mu^{\mathbf{N}}}\right)\ \textit{and almost surely.}$$

Since $(M_t^z(k))_{t\geq 0}$ is a $\left(\mathcal{F}_t\left(\mathbf{w}^{\mathbf{k}}\right)\right)_t$-martingale, for any $0 \leq s < t < \infty,\ n \in \mathbf{N}$, and bounded $\mathcal{F}_s\left(\mathbf{w}^{\mathbf{n}}\right)$-measurable function h on $C_{R^{\mathbf{N}}}$, we have

$$Q_{\mu^{\mathbf{N}}}\left[\overline{\mathbf{M}}_t^z(\mathbf{X})h\right] = \lim_{n\leq k\to\infty} Q_{\mu^{\mathbf{N}}}\left[M_t^z(k)h\right] = \lim_{n\leq k\to\infty} Q_{\mu^{\mathbf{N}}}\left[M_s^z(k)h\right] = Q_{\mu^{\mathbf{N}}}\left[\overline{\mathbf{M}}_s^z(\mathbf{X})h\right],$$

which implies that $\left(\overline{\mathbf{M}}_t^z(\mathbf{X})\right)_{t\geq 0}$ is a $\left(\overline{\mathcal{F}}_t\right)_t$-adapted martingale, where $\overline{\mathcal{F}}_t$ is the completion of $\sigma\left(\bigcup_{k\geq 1} \mathcal{F}_t\left(\mathbf{w}^{\mathbf{k}}\right)\right)$ with respect to $Q_{\mu^{\mathbf{N}}}$.

By the Burkholder-Davis-Gundy inequality,

$$\sup_{z\in R^1} Q_\mu\left[\sup_{s\leq t}\left|\int_0^s I_{(z,\infty)}(\gamma_u)\ dM_u(\gamma)\right|^r\right] < \infty,\ \forall t \in [0,\infty),\ \forall r \in [1,\infty);$$

and further similarly to Step 2 of the proof of Lemma 5.4, we have

$$\sup_{z\in R^1} Q_{\mu^{\mathbf{N}}}\left[\int_{C_{R^1}} \sup_{s\leq t}\left|\int_0^s I_{(z,\infty)}(\gamma_u) dM_u(\gamma)\right|^r Z(\mathbf{w}^{\mathbf{N}})(d\gamma)\right] < \infty,$$

which implies $\left(\overline{\mathbf{M}}_t^z(\mathbf{X})\right)_{t\geq 0}$ is an L^r-martingale, and is *a.s.* continuous in t (note by assumption, $\int_0^t I_{(z,\infty)}(\gamma_u) dM_u(\gamma)$ is continuous in $t \in [0,\infty)$).

It is easy to prove (c.f. Step 2 for the proof of Lemma 5.2) that for any $r \in [2,\infty)$,

$$Q_\mu\left[\sup_{s\leq t}\left|\int_0^s \left(I_{(z_1,\infty)}(\gamma_u) - I_{(z_2,\infty)}(\gamma_u)\right) dM_u(\gamma)\right|^r\right] \leq c_{t,r}|z_1 - z_2|^{\frac{r}{2}},\ \forall (z_1, z_2) \in R^2;$$

where $c_{t,r}$ is a constant depending on t and r. Then similarly to Step 4 of the proof of Lemma 5.4, one can show that almost surely, $(z,t) \to \overline{\mathbf{M}}_t^z(\mathbf{X})$ is continuous, and $z \in R^1 \to \overline{\mathbf{M}}_t^z(\mathbf{X})$ is Hölder continuous of order α for any $\alpha \in \left(0, \frac{1}{2}\right)$ and uniformly in t on every compact interval.

By Step 1 of proving Theorem 5.1, for any $z \in \{a(\cdot) \neq 0\}$,

$$\mathbf{M}_t^z(\mathbf{X}) = \overline{\mathbf{M}}_t^z(\mathbf{X}), \ \forall t \in [0,\infty), \ Q_{\mu^{\mathbf{N}}} \ - a.s.. \tag{5.6}$$

Notice that $\widetilde{\mathbf{L}}_t^z(\mathbf{X})$, $\int_{[z,\infty)}(x-z) \ \mathbf{X}_t(dx) - \int_{[z,\infty)}(x-z) \ \mu(dx)$ and $\int_0^t \left\langle \mathbf{X}_s, I_{(z,\infty)}b(\cdot)\right\rangle \ ds$ are $\mathcal{F}_t(\mathbf{X})$-measurable. So for any $z \in \{a(\cdot) \neq 0\}$, $\mathbf{M}_t^z(\mathbf{X})$ is $\mathcal{F}_t(\mathbf{X})$-measurable. Recall from Section 4, the $\mathcal{H}$-topology is stronger than the weak topology on $M_1\left(C_{R^1}\right)$, and

$$\text{as } n \to \infty, \ \frac{1}{n}\sum_{i=1}^n \delta_{w^i} \to Z\left(\mathbf{w}^{\mathbf{N}}\right) \ \text{ in } \mathcal{H}-\text{topology}, \ Q_{\mu^{\mathbf{N}}} \ - \ a.s. \ \mathbf{w}^{\mathbf{N}};$$

and $\mathbf{X}_t\left(\mathbf{w}^{\mathbf{N}}\right) = (\pi_t)_* Z\left(\mathbf{w}^{\mathbf{N}}\right), \ t \in [0,\infty)$; and

$$\Xi\left(\mathbf{w}^{\mathbf{N}}\right) = \begin{cases} \text{the limit of } \left\{\frac{1}{n}\sum\limits_{i=1}^n \delta_{w^i}\right\}_{n=1}^{\infty} \text{ in } M_1(C_{R^1}) \text{ if it exists,} \\ Z_0 \in M_1(C_{R^1}), \ \text{otherwise,} \end{cases}$$

$$Z\left(\mathbf{w}^{\mathbf{N}}\right) = \Xi\left(\mathbf{w}^{\mathbf{N}}\right), \ \mathbf{X} = \left(\mathbf{X}_t\left(\mathbf{w}^{\mathbf{N}}\right)\right)_{t\geq 0} = \left((\pi_t)_*\Xi\left(\mathbf{w}^{\mathbf{N}}\right)\right)_{t\geq 0}.$$

Combining with $\pi_t : \ \gamma \in C_{R^1} \to \gamma_t \in R^1$ is continuous for any t, we see that as $n \to \infty$,

$$\frac{1}{n}\sum_{i=1}^n \delta_{w_t^i} \to \mathbf{X}_t\left(\mathbf{w}^{\mathbf{N}}\right) \ \text{(in the weak topology)}, \ \forall t \in [0,\infty), \ Q_{\mu^{\mathbf{N}}} \ - \ a.s. \ \mathbf{w}^{\mathbf{N}}. \tag{5.7}$$

Thus $\mathcal{F}_t(\mathbf{X}) \subseteq \overline{\mathcal{F}}_t, \ t \in [0,\infty)$. Use (5.6) to finish the proof of this step.

Step 3. For any $\mu \in M_1\left(R^1\right)$ with $\int_{[0,\infty)} x \ \mu(dx) < \infty$, by the definition of

$$\left((\mathbf{M}_t^z(\mathbf{X}))_{t\geq 0}\right)_{z\in\{a(\cdot)\neq 0\}},$$

for any $\mathbf{w}^{\mathbf{N}} \in C_{R^{\mathbf{N}}}$,

$$\begin{aligned} \widetilde{\mathbf{L}}_t^z\ (\mathbf{X}) \ &= \ \frac{2}{a(z)}\left\{\int_{[z,\infty)}(x-z) \ \mathbf{X}_t(dx) - \int_{[z,\infty)}(x-z) \ \mu(dx) - \right. \\ &\left. \int_0^t \left\langle \mathbf{X}_s, I_{(z,\infty)}b(\cdot)\right\rangle \ ds - \mathbf{M}_t^z(\mathbf{X})\right\}, \\ &\forall t \in [0,\infty), \ \forall z \in \{a(\cdot) \neq 0\}. \end{aligned}$$

Since each term in the above equality is $\mathcal{F}_t(\mathbf{X})$-measurable for any $t \in [0,\infty)$, combining with that random variable $\mathbf{X}$ defined on $\left(C_{R^{\mathbf{N}}}, Q_{\mu^{\mathbf{N}}}\right)$ is of the law P_μ, and Lemma 5.4, we have finished proving Theorem 3.1 with the Tanaka formula for initial measure $\mu \in M_1\left(R^1\right)$ satisfying $\int_{[0,\infty)} x \ \mu(dx) < \infty$. Similarly, one can prove the Tanaka formula in Theorem 3.1 for $\mu \in M_1\left(R^1\right)$ with $\int_{(-\infty,0]} |x| \ \mu(dx) < \infty$. □

References

[1] Aaronson, J.: An introduction to infinite ergodic theory. Mathematical surveys and monographs **50**, AMS Providence RI (1997).

[2] Arnold, L.: Random dynamical systems. Springer-Verlag, Berlin Heidelberg, New York (1998).

[3] Barlow, M. T.: Diffusions on fractals. Lect. Note. Math. **1690**, Springer (1998).

[4] Barlow, M. T., Perkins, E. A.: Brownian motion on the Sierpinski Gasket. Probab. Th. Rel. Fields. **79,** 543-623 (1988).

[5] Bass, R. F., Burdzy, K., Chen, Z. Q.: Pathwise uniqueness for a degenerate stochastic differential equation. Ann. Probab. **35,** 2385-2418 (2007).

[6] Bertoin, J., Le Gall, J. F.: Stochastic flows associated to coalescent processes. Probab. Th. Rel. Fields. **126,** 261-288 (2003).

[7] Cherny, A. S., Engelbert, H. J.: Singular stochastic differential equations. Lect. Note. Math. **1858**, Springer-Verlag (2005).

[8] Cornfeld, I. P., Fomin, S. V., Sinai, Y. G.: Ergodic theory. Translated from the Russian by A. B. Sosinskii. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences] **245.** Springer-Verlag, New York (1982).

[9] Dawson, D. A.: Measure-valued Markov Processes. Lect. Note. Math. **1541**. Springer-Verlag (1993).

[10] Dellacherie, C., Meyer, P. A.: Probabilities and Potential. North-Holland Publishing Company-Amsterdam-New York-Oxford (1978).

[11] Donnelly, P., Kurtz, T.: A countable representation of the Fleming-Viot measure-valued diffusion. Ann. Proba. **24,** 698-742 (1996).

[12] Dubischar, D.: Representation of Markov processes by random dynamical systems. Report **393**, Institut für Dynamische Systeme, Universität Bremen (1997).

[13] Dynkin, E. B.: Three classes of infinite dimensional diffusions. J. Funct. Anal. **86,** 75-110 (1989).

[14] Engelbert, H. J., Schmidt, W.: On solutions of one-dimensional stochastic differential equations without drift. Z. Wahrsch. Verw. Gebiete. **68(3),** 287-314 (1985).

[15] Etheridge, A., March, P.: A note on superprocesses. Prob. Th. Rel. Fields. **89,** 141-147 (1991).

[16] Evans, S. N.: Embedding a Markov chain into a random walk on a permutation group. Comb. Proba. Computing. **13,** 339-351 (2004).

[17] Fleming, W. H., Viot, M.: Some measure-valued Markov processes in population genetics theory. Indiana Univ. Math. J. **28,** 817-843 (1979).

[18] Frisch, U.: Turbulence (The Legacy of A. N. Kolmogorov). Cambridge University Press (1996).

[19] Kallenberg, O.: Foundations of modern probability. Springer (1997).

[20] Karatzas, I., Shreve, S. E.: Brownian motion and stochastic calculus. Springer-Verlag (1988).

[21] Kifer, Y.: Ergodic theory of random transformations. Birkhäuser, Boston. Basel. Stuttgart (1986).

[22] Konno, N., Shiga, T.: Stochastic differential equations for some measure-valued diffusions. Prob. Th. Rel. Fields. **79,** 201-225 (1988).

[23] Kunita, H.: Stochastic flows and stochastic differential equations. Camb. Univ. Press (1990).

[24] Le Gall, J. F.: Spatial branching processes, random snakes and partial differential equations. Birkhäuser (1999).

[25] Le Jan, Y., Raimond, O.: Flows, coalescence and noise. Ann. Probab. **32(2),** 1247-1315 (2004).

[26] Le Jan, Y.: New developments in stochastic dynamics. Proceedings of ICM 2006 Madrid, Vol. **III,** 649-667. European Mathematical Society (2006).

[27] Ma, Z. M., Xiang, K. N.: Superprocesses of stochastic flows. Ann. Probab. **29(1)**, 317-343 (2001).

[28] Majda, A. J., Bertozzi, A. L.: Vorticity and incompressible flow. Cambridge University Press (2002).

[29] Malliavin, P.: Stochastic analysis. Springer-Verlag, Berlin (1997).

[30] Mytnik, L., Xiang, K. N.: Tanaka formulae for (α, d, β)-superprocesses. J. Theoret. Probab. **17(2),** 483-502 (2004).

[31] Perkins, E. A.: Conditional Dawson-Watanabe processes and Fleming-Viot processes. Seminar on Stochastic Processes 1991, S. Port and P. J. Fitzsimmons (eds), 143-156, Birkhäuser (1992).

[32] Quas, A. N.: On representation of Markov chains by random smooth maps. Bull. London. Math. Soc. **23,** 487-492 (1991).

[33] Revuz, D., Yor, M.: Continuous martingales and Brownian motion. Springer-Verlag (1990).

[34] Sharpe, M.: General theory of Markov processes. Academic Press, INC (1988).

[35] Stroock, D. W., Varadhan, S. R. S.: Multidimensional diffusion processes. Springer-Verlag (1979).

[36] Viana, M.: Dynamics: probabilistic and geometric perspective. Proceedings of the International congress of Mathematicians, Berlin, Documenta Mathematica. Extra volume ICM 1998. **I,** 557-578 (1998).

[37] Wang, B., Xiang, K. N.: Measure evolution for “stochastic flows”. Stochastics and Dynamics **8(2),** 1-37 (2008).

[38] Wang, B., Xiang, K. N., Yang, X. Q.: On a class of measure-valued processes: singular cases. Science in China Ser. A. **49(10),** 1315-1326 (2006).

[39] Xiang, K. N.: On Tanaka formulae for (α, d, β)-superprocesses. Sci. China Ser. A. **48(9),** 1194-1208 (2005).