Monochromatic and Heterochromatic Subgraphs in Edge-Colored Graphs - A Survey *

Mikio Kano[1] and Xueliang Li[2]

[1]Department of Computer and Information Sciences, Ibaraki University
Hitachi, Ibaraki, 316-8511, Japan. Email: kano@mx.ibaraki.ac.jp

[2]Center for Combinatorics and LPMC-TJKLC, Nankai University
Tianjin 300071, P.R. China. Email: lxl@nankai.edu.cn

Abstract

Nowadays the term monochromatic and heterochromatic (or rainbow, multicolored) subgraphs of an edge-colored graph appeared frequently in literature, and many results on this topic have been obtained. In this paper, we survey results on this subject. We classify the results into the following categories: vertex-partitions by monochromatic subgraphs, such as cycles, paths, trees; vertex partition by some kinds of heterochromatic subgraphs; the computational complexity of these partition problems; some kinds of large monochromatic and heterochromatic subgraphs. We have to point out that there are a lot of results on Ramsey type problem of monochromatic and heterochromatic subgraphs. However, it is not our purpose to include them in this survey because this is slightly different from our topics and also contains too large amount of results to deal with together. There are also some interesting results on vertex-colored graphs, but we do not include them, either.

Keywords: monochromatic, heterochromatic, edge-colored graph, vertex-partition, cycle, path, tree

AMS subject classification 2000: 05C

*Supported by NSFC, PCSIRT and the “973” program.

1 Introduction

The study of monochromatic and heterochromatic subgraphs of an edge-colored graph may be dated back to the 60's or 70's of the last century. But, the development was not so fast. Since the 90's, many important results appeared. Most of the impressive results were obtained by using *asymptotic method* [44]. Namely, in an edge-colored graph G, first find a sufficiently large monochromatic and dense subgraph H having the required structural properties that will be used in the last step for adjustment. Then remove the vertices of the dense subgraph H, and greedily remove a number, which depends on the number of colors, of vertex-disjoint monochromatic subgraphs (cycles, paths, trees) from the remaining graph until the number of leftover vertices is much smaller than the number of vertices associated to H. Finally after some adjustment with respect to H, find a monochromatic subgraph spanning the remaining vertices of H. Hence we can obtain the desired spanning monochromatic subgraph of G. The Regularity Lemma plays a central role in the process.

However, many unsolved problems or conjectures are still open. As one knows, an asymptotic method does not like a canonical combinatorial method. The former usually deals with cases of a large size, and the later aims to solve all cases, small and large. The former gives us a support for an open problem, however, if one wants to solve the whole problem, asymptotic method does not seem to work. This can be found later in the sequel, for instance, the conjecture that the cycle and tree partition number of an r-edge-colored complete graph is r and $r-1$, respectively. In this paper, we will survey results in this field. Open problems or conjectures from literature are listed. We have to point out that there are a lot of results on Ramsey type problem of monochromatic and heterochromatic subgraphs. However, it is not our purpose to include them in this survey, because their taste and proof techniques are slightly different from those of the topics dealt with in this survey and the length of the survey is limited. There are also some interesting results on vertex-colored graphs [60], we do not survey them, either.

We use Chartrand and Lesniak [20] for terminology and notations not defined here and consider only simple graphs, which have neither loops nor multiple edges, unless otherwise stated.

Let $G = (V, E)$ be a graph with vertex set $V = V(G)$ and edge set $E = E(G)$. Then $|V|$ and $|E|$ are called the *order* and the *size* of G, respectively. By an *edge coloring* of G we mean a function $col : E \to \{1, 2, \cdots, r\}$, and if the function col is surjective, it is called an *r-edge coloring* of G. If G is assigned such colorings, we say that G is an *edge-colored graph* or *r-edge-colored graph*. If the edges of G is colored so that no color is appeared in more than k edges, we refer to this as a *k-bounded coloring*. A subgraph of an edge-colored graph is called *monochromatic* if all of its edges have the same color, and called *heterochromatic* if all of its edges have distinct colors. A heterochromatic subgraph is also called *rainbow, multicolored, polychromatic* or *colorful*. The complete k-partite graph $K(n_1, n_2, \ldots, n_k)$ has the vertex set $V_1 \cup V_2 \cup \ldots \cup V_k$ such that $V_i \cap V_j = \emptyset$ and $|V_i| = n_i$ for every $1 \leq i < j \leq k$, and the edge set $\{x_i x_j \mid x_i \in V_i,\ x_j \in V_j,\ 1 \leq i < j \leq k\}$.

Erdős, Gyárfás, and Pyber [32] introduced the following notions. The *monochromatic tree partition number* of an r-edge-colored graph G, denoted by $monotree_r(G)$, is the minimum number k such that whenever the edges of G are colored with r colors, the vertices of G can be covered by at most k vertex-disjoint monochromatic trees. The *monochromatic path partition number* and *monochromatic cycle partition number* of an r-edge-colored graph G are defined analogously. Also the *heterochromatic tree partition number* of an r-edge-colored graph G, denoted by $hetetree_r(G)$, *heterochromatic path partition number* and *heterochromatic cycle partition number* can be defined similarly by using heterochromatic subgraphs instead of monochromatic subgraphs.

In Sections 2 and 3, we will focus on vertex-partition problems by monochromatic paths, cycles, and trees including the monochromatic partition numbers defined above. Then consider similar problems of heterochromatic paths, cycles and trees. In Sections 4 and 5, we survey the results on the existence of some kind of monochromatic and heterochromatic subgraphs. In the last section, we give some other results. Heterochromatic subgraphs in random colored graph, proper colored subgraphs, subgraphs of given color pattern will be discussed here.

2 Monochromatic Vertex Partitions

In this section we survey results concerning vertex coverings and vertex partitions by monochromatic subgraphs in edge-colored graphs.

We discuss vertex partition problems with respect to monochromatic path, cycle, k-regular subgraph and tree, respectively. Almost all these vertex partition problems employ the asymptotic method as described in the introduction. The last part of this section is devoted to the corresponding optimization problems of these partitions and their computational complexity.

2.1 Partitions by Monochromatic Paths

We begin with vertex partition by monochromatic paths in 2-edge-colored graphs. A graph is 2-*edge-colored* if each edge is colored either red or blue. So its monochromatic path can be called a *red path* or a *blue path.*

Theorem 1 *[39, 41] Every 2-edge-colored complete graph K_n contains either a monochromatic hamiltonian path or vertex-disjoint one red path and one blue path that together cover the vertices of K_n.*

This interesting result was mentioned by Gerencsér and Gyárfás in a footnote in [39]. Its proof shown in Gyárfás [41] gives an algorithm, which enables us to find the required paths in the above Theorem 1 in $O(n)$ time, consequently, the algorithm gives us a monochromatic path of length at least $n/2$ in $O(n)$ time. Later, Gyárfás, Jagota and Schelp [43] obtained a similar theorem for 2-edge-colored nearly complete graphs.

Theorem 2 *[43] Let $n \geq 5$ be an integer and G be a graph obtained from the complete graph K_n by removing at most $\lfloor n/2 \rfloor$ edges. Then every 2-edge-colored graph G contains vertex-disjoint one red path and one blue path that together cover the vertices of K_n.*

In 1978, Rado [70] obtained the following result.

Theorem 3 *[70] Let $\Gamma = (V, E)$ be an infinite directed graph such that $A \subseteq V$, $|A| \leq \aleph_0 \leq |V|$ and $E \subseteq A \times V$. Suppose that, for every $x \in A$, $|\{y \in V : (x, y) \notin E\}| < |V|$, and every arc of Γ receives a color from the color set I. Let ω denote the least infinite ordinal. Then there exists a subset $J \subseteq I$ such that for every $j \in J$, letting $m_j \in L = \{1, 3, 5, \ldots\} \cup \{\omega\}$, the vertices $x_j(\nu) \in V$ for $0 \leq \nu < m_j$ satisfy the following:*

(i) every $x \in A$ occurs among the $x_j(\nu)$,

(ii) $x_i(\mu) = x_j(\nu)$ implies $(i, \mu) = (j, \nu)$,

(iii) if $j \in J$ and $0 < \nu < m_j$ and ν is odd, then

$$(x_j(\nu - 1), x_j(\nu)) \quad \textit{and} \quad (x_j(\nu + 1), x_j(\nu))$$

are two arcs colored with j.

The following theorem, which generalized Theorem 1 to r-edge-colored countably infinite complete graph K_∞, can be easily obtained from Theorem 3 by putting $A = V$; $|V| = \aleph_0$; $E = V \times V - \{(x, x) : x \in V\}$; $|I| = r$ and making the color of every edge independent of its orientation.

Theorem 4 *If the edges of the countably infinite complete graph K_∞ are colored with r colors, then the vertices of K_∞ can be covered by at most r vertex-disjoint finite or one-way infinite monochromatic paths.*

Then, it is natural to ask whether we have a similar result in finite complete graphs. Gyárfás made the following conjecture.

Conjecture 1 *[42] The vertices of every r-edge-colored complete graph K_n can be covered by at most r vertex-disjoint monochromatic paths.*

It is easy to see that Conjecture 1 is true for $r = 2$ (see Theorem 1) but for $r = 3$ it seems to be difficult. It is worth considering the following weaker versions.

Conjecture 2 *[42] The vertices of every r-edge-colored complete graph K_n can be covered by at most r monochromatic paths, which are not necessary vertex-disjoint.*

Conjecture 3 *[42] There exists a function $f(r)$ with the following property: The vertices of every r-edge-colored complete graph K_n can be covered by at most $f(r)$ vertex-disjoint monochromatic paths.*

Conjecture 2 is open even for $r = 3$, on the other hand, Conjecture 3 was solved as explained in the next subsection. For general r, Gyárfás proved the following theorem, which is weaker than Conjecture 2 or Conjecture 3.

Theorem 5 *[42] There exists a function $f(r)$ with the following property: The vertices of every r-edge-colored complete graph K_n can be covered by at most $f(r)$ monochromatic paths.*

2.2 Partitions by Monochromatic Cycles

In this subsection, for convenience, we regard a vertex K_1 or an edge K_2 as a cycle, and call such a cycle and a usual cycle a *general cycle*, that is, a general cycle of order at least three is a usual cycle, and a general cycle of order one or two is also allowed. The cycle partition number is defined by using general cycles.

Like the above discussion for partitions by monochromatic paths, we first give a result of a partition by monochromatic cycles in 2-edge-colored graphs. The following theorem was obtained by Gyárfás in [41].

Theorem 6 *[41] The vertices of a 2-edge-colored complete graph K_n can be covered by one red general cycle and one blue general cycle that have at most one vertex in common.*

In 1998, Łuczak, Rödl and Szemerédi [66] gave a stronger theorem.

Theorem 7 *[66] There exists a constant n_0 such that for each $n \geq n_0$, the vertices of every 2-edge-colored complete graph K_n can be covered by vertex-disjoint one red general cycle and one blue general cycle.*

Erdős, Gyárfás and Pyber [32] considered a partition by monochromatic cycles in r-edge-colored complete graphs.

Theorem 8 *[32] The vertices of every r-edge-colored complete graph K_n can be covered by at most $cr^2 \log r$ vertex-disjoint monochromatic general cycles.*

Note that Theorem 8 solved Conjecture 3 in a stronger form since every cycle can be covered by a path. Theorem 8 implies that the cycle partition number of r-edge-colored complete graphs depends only on r and is no more than $cr^2 \log r$. In [44] Gyárfás, Ruszinkó, Sárközy and Szemerédi gave a significant improvement to Theorem 8 for large n.

Theorem 9 *[44] For every integer $r \geq 2$, there exists a constant $n_0 = n_0(r)$ such that for every $n \geq n_0$, the vertices of every r-edge-colored complete graph K_n can be covered by at most $100r \log r$ vertex-disjoint monochromatic general cycles.*

In [32], Erdős, Gyárfás and Pyber gave the following example. Consider a partition $A_1 \cup A_2 \cup \cdots \cup A_n$ of the vertices of a complete graph, and, for $x \in A_i$, $y \in A_j$, $i \leq j$, color the edge xy with color i. If the sequence $|A_i|$ grows fast enough, then the vertices of this r-edge-colored complete graph cannot be covered by less than r vertex-disjoint monochromatic paths. So the monochromatic path partition number is at least r. This implies that the monochromatic cycle partition number is also at least r. They conjectured that this example is best possible and Theorem 8 can be sharpened as follows.

Conjecture 4 *[32] The cycle partition number of an r-edge-colored complete graph is r.*

The special case $r = 2$ of this conjecture was asked earlier by Lehel, and proved for large n by Łuczak, Rödl and Szemerédi [66] (see Theorem 7). Some special cases for $r = 2$ have been solved by Ayel [10].

Erdős, Gyárfás and Pyber [32] also raised the question whether the cycle partition number for the complete bipartite graph $K_{n,n}$ is independent of n. Haxell [49] settled it affirmatively.

Theorem 10 *[49] Let a positive integer r be given. Let ε be a real number such that*

$$\frac{1}{16r} < \varepsilon < \frac{1}{7r}(1 - \frac{1}{r^3})(\frac{4}{5} - \frac{1}{r^2}),$$

and let $s \geq 10$ be an integer such that

$$\frac{1}{1-\varepsilon} - (1-\varepsilon)^{1-1/s} - 2\varepsilon^{1-1/s} > 0.$$

Then the vertices of every r-edge-colored complete bipartite graph $K_{n,n}$ can be covered by at most $2r(s+3)\log r + 3r^2$ vertex-disjoint monochromatic general cycles.

Theorem 10 shows that the cycle partition number of an r-edge-colored $K_{n,n}$ is at most $O((r \log r)^2)$ for large r. Notice that if the requirement that the cycles are vertex-disjoint is dropped, then as shown in [32], an r-edge-colored $K_{n,n}$ can be covered by $O(r^2)$ monochromatic cycles.

2.3 Partitions by Monochromatic k-Regular Subgraphs

Notice that cycles can be viewed as 2-regular graphs. So the monochromatic cycle partition problem can be naturally generalized to k-regular subgraph partition problem. In [73], Sárközy and Selkow obtained the following result on partitioning the vertices of an r-edge-colored complete graph into connected monochromatic k-regular subgraphs.

Theorem 11 *[73] There exists a constant c such that for any $r, k \geq 2$, the vertices of every r-edge-colored complete graph K_n can be covered by at most $r^{c(r \log r + k)}$ vertex-disjoint connected monochromatic k-regular subgraphs and vertices.*

The necessity of including vertices in the partition follows from a coloring in which there is a vertex v such that the edges incident with v are colored red and all the other edges are colored blue. Sárközy and Selkow gave a similar theorem for complete bipartite graphs.

Theorem 12 *[73] There exists a constant c such that for any $r, k \geq 2$, the vertices of every r-edge-colored complete bipartite graph $K_{n,n}$ can be covered by at most $r^{c(r \log r + k)}$ vertex-disjoint connected monochromatic k-regular subgraphs and vertices.*

Sárközy and Selkow mentioned in [73] that they could get $c = 200$ in Theorems 11 and 12, but the details were omitted since they thought that it was far from optimal constant.

2.4 Partitions by Monochromatic Trees

There are many results on partition or cover by monochromatic trees. We begin with some results given by Erdős, Gyárfás and Pyber in [32]. They claimed that the monochromatic tree cover number of an r-edge-colored complete graph is at most r since the monochromatic stars at any vertex give a good partition in many colorings. They also gave the following example to show that the monochromatic tree cover number of an r-edge-colored complete graph is at least $r-1$, when $r-1$ is a prime power. Consider a complete graph with vertex set identified with the points of an affine plane of order $r-1$. Color the edge pq with color i $(1 \leq i \leq r)$ if the line through p and q is in the ith parallel class. This example shows that the following conjecture, if true, is best possible when $r-1$ is a prime power.

Conjecture 5 *[32] The monochromatic tree partition number of an r-edge-colored complete graph is $r-1$, where $r \geq 2$.*

The case $r = 2$ in Conjecture 5 is equivalent to the fact that for any graph G, either G or its complement is connected, an old remark of Erdős and Rado. The case $r = 3$ is settled by Erdős, Gyárfás and Pyber in the same paper [32].

Theorem 13 *[32] The monochromatic tree partition number of a 3-edge-colored complete graph is 2.*

A weaker form of Conjecture 5 is that the vertices of an r-edge-colored complete graph can be covered by $r-1$ monochromatic trees, which are not necessary to be vertex-disjoint. This is equivalent to the following conjecture of Lovász and Ryser (see, e.g., Füredi [37]). An r-partite intersecting hypergraph has a transversal (blocking set) of at most $r-1$ elements. (This is proved by Tuza for $r \leq 5$ in [77].) Conjecture 5 implies that an r-edge-colored complete graph K_n contains a monochromatic tree with at least $n/(r-1)$ vertices. This consequence was known to be true in [13, 36, 40].

The monochromatic tree partition number seems to be more under control for infinite graphs. Hajnal, Komjáth, Soukup and Szalkai [47] proved that the monochromatic tree partition number of an r-edge-colored infinite complete graph is at most r. Nagy and Szentmiklóssy proved Theorem 13 for infinite graphs. For finite complete graphs, Haxell and Kohayakawa [48] proved the next theorem.

Theorem 14 *[48] Let $r \geq 1$ and $n \geq 3r^4 r!(1-1/r)^{3(1-r)} \log r$ be integers. Then the vertices of every r-edge-colored complete graph K_n can be covered by r vertex-disjoint monochromatic trees with different colors and of radius at most 2.*

Theorem 14 implies that the monochromatic tree partition number for an r-edge-colored complete graph K_n is at most r provided n is sufficiently large with respect to r. It was also mentioned in [49] that the following theorem can be proved by similar arguments given in the proof of Theorem 14.

Theorem 15 *If n is sufficiently large, then the monochromatic tree partition number of an r-edge-colored complete bipartite graph $K_{n,n}$ is at most $2r$.*

In [58], Kaneko, Kano and Suzuki determined the monochromatic tree partition number for a 2-edge-colored complete multipartite graph $K(n_1, n_2, \cdots, n_k)$.

Theorem 16 *Let n_1, n_2, $\cdots$, n_k $(2 \leq k)$ be integers such that $1 \leq n_1 \leq n_2 \leq \cdots \leq n_k$, and let $n = n_1 + n_2 + \cdots + n_{k-1}$ and $m = n_k$. Then the monochromatic tree partition number of 2-edge-colored complete k-partite graph $K(n_1, n_2, \cdots, n_k)$ is*

$$\left\lfloor \frac{m-2}{2^n} \right\rfloor + 2.$$

Using the proof technique in [52], another (more natural) proof of Theorem 16 could be given.

2.5 Computational Complexity

Now we discuss optimization problems on the vertex partition of edge-colored graphs: Given an edge-colored graph G, find the minimum number of vertex-disjoint monochromatic trees, cycles and paths, respectively, which cover the vertices of G. For convenience, we simply call these three problems the *PGMT, PGMC and PGMP problems,* respectively, since there are problems of partitioning a graph into monochromatic trees, cycles and paths. For a fixed integer r, the PGMT, PGMC, PGMP problem for an r-edge-colored graph is addressed as the *r-PGMT, r-PGMC and r-PGMP problem,* respectively.

The following facts are easily seen: If a given graph G is properly edge-colored, then both the PGMT and the PGMP problems are equivalent to finding a maximum matching, which can be solved in polynomial time (see [38]). By transforming the set cover problem to each of the problems in polynomial time, Jin and Li [53] showed the following theorems.

Theorem 17 *[53] The PGMT, PGMC and PGMP problems are NP-complete.*

Theorem 18 *[53] There is no constant factor approximation algorithm for any of the PGMT, PGMC and PGMP problems unless $P = NP$.*

Jin and Li [54] and Jin, Kano, Li and Wei [52] gave the following theorem for r-PGMT, r-PGMC and r-PGMP problems.

Theorem 19 *[54] For any fixed integer $r \geq 5$, the r-PGMT, r-PGMC and r-PGMP problems are NP-complete.*

Theorem 20 *[52] Both 2-PGMC and 2-PGMP problems are NP-complete for complete and complete bipartite graphs. Therefore the 2-PGMC and 2-PGMP problems are NP-complete in general.*

Theorem 21 *[52] The 2-PGMT problem is NP-complete for bipartite graphs.*

Theorem 22 *[52] The 2-PGMT problem can be solved in polynomial time for complete bipartite and complete multipartite graphs.*

Thus the problem of determining whether the r-PGMT, r-PGMC and r-PGMP problems are NP-complete for $r = 3$ and 4 is worth mentioning.

3 Heterochromatic Subgraph Partitions

In this section, we consider problems of vertex partition and edge partition by heterochromatic subgraphs in edge-colored graphs.

3.1 Vertex Partitions

When we consider problems of vertex partition by heterochromatic subgraphs, we mainly try to find the minimum number of vertex-disjoint heterochromatic trees, cycles, paths, respectively, which cover the vertices of a given edge-colored graph. The decision versions of these three problems are addressed as the *minimum heterochromatic tree, cycle, path partition problem*, respectively. On the other hand, for a given graph G, the minimum number of vertex-disjoint heterochromatic trees, cycles, paths which cover the vertices of G for all its r-edge colorings is called the *heterochromatic tree, cycle, path number*, respectively.

If all the edges of a graph G are colored with the same one color, then the minimum heterochromatic tree partition problem is equivalent to finding a maximum matching, which can be solved in polynomial time (see [59]). If all the edges of G are colored with distinct $|E(G)|$ colors, then a heterochromatic tree is nothing but an usual tree, and so this case is easy.

Li and Zhang [64] studied the complexity of the heterochromatic tree, cycle and path partition problems.

Theorem 23 *[64] The minimum heterochromatic tree (cycle) partition problem is NP-complete, and there does not exist constant factor approximation algorithm for it. Actually, the minimum heterochromatic tree partition problem is NP-complete for bipartite graphs.*

Theorem 24 *[64] The minimum heterochromatic path partition problem is NP-complete for 2-edge-colored graphs, and therefore, it is NP-complete for general graphs.*

Theorem 24 implies that the minimum heterochromatic tree partition problem is also NP-complete for 2-edge-colored graphs, since in this case any heterochromatic tree has at most two edges and therefore is a path. However, for complete bipartite graphs, Chen, Jin, Li and Tu [21] gave an explicit formula for the heterochromatic tree partition number of an r-edge-colored complete bipartite graph $K_{m,n}$. Notice that it is clear that the heterochromatic tree partition number of r-edge-colored star $K_{1,n}$ is $n - r + 1$.

Theorem 25 *[21] Let n, m and r be integers such that $2 \le m \le n$, $1 \le r \le mn$. Then the heterochromatic tree partition number of an r-edge-colored complete bipartite graph $K_{m,n}$ is*

$$hetetree_r(K_{m,n}) = \begin{cases} n & \text{if } 1 \le r \le n; \\ 1 & \text{if } m(n-1)+1 \le r \le mn; \\ 2 & \text{if } m = n \text{ and } r = n^2 - 2n + 2; \\ n - \lfloor r/m \rfloor & \text{if } m+1 \le r \le m(n-1) \\ & \quad \text{and } r \equiv 0, 1 \pmod{m}; \\ n - \lfloor r/m \rfloor - 1 & \text{otherwise.} \end{cases}$$

Using a different proof method, Jin and Li [56] get the heterochromatic tree partition number of an r-edge-colored complete graph.

Theorem 26 *[56] Let $3 \le n$, $2 \le r \le \binom{n}{2}$ and $\binom{t}{2} + 2 \le r \le \binom{t+1}{2} + 1$. Then $hetetree_r(K_n) = \lceil (n-t)/2 \rceil$.*

However, to give the heterochromatic tree partition number for all complete multipartite graphs is still under our consideration.

3.2 Edge Partitions

Here we consider a heterochromatic tree partition of the edge set of a properly edge-colored complete graph. Constantine [26] proved the following result on the existence of a proper edge coloring of complete graphs whose edges can be partitioned into heterochromatic spanning trees.

Theorem 27 *[26] For $n \neq 1, 3$, K_{2n} can be properly edge-colored with $2n-1$ colors in such a way that the edges can be partitioned into edge-disjoint heterochromatic isomorphic spanning trees.*

Yuster [80] gave the following degree condition for a properly edge-colored graph to have a heterochromatic H-factor, each of whose components is isomorphic to H.

Theorem 28 *[80] Let H be a graph and $n \geq 2$ be an integer such that $|H|$ divides n. Then there exists an integer $k = k(H)$ such that every properly edge-colored graph of order n and with minimum degree at least $(1-1/\chi(H))n+k$ has a heterochromatic H-factor.*

Suppose that the edges of the complete K_{2n} are colored with $2n-1$ colors in such a way that the edges of any single color form a perfect matching. Such a coloring is called a *factorization*. Observe that every complete graph K_{2n} with any factorization contains a heterochromatic spanning tree: namely, the star $K_{1,2n-1}$ at any vertex. Indeed, the edges of this K_{2n} can be partitioned into $2n-1$ heterochromatic trees $K_{1,2n-1}, \cdots, K_{1,2}, K_{1,1}$. On the other hand, it is well-known that K_{2n} can be partitioned into n spanning trees. So Brualdi and Hollingsworth [17] made the following conjecture.

Conjecture 6 *[17] If the complete graph K_{2n} has a coloring of factorization, then the edges of K_{2n} can be partitioned into n heterochromatic spanning trees.*

They also proved the following theorem.

Theorem 29 *[17] If the complete graph K_{2n} $(n \geq 3)$ has a coloring of factorization, then there exist two edge-disjoint heterochromatic spanning trees.*

Kaneko, Kano and Suzuki [57] extended Theorem 29 to properly edge-colored complete graphs and gave the following theorem and conjecture.

Theorem 30 *[57] Every properly edge-colored complete graph K_n $(n \geq 6)$ has three edge-disjoint heterochromatic spanning trees.*

Conjecture 7 *[57] Every properly edge-colored complete graph K_n $(n \geq 6)$ has $\lfloor n/2 \rfloor$ edge-disjoint heterochromatic spanning trees. In particular, if n is even then the edges of K_n can be partitioned into $n/2$ heterochromatic spanning trees.*

If the edges of a graph G are colored by r colors $1, 2, \ldots, r$, then its *color distribution* $(a_1, a_2, \cdots, a_r)$ means that the number of edges with color i is equal to a_i for every $1 \leq i \leq r$. In [1], Akbari and Alipourn generalized Theorem 29 as follows.

Theorem 31 *[1] If the r-edge-colored complete graph K_n has a color distribution $(a_1, \cdots, a_r)$ with $1 \leq a_1 \leq \cdots \leq a_r \leq (n+3)/2$ and $r \geq n-1$, then K_n has a heterochromatic spanning tree.*

Theorem 32 *[1] Suppose that the r-edge-colored complete graph K_n has a color distribution $(a_1, \cdots, a_r)$ with $2 \leq a_1 \leq \cdots \leq a_r \leq (n+1)/2$. If T is a non-star heterochromatic spanning tree of K_n, then $K_n - T$ has a heterochromatic spanning tree, where T is regarded as an edge subset.*

Theorem 33 *[1] If the complete graph K_n is r-edge-colored so that its color distribution $(a_1, \cdots, a_r)$ satisfies $1 \leq a_1 \leq \cdots \leq a_r \leq n/2$, then K_n has two edge-disjoint heterochromatic spanning trees.*

Theorem 34 *[1] If the complete graph K_n, $n \geq 3$, is r-edge-colored and $r \geq \binom{n-2}{2} + 2$, then K_n has a heterochromatic spanning tree.*

Theorem 35 *[1] If the complete graph K_n, $n \geq 6$, is r-edge-colored and $r \geq \binom{n-2}{2} + 3$, then K_n has two edge-disjoint heterochromatic spanning trees.*

Brualdi and Hollingsworth [18] found inequalities and major conditions on *color distributions* of the complete bipartite graph $K_{n,n}$, which guarantee the existence of a partitioning the edges into heterochromatic subgraphs of sizes $2n-1, 2n-3, \cdots, 3, 1$. Let $p = (p_1, p_2, \cdots, p_k)$ and $q = (q_1, q_2, \cdots, q_k)$ be two sequences of nonnegative integers. Then p *majorizes* q, denoted by $p \succeq q$, provided that when the subscripts are re-ordered so that $p_1 \leq p_2 \leq \cdots \leq p_k$ and $q_1 \leq q_2 \leq \cdots \leq q_k$, p and q satisfy

$$p_i + p_{i+1} + \cdots + p_k \geq q_i + q_{i+1} + \cdots + q_k \qquad \text{for all} \ \ i = 1, 2, \cdots, k,$$

with equality holding when $i = 1$. Observe that two color distributions for a given graph G need not be sequences of the same length. We shall sometimes wish to apply the majorization order to color distributions: in cases in which the sequences are of different lengths, it is to be understood that the shorter sequence has been padded with zeroes so that the two sequences may be compared.

Theorem 36 *[18] If the complete bipartite graph $K_{n,n}$ is p-edge-colored with color distribution $(a_1, a_2, \cdots, a_p)$, then $K_{n,n}$ admits an edge partition into heterochromatic subgraphs of sizes $2n-1$, $2n-3$, $\cdots$, 3, 1 if and only if*

$$(a_1, a_2, \cdots, a_p) \preceq (1, 1, 2, 2, \cdots, n-1, n-1, n). \tag{1}$$

Moreover, if $K_{n,n}$ is p-edge-colored with color distribution $(a_1, a_2, \cdots, a_p)$ satisfying (1), then it is possible to partition the edges of $K_{n,n}$ into heterochromatic forests of sizes $2n-1, 2n-3, \cdots, 3, 1$.

Furthermore, Brualdi and Hollingsworth [18] obtained a necessary and sufficient condition using color distribution for an r-edge-colored complete bipartite graph to have a heterochromatic spanning tree as follows.

Theorem 37 *[18] Every $(2n-1)$-edge-colored complete bipartite graph $K_{n,n}$ with color distribution $(a_1, a_2, \cdots, a_{2n-1})$ such that $a_1 \leq \cdots \leq a_{2n-1}$ has a heterochromatic spanning tree if and only if for every integer k with $k \leq 2n-1$, it follows*

$$\sum_{i=1}^{k} a_i > k^2/4.$$

Brualdi and Hollings made the following conjecture, which strengthens Theorem 36.

Conjecture 8 *[18] The edges of every proper p-edge-colored complete bipartite graph $K_{n,n}$ with color distribution $(a_1, \cdots, a_p)$ satisfying (1) can be partitioned into n heterochromatic trees of sizes $2n-1, 2n-3, \cdots, 3, 1$.*

4 Monochromatic Subgraphs

In this section we survey results on the existence of some monochromatic subgraphs. There are a lot of papers that deal with monochromatic subgraphs, and most of them discuss Ramsey type problems, which form an important subject in graph theory. But as we already claimed, we do not survey these results here.

First, we consider the size of a largest monochromatic connected subgraph. It is well-known that every 2-edge-colored complete graph K_n has a monochromatic connected spanning subgraph. Gyárfás [40] generalized this result to an r-edge-colored complete graph K_n, and obtained the next theorem.

Theorem 38 *[40] Every r-edge-colored complete graph K_n has a monochromatic connected subgraph with order at least $n/(r-1)$.*

In [69], Pyber, Rödl and Szemerédi obtained the following condition for the existence of a monochromatic k-regular subgraph in an r-edge-colored complete graph K_n.

Theorem 39 *[69] There exists $\varepsilon_r > 0$ such that every r-edge-colored complete graph K_n has a monochromatic k-regular subgraph with $k \geq \varepsilon_r n$.*

From Erdős and Gallai's result [31] on the relationship between the number of edges and long paths and cycles, one can immediately get the following.

Theorem 40 *Let G be an r-edge-colored graph of order n and size m. If $m \geq rn$, then G has a monochromatic path of length at least $\lceil (2m)/(rn) \rceil$, and a monochromatic cycle of length at least $\lceil (2m)/r(n-1) \rceil$.*

It was shown in [16] that the above bounds are best possible for general graphs. However, for complete graphs, Faudree and Saito claimed that the bound can be improved a lot, in the 2006 International Workshop on Discrete Mathematics and its Applications at Hitach of Japan.

Erdős and Galvin [30] studied the upper density and the strong upper density of monochromatic paths in an edge-colored infinite complete graph K_ω. Let $V(K_\omega) = \{1, 2, \cdots, n, \cdots\}$. The *upper density* of a subgraph G of K_ω is defined as

$$\limsup_{n \to \infty} \frac{|V(G) \cap \{1, 2, \cdots, n\}|}{n}.$$

For an infinite path $P = (x_1, x_2, x_3, \cdots)$, the *strong upper density* of P is

$$\limsup_{n \to \infty} \frac{f(n)}{n} \quad \text{where } f(n) = \sup\{m : \{x_1, \cdots, x_m\} \subseteq \{1, \cdots, n\}\}.$$

Theorem 41 *[30] The edges of K_ω can be colored with r colors so that every monochromatic infinite path has upper density $\leq 2/r$. Moreover, the edges of K_ω can be colored with two colors so that every monochromatic infinite path has upper density $\leq 8/9$.*

Theorem 42 *[30] (i) The edges of K_ω can be colored with three colors so that every monochromatic infinite path has strong upper density 0.*

(ii) The edges of K_ω can be colored with two colors so that every monochromatic infinite path has strong upper density $\leq 2/3$.

(iii) If the edges of K_ω are colored with two colors, then there is a monochromatic infinite path with strong upper density $\geq 1/(3+\sqrt{3})$.

Theorem 43 *[30] Let $r \geq 1$ be an integer, and let $\alpha_1, \cdots, \alpha_r$ be real numbers such that $\alpha_1 + \cdots + \alpha_r = 1$. If the edges of K_ω are colored with r colors $1, 2, \cdots, r$, then either, for every i, there is a monochromatic infinite path of color i with upper density α_i, or else, for some j, there is a monochromatic infinite path of color j with upper density $> \alpha_j$.*

Akiyama et al. [3] showed that every cubic graph has a 2-edge coloring such that every monochromatic component is a path. In 1984, Bermond, Fouquet, Habib and Péroche [11] conjectured that every cubic graph has a 2-edge coloring such that each monochromatic component is a path of length at most 5. Note that they [11] pointed out that the number 5 cannot be replaced by 4, but the two cubic graphs of order 6 are the only known cubic connected graph for which 5 cannot be replaced by 4. Some weaker versions of this conjecture were obtained by Jackson and Wormald [51] and Aldred and Wormald [2]. Finally in 1999, Thomassen [76] settled this conjecture as follows by considering an orientation of the given graph.

Theorem 44 *[76] Let G be a connected graph with maximum degree at most 3. Then the edges of G can be colored with two colors so that every monochromatic component is a path of length at most 5.*

In [45], Gyárfás and Simonyi studied some monochromatic subgraphs in Gallai coloring of a complete graph K_n, where *Gallai coloring* is an edge coloring of a complete graph in which no triangle is heterochromatic. The following Theorem 45 was conjectured in [12]. A *broom* consists of a path and a star such that an end-vertex of the path is identified with the center of the star.

Theorem 45 *[45] In every Gallai coloring of a complete graph, there exists a monochromatic spanning broom.*

Theorem 46 *[45] In every Gallai coloring of a complete graph, there exists a monochromatic spanning tree with diameter at most four.*

Theorem 47 *[45] In every Gallai coloring of a complete graph K_n, there exist a vertex v and a color i such that the number of edges incident with v and colored with i is at least $2n/5$.*

5 Heterochromatic Subgraphs

In this section, we consider heterochromatic subgraphs. For example, we deal with heterochromatic Hamiltonian cycles, heterochromatic spanning trees and other kinds of heterochromatic subgraphs.

5.1 Heterochromatic Hamiltonian Cycle

The problem to find a large bound $k = k(n)$ such that every k-bounded edge-colored complete graph K_n contains a heterochromatic Hamiltonian cycle was mentioned in Erdős, Nesetril and Rödl [33]. They addressed it as an Erdős-Stein problem and showed that k can be any constant.

Theorem 48 *[46] There exists a constant number c such that if $n \geq ck^3$ then every k-bounded edge-colored complete graph K_n has a heterochromatic Hamiltonian cycle.*

The above Theorem 48 was obtained by Hahn and Thomassen [46] and implies that k could grow as fast as $n^{1/3}$ to guarantee that a k-bounded edge-colored K_n contains a heterochromatic Hamiltonian cycle. It was conjectured in [46] that the growth rate of k could in fact be linear. In unpublished work Rödl and Winkler in 1984 improved the growth rate of k to $\sqrt{n}$. Frieze and Reed [35] made further progress.

Theorem 49 *[35] There exists a constant number c such that if n is sufficiently large and $k \leq n/(c \ln n)$, then every k-bounded edge-colored complete graph K_n contains a heterochromatic Hamiltonian cycle.*

In [4], Albert, Frieze and Reed improved Theorem 49 and proved the conjecture of [46].

Theorem 50 *[4] Let $c < 1/32$. If n is sufficiently large and $k \leq \lceil cn \rceil$, then every k-bounded edge-colored complete graph K_n contains a heterochromatic Hamiltonian cycle.*

Albert, Frieze and Reed also obtained a similar result in the case of directed graphs.

Theorem 51 *[4] Let $c < 1/64$. If n is sufficiently large and $k \leq \lceil cn \rceil$, then every k-bounded edge-colored complete digraph $\overrightarrow{K_n}$ contains a heterochromatic Hamiltonian directed cycle.*

Hahn and Thomassen [46] considered the heterochromatic Hamiltonian path in infinite complete graph K_ω.

Theorem 52 *[46] Assume that the infinite complete graph K_ω is edge-colored so that each monochromatic subgraph is locally finite. Then it either contains K^* or for every vertex v, there exists a heterochromatic Hamiltonian path starting at v, where K^* denotes the complete graph with vertex set $\omega = \{1, 2, \cdots\}$ and whose edge coloring $col : \omega \times \omega \rightarrow \omega$ satisfies $col(ij) = j$ if $i < j$.*

Corollary 53 *[46] If K_ω is edge-colored so that no color is used more than k times for a fixed k, then it contains a heterochromatic Hamiltonian path.*

Erdős and Tuza [34] generalized Theorem 52 as follows.

Theorem 54 *[34] Every edge-colored infinite complete graph K_ω contains a heterochromatic Hamiltonian path, provided that the edges of a single color incident with a vertex are of measure 0, in particular, if they are finitely many.*

5.2 Heterochromatic Spanning Tree

Here we give some sufficient conditions for the existence of heterochromatic spanning trees. Recall that Theorem 29, Conjecture 6, Theorem 30, Conjecture 7, Theorems 31 through 35 and Theorem 37 are all about this problem.

An edge coloring of a graph G is called an *edge coloring with complete bipartite decomposition* if each color class forms a complete bipartite subgraph of G. In unpublished work de Caen conjectured that if a complete graph K_n is edge-colored with complete bipartite decomposition using $n-1$ colors, then K_n has a heterochromatic spanning tree. In [6], Alon, Brualdi and Shader proved the following stronger result.

Theorem 55 *[6] Every complete graph K_n having an edge coloring with bipartite decomposition contains a heterochromatic spanning tree.*

Suzuki [75] gave a necessary and sufficient condition for the existence of a heterochromatic spanning tree in an edge-colored graph.

Theorem 56 *[75] An edge-colored connected graph G of order n has a heterochromatic spanning tree, if and only if, for any k colors ($1 \leq k \leq n-2$), the removal of all the edges colored with these k colors from G results in a graph having at most $k+1$ components.*

Using Theorem 56 and the fact that for any partition $D_1 \cup D_2 \cup \cdots \cup D_s$ of the vertices of a complete graph K_n ($2 \leq s \leq n$) the total number of edges whose two ends belong to two different D_i is no less than $\binom{n}{2} - \binom{n-(s-1)}{2} > \frac{n}{2}(s-2)$, Suzuki proved the following theorem.

Theorem 57 *[75] The $(n/2)$-bounded edge-colored complete graph K_n has a heterochromatic spanning tree.*

Jin and Li [55] generalized Theorem 56 to the following theorem since by taking $k = n-1$ in Theorem 58, Theorem 56 is obtained.

Theorem 58 *[55] Let (G, col) be an edge-colored connected graph and $1 \leq k \leq |G| - 1$ be an integer. Then G has a spanning tree with at least k colors if and only if for any color subset $S \subseteq \{col(e) : e \in E(G)\}$, it holds that*

$$\omega(G - E_S) \leq n - k + |S|,$$

where E_S is the set of edges with colors in S, and $\omega(G - E_S)$ denotes the number of components in $G - E_S$.

Akbari and Alipour [1] gave another necessary and sufficient condition for the existence of a heterochromatic spanning tree in an edge-colored connected graph.

Theorem 59 *[1] An edge-colored connected graph G has a heterochromatic spanning tree if and only if for every partition of $V(G)$ into t parts, where $1 \leq t \leq |V(G)|$, there exist at least $t - 1$ edges with distinct colors that join different partition sets.*

We conclude this subsection with some results on the complexity problem of finding a spanning tree with as many different colors as possible, and of finding one with as few different colors as possible, most of which was obtained by Broersma and Li [15].

Theorem 60 *[15] Finding a spanning tree with as many different colors as possible is equivalent to finding a common independent set of maximum cardinality in two matroids, in particular, there is a polynomial time algorithm.*

Theorem 61 *[15] Finding a spanning tree with as few different colors as possible is NP-hard.*

Broersma and Li used the minimum dominating set problem to prove Theorem 61. Later, Chang and Leu [19] proved Theorem 61 in a different way. They showed that finding a spanning tree with as few different colors as possible is NP-complete even for complete graphs, by using set cover problem, which implies that there is no constant factor approximation algorithm unless $P = NP$.

5.3 Large Heterochromatic Subgraphs

Now we discuss large heterochromatic subgraphs, which are not necessary to be spanning as before, for example, we consider large heterochromatic cliques, trees, paths, cycles, matchings, and others.

In order to explain the results in this section, we need some new notation and definitions. Let (G, col) be an edge-colored graph. The set of colors appeared in G is denoted by $col(G) = \{col(e) : e \in E(G)\}$. For a vertex v of G, the *color neighborhood* $CN(v)$ of v is defined as the set $\{col(e) : e \text{ is incident with } v\}$ of colors, and the *color degree* of v is defined as $\deg_G^{col}(v) = |CN(v)|$. For a vertex set $S \subseteq V(G)$, let $CN(S) = \cup_{x\in S} CN(x)$.

Let (G, col) be an edge-colored bipartite graph with bipartition (X, Y). For a vertex set $S \subseteq X$ or Y, a *color representative neighborhood* of S is defined as a subset $T \subseteq N_G(S)$ such that there exist $|T|$ edges between S and T that are incident with distinct vertices of T and have distinct colors. A *maximum color representative neighborhood* $ColRepNei(S)$ of S is such a maximum subset T.

We first consider under what condition an edge-colored graph has a longer heterochromatic path. From Erdös and Gallai's result [31] on the relationship between the number of edges and long paths and cycles, one can immediately get the following.

Theorem 62 *Every r-edge-colored graph G of order n has a heterochromatic path of length at least $\lceil (2r)/n \rceil$, and a heterochromatic cycle of length at least $\lceil (2r)/(n-1) \rceil$.*

Broersma, Li, Woeginger and Zhang [16] obtained the following results.

Theorem 63 *[16] Let G be an edge-colored graph. If $\deg_G^{col}(x) \geq k$ for every vertex x of G, then for every vertex v of G, there exists a heterochromatic path starting at v and of length at least $\lceil (k+1)/2 \rceil$.*

Theorem 64 *[16] Let G be an edge-colored graph and $s \geq 2$ be an integer. If $|CN(x) \cup CN(y)| \geq s$ for every pair of vertices x and y of G, then G contains a heterochromatic path of length at least $\lceil s/3 \rceil + 1$.*

Chen and Li [22] found a condition for an edge-colored graph to have a longer heterochromatic path under the conditions in Theorem 63.

Theorem 65 *[22] Let G be an edge-colored graph and $k \geq 1$ be an integer. If $\deg_G^{col}(x) \geq k$ for every vertex x of G, then there exists a heterochromatic path of length at least $\lceil (3k)/5 \rceil + 1$. Moreover, if $1 \leq k \leq 7$, there exists a heterochromatic path of length at least $k - 1$.*

Later, Chen and Li improved Theorem 65 as follows.

Theorem 66 *[23] Let G be an edge-colored graph and $k \geq 8$ be an integer. If $\deg_G^{col}(x) \geq k$ for every vertex x of G, then there exists a heterochromatic path of length at least $\lceil (2k)/3 \rceil + 1$.*

The best lower bound for the length of a longest heterochromatic path in an edge-colored graph with minimum color degree at least k could be $k-1$, which is checked for $k \leq 9$. However, general case is still open. If it is true, it would be best possible. In fact, let Q_{k-1}^* be an edge-colored graph, whose vertices are the ordered $(k-1)$-tuples of 0's and 1's; two vertices are joined by an edge if and only if they differ in exactly one coordinate or they differ in all coordinates. An edge is colored with color $j \in \{1, 2, \ldots, k-1\}$ if its two ends differ in exactly the jth coordinate, or with color k if its two ends differ in all the coordinates. Then it is not difficult to check that Q_{k-1}^* is the desired graph, whose longest heterochromatic path has length exactly $k-1$. Another class of such graphs is given as follows: Suppose the complete graph K_{2n} with vertex set $\{v_0, v_1, v_2, \ldots, v_{2n-1}\}$ is properly edge-colored with $2n-1$ colors such that the edge (v_i, v_j) is colored with color $(i+j) \pmod{2n-1}$ if $0 \leq i \neq j \leq 2n-2$ and the edge (v_i, v_{2n-1}) is colored with color $2i \pmod{2n-1}$ if $0 \leq i \leq 2n-2$. Then it is not hard to check that the longest heterochromatic path in this edge-colored graph is of length $2n-2$.

Chen and Li [24] improved the result of Theorem 64 as follows.

Theorem 67 *[24] Let G be an edge-colored graph. If $|CN(x) \cup CN(y)| \geq s$ for every pair of vertices x and y of G, then G has a heterochromatic path of length at least $\lceil (s+1)/2 \rceil$.*

Chen and Li also showed that the lower bound in Theorem 67 is best possible by the following example. Let $s \geq 1$ be an integer. If s is even, let G_s be the graph obtained from the complete graph $K_{(s+4)/2}$ by deleting an edge; if s is odd, let G_s be the complete graph $K_{(s+3)/2}$. Then, color all the edges of G_s with different colors. So, for any $s \geq 1$ we have that $|CN(x) \cup CN(y)| \geq s$ for any pair of vertices x and y in G, and every longest heterochromatic path in G is of length $\lceil (s+1)/2 \rceil$.

Broersma, Li, Woeginger and Zhang [16] also considered the complexity of finding a path between two given vertices with as few different colors as possible.

Theorem 68 *[16] Let G be an edge-colored graph and s and t be two vertices of G. Then finding a path between s and t with as few different colors as possible in G is NP-complete.*

Theorem 68 was proved by using 3-SAT problem in [16]. Wirth [78] also proved Theorem 68 by using the "red and blue set cover" problem, which implies that there exists no constant factor approximation algorithm unless $P = NP$.

From the proof of Theorem 68 in [16], which gives another proof to the NP-completeness for the problem of finding a spanning tree with as few different colors as possible, Theorem 61 could be obtained.

Chou, Manoussakis, Megalalaki, Spyratos and Tuza [25] obtained the following results of alternating paths, whose edges are alternately red and blue in 2-edge-colored graphs.

Theorem 69 *[25] For a 2-edge-colored graph G and three given vertices x, y and z, finding an alternating path between x and y through z is NP-complete. For complete graphs, this problem has a polynomial time algorithm, whose time complexity is $O(n^3)$.*

We now turn our attention from heterochromatic paths to heterochromatic cycles. Rodl and Tuza [71] showed by probabilistic technique that there exist graphs G with arbitrarily large girth such that every proper edge coloring of G contains a heterochromatic cycle.

Theorem 70 *[71] Let $t \geq 1$ be an integer and d be a real number such that $0 < d < 1/(2t+1)$, and let n be an integer relatively large with respect to t and d. Then there exists a graph G of order n with girth at least $t+2$ such that for any i, $2t+1 < i < n^d$, every proper edge coloring of G contains a heterochromatic cycle of length i.*

In [16], Broersma, Li, Woeginger and Zhang considered under what conditions there is a heterochromatic triangle or a heterochromatic quadrilateral in an edge-colored graph.

Theorem 71 *[16] Let G be an edge-colored graph of order $n \geq 4$, such that $|CN(x) \cup CN(y)| \geq n-1$ for every pair of vertices x and y of G. Then G has at least one heterochromatic triangle or heterochromatic quadrilateral.*

Alexeev [5] also got some results on long heterochromatic cycles.

Theorem 72 *[5] Let $k \geq 3$ be an odd integer. If an edge-colored complete graph does not contain a heterochromatic cycle of order k, then it contains no heterochromatic cycle of order m for all sufficiently large m; in particular, $m \geq 2k^2$ suffices.*

For even integer k, he showed two examples which demonstrate that a similar result does not hold [5].

Montellano-Ballesteros [68] considered heterochromatic edge-cuts, and gave the number of colors could be used in the edge coloring of a graph G such that there exists no heterochromatic edge-cuts, whose edges have all distinct colors.

Theorem 73 *[68] Let G be a 2-edge connected graph. Then in every edge coloring of G using at least $|E(G)| - |V(G)| + 2$ colors, there exists at least one heterochromatic edge-cut in G.*

We conclude this subsection with heterochromatic matchings. Unlike usual maximum matchings, finding a maximum heterochromatic matching in an edge-colored graph is NP-complete (see [38], page 203, Multiple Choice Matching Problem). By our experience, this means that to find a good necessary and sufficient condition for the existence of perfect heterochromatic matchings would be hopeless.

Woolbright and Fu [79] showed that in any complete graph of even order with edge coloring by perfect matchings, there exists a heterochromatic perfect matching.

Theorem 74 *[79] Every properly (2n-1)-edge-colored complete graph K_{2n}, $n \geq 3$, has a a heterochromatic perfect matching.*

In [29], El-Zanati, Plantholt, Sissokho and Spence gave a similar theorem on complete uniform hypergraphs which are edge-colored by perfect matchings.

Theorem 75 *[29] For $n \geq 3$ and $r \geq 2$, if a complete r-uniform hypergraph on rn vertices $K_{rn}^{(r)}$ is edge-colored by perfect matchings, then there exists a heterochromatic perfect matching.*

In [63], Li and Xu generalized Theorems 74 and 75 to any properly edge-colored complete graph and complete uniform hypergraph.

Theorem 76 *[63] For any proper edge coloring of the complete r-uniform hypergraph $K_{rn}^{(r)}$ with $n \geq 3$ and $r \geq 2$, there is a heterochromatic perfect matching.*

From Theorem 76 we can easily get Theorem 74 as a corollary.

Many conditions were also given for the existence of large heterochromatic matchings in edge-colored bipartite graphs and general graphs [79]. Hu and Li [50] obtained some sufficient conditions. Because the conditions are complicated and not useful, we omit their details. Li, Li, Liu and Wang [61] studied heterochromatic matchings in edge-colored bipartite graphs under a condition related to maximum color representative neighborhoods of subsets of vertices.

Theorem 77 *[61] Let (G, col) be an edge-colored bipartite graph with bipartition (X, Y). If $|ColRepNei(S)| \geq |S|$ for all $S \subseteq X$, then G has a heterochromatic matching of size at least $|X|/3$.*

Theorem 78 *[61] Let (G, col) be an edge-colored bipartite graph with bipartition (X, Y) such that $|X| = |Y| = n$. If $|ColRepNei(S)| \geq |S|$ for all $S \subseteq X$ and $S \subseteq Y$, then G has a heterochromatic matching of size at least $(3n - 1)/8$.*

Li and Wang [62] studied the heterochromatic matchings in general graphs under a color degree condition.

Theorem 79 *[62] Let $k \geq 3$ and G be an edge-colored graph with maximum degree $\triangle$. If $\deg_G^{col}(x) \geq k$ for every vertex x of G, then G has a heterochromatic matching of size at least $(2k)/3$. Moreover there is an $O(k\triangle)$-time algorithm to construct such a heterochromatic matching.*

Here we point out a strong relation of heterochromatic matchings with the transversals of Latin squares ([74]).

Suppose that L is an $n \times n$ matrix such that each cell is assigned one of the n symbols $1, 2, \cdots, n$. If each row and each column of L contains each symbol exactly once, then L is called a *Latin square*. A *transversal* of a Latin square L is a set of n cells of L such that no two of the cells are taken from the same row or same column. A *partial transversal* of L is a subset of a transversal. A (partial) transversal of L is called *Latin* if no two of the cells have the same symbol. A (partial) transversal of an $m \times n$-matrix is defined similarly.

Notice that an $n \times n$ Latin square corresponds to a proper edge coloring of the complete bipartite $K_{n,n}$ with n colors, a transversal of a Latin square corresponds to a perfect matching of $K_{n,n}$ and a Latin transversal of a Latin square corresponds to a heterochromatic perfect matching.

On the transversals of Latin square, Ryser [72] conjectured the following.

Conjecture 9 *[72] Every Latin square of odd order has a Latin transversal.*

Moreover, Stein [74] and Brualdi [28] conjectured that

Conjecture 10 *Every Latin square of order n has a partial Latin transversal of size at least $n-1$.*

The conjectures remain unsolved. There are many publications on the lower bound for the size of a maximum transversals.

At the end of this section, we would like to claim that many other kinds of heterochromatic subgraphs have also been discussed, but most of them are of Ramsey type.

6 Some Other Results

We will here mainly study random graphs with random coloring. The study of random graphs was begun by Erdős and Rényi in 1960s. In [27], Cooper and Frieze gave a threshold for the random graph $G_{n,m}$ to be in $\mathcal{AR}_k = \{G :$ every edge coloring of G with no color appears more than k times contains a heterochromatic Hamiltonian cycle$\}$.

Theorem 80 *[27] If we express m by n and k into $m = n(\log n + (2k-1)\log\log n + c_n)/2$ and $\lambda = e^{-c}$, then*

$$\begin{aligned}\lim_{n\to\infty} \mathbf{Pr}(G_{n,m} \in \mathcal{AR}_k) &= \begin{cases} 0 & c_n \to -\infty \\ \sum_{i=0}^{k-1} \frac{e^{-\lambda}\lambda^i}{i!} & c_n \to c \\ 1 & c_n \to \infty \end{cases} \\ &= \lim_{n\to\infty} \mathbf{Pr}(G_{n,m} \in \mathcal{B}_k),\end{aligned}$$

where $\mathcal{B}_k = \{G : G$ has at most $k-1$ vertices of degree less than $2k\}$.

Li and Zheng [65] investigated the problems of having a monochromatic matching of size k, clique of order k or tree of order k and having a heterochromatic matching of size k, clique of order k or tree of order k and obtained the threshold functions for them in the following probabilistic model of random graphs. Let K_n be the complete graph with vertex set $V = \{1, 2, \cdots, n\}$ and $1, 2, \cdots, r = r(n)$ be r different colors. Send $1, 2, \cdots, r$ to the edges of K_n randomly and equiprobably, which means each edge is colored in $i (1 \le i \le r)$ with probability $1/r$. Thus a random graph K_n^r is produced. This kind of random graph is somehow interesting, but the results obtained so far is not deep. The purpose to put forward the concept here is to hope more solid work coming later. Here we list some of the results in [65].

Theorem 81 *[65] If $r \geq 1$ is fixed, then almost every K_n^r has a monochromatic matching of size k for every $1 \leq k \leq n/2$. If $k \geq 2$ is fixed, then*

$$\left(\frac{n!}{(n-2k)!2^k k!}\right)^{\frac{1}{k-1}}$$

is the threshold function for the property that K_n^r has a monochromatic matching of size k.

Theorem 82 *[65] If $r \geq n/c_1$, where $c_1 < e$ is a constant, then almost no K_n^r has a monochromatic perfect matching. On the other hand, if $r \leq n/(\log n + c_2(n))$, where $c_2(n) \to \infty$, then almost every K_n^r has a monochromatic perfect matching.*

Theorem 83 *[65] If $1 \leq k \leq n^{1-\epsilon}$ and $k \leq r$, where $0 < \epsilon < 1$ is an arbitrarily small constant, then almost every K_n^r contains a heterochromatic matching of size k.*

Theorem 84 *[65] If $r \geq c_3 n$, where $c_3 > 1$ is a constant, then almost no K_n^r contains a monochromatic spanning tree.*

Theorem 85 *[65] If r is fixed, then almost every K_n^r contains a monochromatic tree of order k for any $2 \leq k \leq n$. If $2 \leq k \leq \log n$ and $r \geq k-1$, then almost every K_n^r contains a heterochromatic tree of order k.*

In [14], Bollobás and Erdős gave a conjecture on properly colored Hamiltonian cycle.

Conjecture 11 *[14] Every edge-colored complete graph K_n with minimum color degree at least $\lfloor n/2 \rfloor$ has a properly colored Hamiltonian cycle.*

In [7], Alon and Gutin got the following result, which is the best known result toward the conjecture for many years.

Theorem 86 *[7] Given $\varepsilon > 0$, there exists a positive integer n_0 such that for all $n > n_0$ and $m \leq (1 - 1/\sqrt{2} - \varepsilon)n$, every edge coloring of K_n with minimum color degree at least m has a properly colored Hamiltonian cycle.*

In [8], Alon, Jiang, Miller and Pritikin generalized Theorem 86.

Theorem 87 *[8] Let G be a graph with n vertices, maximum degree at most d, and suppose that $n > 216(3m+2d)^7(d+1)^{20}m$. Then every edge-colored K_n with minimum color degree at least m contains a properly colored copy of G.*

There are also many other results on proper-colored subgraphs, but most of them are of Ramsey type. For example, [8, 9] gave some Ramsey type results on proper-colored cycles, cliques, etc.

In [67], Manoussakis, Spyratos and Tuza investigated the existence of (s, t)-cycles, i.e., cycles of length $s+t$ in which s consecutive edges are colored red and the remaining t edges are colored blue in a 2-edge-colored complete graphs K_n, $n \geq 3$.

Theorem 88 *[67] Let s, t be positive integers such that $s \geq t$. If n is large enough, i.e., $n \geq n_0(s)$ for some function $n_0(s)$, then a 2-edge-colored complete graph K_n contains an (s, t)-cycle if, and only if*

(i) K_n contains a monochromatic path of length t in each color; and

(ii) in the case where both s and t are odd, K_n does not have a complete bipartite coloring, i.e., none of the two color classes induces in K_n a complete bipartite spanning subgraph.

Manoussakis, Spyratos and Tuza [67] investigated Theorem 6, by showing that there exists just one family of colorings for which no $(s, n-s)$-cycle occurs.

Theorem 89 *[67] Any 2-edge-colored complete graph K_n, $n \neq 5$, contains some Hamiltonian $(s, n-s)$-cycle C, $1 \leq s \leq n-1$, unless n is even and the edge coloring is a complete bipartite coloring with bipartition classes of equal size.*

Theorem 90 *[67] Let K_n be a 2-edge-colored complete graph admitting a monochromatic Hamiltonian cycle. Then K_n contains also an $(s, 1)$-cycle C, for*

(i) each even s, $2 \leq s \leq n-2$, and

(ii) each odd s, $3 \leq s \leq n-2$, unless n is even and the edge coloring is a complete bipartite coloring with bipartition classes of equal size.

Acknowledgement. The authors would like to thank Dr. He Chen for her careful checking of the manuscript. The authors are also grateful to the referees for helpful comments and suggestions.

References

[1] S. Akbari and A. Alipour, Multicolored trees in complete graphs, *J. Graph Theory* **54**(2006), 221–232.

[2] R.E.L. Aldred and N.C. Wormald, More on the linear k–arboricity of regular graphs, *Australas. J. Combin.* **18** (1998), 97–104.

[3] J. Akiyama, G. Exoo and F. Harary, Covering and packing in graphs III, Cyclic and acyclic invariants, *Math. Slovaca* **30** (1980), 405–417.

[4] M. Albert, A. Frieze and B. Reed, Multicoloured Hamilton cycles, *Electronic J. Combin.* **2**(1995), ♯R10.

[5] B. Alexeev, On lengths of rainbow cycles, *Electron. J. Combin.* **13**(2006), $\sharp R105$.

[6] N. Alon, R.A. Brualdi and B.L. Shader, Multicolored forests in bipartite decompositions of graphs, *J. Combin. Theory, Ser.B* **53**(1991), 143–148.

[7] N. Alon, G. Gutin, Properly colored Hamiltonian cycles in edge-colored complete graphs, *Random Structures Algorithms* **11**(1997) 179–186.

[8] N. Alon, T. Jiang, Z. Miller and D. Pritikin, Properly colored subgraphs and rainbow subgraphs in edge colorings with local constraints, *Random Structures Algorithms* **23**(2003), no. 4, 409–433.

[9] M. Axenovich, T. Jiang and Zs. Tuza, Local anti-Ramsey numbers of graphs, *Combin. Probab. Comput.* **12**(2003), no. 5–6, 495–511.

[10] J. Ayel, ”Sur l'existence de deux cycles supplémuntaires unicolorés disjoints et de couleurs différentes dans un graphe complet bicolore,” Thesis, University of Grenoble, 1979.

[11] J.C. Bermond, J.L. Fouquent, M. Habibi, and B. Ṕeroche, On linear k-arboricity, *Discrte Math.* **52** (1984) 123–132.

[12] A.Bialostocki and W. Voxman, On monochromatic-rainbow generalizations of two Ramsey type theorems. *Ars Combin.* **68** (2003), 131–142.

[13] J. Bierbrauer and P. Dierker, Monochromatic connected subgraphs in a multicoloring of the complete graph, Preprint.

[14] B. Bollobás and P. Erdős, Alternating Hamiltonian cycles, *Isr. J. Math.* **23**(1976) 126–131.

[15] H.J. Broersma and X. Li, Spanning trees with many or few colors in edge-colored graphs, *Discuss. Math. Graph Theory* **17**(1997), 259–269.

[16] H.J. Broersma, X. Li, G. Woeginger and S. Zhang, Paths and cycles in colored graphs, *Australasian J. Combin.* **31**(2005), 297–309.

[17] R.A. Brualdi and S. Hollingsworth, Multicolored trees in complete graphs, *J. Combin. Theory, Ser.B* **68**(1996), 310–313.

[18] R.A. Brualdi and S. Hollingsworth, Multicolored forests in complete bipartite graphs, *Disrete Math.* **240**(2001), 239–245.

[19] R.S. Chang and S.J. Leu, The minimum labelling spanning trees, *Inform. Process. Lett.* **63**(1997), 277–282.

[20] G. Chartrand and L. Lesniak, *Graphs and Combinatorics* (third edition), Chapman & Hall, New York (1996).

[21] H. Chen, Z.M. Jin, X. Li and J.H. Tu, Heterochromatic tree partition numbers for complete bipartite graphs, *Discrete Math.*, in press, doi:10.1016/j.disc.2007.07.085.

[22] H. Chen and X. Li, Long heterochromatic paths in edge-colored graphs, *Electron. J. Combin.* **12**(2005), ♯R33.

[23] H. Chen and X. Li, Color degree and color neighborhood union conditions for long heterochromatic paths in edge-colored graphs, arXiv:math.CO/0512144 v1 7 Dec 2005.

[24] H. Chen and X. Li, Color neighborhood union conditions for long heterochromatic paths in edge-colored graphs, *Electron. J. Combin.* **14**(2007), ♯R77.

[25] W.S. Chou, Y. Manoussakis, O. Megalalaki, M. Spyratos and Zs. Tuza, Paths through fixed vertices in edge-colored graphs, *Math. Inf. Sci. Hun.* **32**(1994), 49–58.

[26] G.M. Constantine, Colorful isomorphic spanning trees in complete graphs, *Ann. Combin.* **9**(2005), 163–167.

[27] C. Cooper and A. Frieze, Multicoloured Hamilton cycles in random graphs; an anti-Ramsey threshold, *Electron. J. Combin.* **2**(1995), ♯R19.

[28] J. Dénes and A.D. Keedwell, Latin Squres: New Developments in the Theory and Applications, *Ann. Discrete Math.* **46**, North-Holland, New York, 1991.

[29] S.I. El-Zanati, M.J. Plantholt, P.A. Sissokho and L.E. Spence, On the existence of rainbow 1-factor in 1-factorizaions of $K_{rn}^{(r)}$, J. Combin. Des., preprint.

[30] P. Erdős and F. Galvin, Monochromatic infinite paths, *Discrete Math.* **113**(1993), 59–70.

[31] P. Erdős and T. Gallai, On maximal paths and circuits of graphs, *Acta. Math. Acad. Sc. Hungar.* **10**(1959), 337–356.

[32] P. Erdős, A. Gyárfás and L. Pyber, Vertex coverings by monochromatic cycles and trees, *J. Combin. Theory Ser.B* **51**(1991), 90–95.

[33] P. Erdős, J. Nesetril and V. Rödl, Some problems related to partitions of edges of a graph, *Graphs and Other Combinatorial Topics*, Teubner, Leipzig (1983), 54–63.

[34] P. Erdős and Zs. Tuza, Rainbow Hamiltonian paths and canonically colored subgraphs in infinite complete graphs, *Math. Pannon* **1**(1990), no.1, 5–13.

[35] A. Frieze and B. Reed, Polychromatic Hamilton cycles, *Discrete Math.* **118**(1993), 69–74.

[36] Z. Füredi, Maximum degree and fractional matchings in uniform hypergraphs, *Combinatorica* **1**(1981), 155–162.

[37] Z. Füredi, Matchings and covers in hypergraphs, *Graphs Combin.* **4** (1988), 115–206.

[38] M.R. Garey and D.S. Johnson, *Computers and Intractability,* W.H. Freeman, San Francisco, 1979.

[39] L. Gerencsér and A. Gyárfás, On Ramsey-type problems, *Ann. Univ. Sci. Bud. de Rol. Eötvös Sect. Math.* **10**(1967), 167–170.

[40] A. Gyárfás, Partition coverings and blocking sets in hypergraphs (in Hungarian), *Tanulmányok-MTA Számitástechn. Automat. Kutató Int. Budapest* **71**(1977), 62 pp.

[41] A. Gyárfás, Vertex coverings by monochromatic paths and cycles, *J. Graph Theory* **7**(1983), 131–135.

[42] A. Gyárfás, Covering complete graphs by monochromatic paths, *in Irregularities of Partitions, Algorithms and Combinatorics, Springer-Verlag* **8** (1989), 89–91.

[43] A. Gyárfás, A. Jagota and R.H. Schelp, Monochromatic path covers in nearly complete graphs, *J. Combin. Math. Combin. Comput.* **25**(1997), 129–144.

[44] A. Gyárfás, M. Ruszinkó, G.N. Sárközy and E. Szemerédi, An improved bound for the monochromatic cycle partition number, *J. Combin. Thery Ser.B* **96**(2006) 855–873.

[45] A. Gyárfás, G. Simonyi, Edge colorings of complete graphs without tricolored triangles, *J. Graph Theory* **46**(2004), 211–216.

[46] G. Hahn and C. Thomassen, Path and cycle sub-Ramsey numbers and an edge coloring conjecture, *Discrete Math.* **62**(1986), 29–33.

[47] A. Hajnal, P. Komjáth, L. Soukup, and I. Szalkai, Decompositions of edge-colored infinite complete graphs, *Colloq. Math. Soc. János Bolyai* **52**(1987), 277–280.

[48] P.E. Haxell and Y. Kohayakawa, Partitioning by monochromatic trees, *J. Combin. Theory, Ser.B* **68**(1996), 218–222.

[49] P.E. Haxell, Partititoning complete bipartite graphs by monochromatic cycles, *J. Combin. Theory, Ser.B* **69**(1997), 210–218.

[50] L. Hu and X. Li, Sufficient conditions for the existence of perfect heterochromatic matchings in colored graphs, *Discrete Geometry, Combinatorics and Graph Theory, 7th China-Japan Conference, CJCDGCGT 2005, Lecture Notes in Computer Science* **Vol.4381** (2007).

[51] T. Jiang and D.B. West, Edge colorings of complete graphs that avoid polychromatic trees, *Discrete Math.* **274**(2005), 137–145.

[52] Z.M. Jin, M. Kano, X. Li and B. Wei, Partitioning 2-edge-colored complete multipartite graphs into monochromatic cycles, paths and trees, *J. Comb. Optim.* **11**(2006), 445–454.

[53] Z.M. Jin and X. Li, The complexity for partitioning graphs by monochromatic trees, cycles and paths, *Int. J. Comput. Math.* **81**(2004), 1357–1362.

[54] Z.M. Jin and X. Li, Vertex partitions of r-edge-colored graphs, *Appl. Math. J. Chinese Univ. Ser.B.* **23(1)**(2008), 120–126.

[55] Z.M. Jin and X. Li, Spanning trees with many colors in edge-colored graphs, *J. Xinjiang University*, **23**(Supp.)(2006), 12–14.

[56] Z.M. Jin and X. Li, Partitioning complete graphs by rainbow trees, arXiv:0711.2849 [math.CO] 19 Nov 2007.

[57] A. Kaneko, M. Kano and K. Suzuki, Three edge disjoint multicolored spanning trees in complete graphs, Preprint 2003.

[58] A. Kaneko, M. Kano and K. Suzuki, Partitioning complete multipartite graphs by monochromatic trees, *J. Graph Theory* **48**(2005), 133–141.

[59] E.L. Lawler, Combinatorial Optimization: Networks and Matroids, Holt, Tinehart and Winston, New York, 1976.

[60] H. Li, A generalization of Gallai-Roy theorem, *Graphs Combin.* **17**(2001), 681–685.

[61] H. Li, X. Li, G. Liu and G. Wang, The heterochromatic matchings in edge-colored bipartite graphs, *Ars Combin.*, in press.

[62] H. Li and G. Wang, Color degree and heterochromatic matchings in edge-colored bipartite graphs, *Utilitas Math.*, to appear.

[63] X. Li and Z.X. Xu, On the existence of a rainbow 1-factor in proper coloring of $K_{rn}^{(r)}$, arXiv:0711.2847 [math.CO] 19 Nov 2007.

[64] X. Li and X.Y. Zhang, On the minimum monochromatic or multicolored subgraph partition problems, *Theoretical Computer Science* **385**(2007), 1-10.

[65] X. Li and J. Zheng, Monochromatic and heterochromatic subgraphs problems in a randomly colored graph, Preprint 2005 (arXiv:0711.3827 [math.CO] 24 Nov 2007).

[66] T. Łuczak, V. Rödl and E. Szemerédi, Partitioning two-coloured complete graphs into two monochromatic cycles, *Probab. Combin. Comput.* **7**(1998), 423–436.

[67] Y. Manoussakis, M. Spyratos and Zs. Tuza, Cycles of given color paterns, *J. Graph Theory* **21**(1996), no.2, 153–162.

[68] J.J. Montellano-Ballesteros, An anti-Ramsey theorem on edge-cuts, *Discuss. Math. Graph Theory* **26**(2006), 19–21.

[69] L. Pyber, V. Rödl and E. Szemerédi, Dense graphs without 3-regular subgraphs, *J. Combin. Theory, Ser.B* **63**(1995), 41–54.

[70] R. Rado, Monochromatic paths in graphs, *Ann. Discrete Math.* **3**(1978), 197–194.

[71] V. Rödl and Zs. Tuza, Rainbow subgraphs in properly edge-colored graphs, *Random Structures Algorithms* **3**(1992), 175–182.

[72] H.J. Ryser, Neuere probleme der kombinatorik, in "Vortrage uber Kombinatorik Oberwolfach", *Mathematiches Forschungsinstitut Oberwolfach*, pp24–29, Juli 1967.

[73] G.N. Sárközy and S.M. Selkow, Vertex partitions by connected monochromatic k-regular graphs, *J. Combin. Theory, Ser.B* **78**(2000), 115–122.

[74] S.K. Stein, Transversals of Latin squares and their generalizations, *Pacific J. Math.* **59(2)**(1975), 567–575.

[75] K. Suzuki, A necessary and sufficient contidion for the existence of a heterochromatic spanning tree in a graph, *Graphs Combin.* **22**(2006), 261–269.

[76] C. Thomassen, Two-coloring the edges of a cubic graph such that each monochromatic component is a path of length at most 5, *J. Combin. Theory, Ser.B* **75**(1999), 100–109.

[77] Zs. Tuza, Some special cases of Rysers Conjecture (1978), Preprint.

[78] Hans-Christoph Wirth, Multicriteria approximation of network design and network upgrade problems, Chapter 2, Ph.D. Dissertation, Bayerischen Julius-Maximilians Universität, Wüzburg, March 2001.

[79] D.E. Woolbright and H.L. Fu, On the existence of rainbows in 1-factorizations of K_{2n}, *J. Combin. Des.* **6**(1998), 1–20.

[80] R. Yuster, Rainbow H-factors, *Electron. J. Combin.* **13**(2006), $\sharp R13$.