MEASURE EVOLUTION FOR "STOCHASTIC FLOWS"

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In this paper we study how σ-finite measures on R^d evolve under a class of "stochastic flows" associated to stochastic differential equations with (resp. without) jumps in R^d. First we show the related measure evolution processes are càdlàg (resp. continuous), strongly Markovian and weakly Fellerian. Then we extend the existing results on incompressibility in Harris [8] and Kunita [14], and prove strong Markov property of the process describing how compact subsets evolve under incompressible "stochastic flows" under a certain condition.[1]

Keywords: Measure evolution; stochastic flow; incompressibility; compact subset

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Notation. For any separable metric space E, the following notations will be used.

$\mathcal{M}(E)$ (*resp.* $\mathcal{P}(E)$) : *all finite* (*resp. probability*) *measures on* E *endowed with the weak topology.*

$C_b(E)$ (*resp.* $\mathcal{B}_b(E)$) : *all bounded continuous* (*resp. measurable*) *functions on* E.

$C_c(E)$: *all bounded continuous functions on* E *with compact supports.*

$\|\cdot\|$: *the uniform norm on* $\mathcal{B}_b(E)$.

$$f^{\otimes k} \ : \ f^{\otimes k}(x_1, \cdots, x_k) = \prod_{i=1}^{k} f(x_i), \ \forall (x_1, \cdots, x_k) \in E^k; \ \forall f \in \mathcal{B}_b(E), \ \forall k \geq 1.$$

μ^k : *the* $k-$*fold product of a measure* μ *on* E, $\forall\ k \geq 1$.

$\langle \mu, f \rangle = \mu(f) = \int_E f d\mu$: *the integral of a measurable function* f *against a measure* μ *on* E *provided it exists.*

$F_{f,k}(\mu) = \langle \mu^k, f \rangle = \mu^k(f)$: *for any measure* μ *on* E *and any measurable function* f *on* E^k *provided the integral exists.*

$C_p(\mathcal{P}(E))$ (*resp.* $\mathcal{B}_p(\mathcal{P}(E))$) : *all functions* F *on* $\mathcal{P}(E)$ *of forms* $F = F_{f,k}$, $f \in C_b\left(E^k\right) \left(resp.\ \mathcal{B}_b\left(E^k\right)\right),\ k \geq 1$.

D_E (*resp.* C_E) : *all càdlàg* (*resp. continuous*) *maps from* $[0, \infty)$ *to* E *endowed with the Skorohod* (*resp. compact* $-$ *open*) *topology.*

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1. Introduction

Recall a stochastic flow $(\psi_t)_{t\geq 0}$ of measurable maps for a càdlàg Markov process on R^d is a stochastic process on the space of measurable maps from R^d to R^d satisfying

$$\psi_{t+s}(x)(\omega) = \psi_t(\psi_s(x)(\omega))(\theta_s\omega) \ \ a.\ s.\ -\omega \ for\ any\ x \in R^d,\ t, s \geq 0,$$
$$the\ 1-point\ motion\ \left((\psi_t(x))_{t\geq 0}, x \in R^d\right)\ is\ the\ given\ càdlàg\ Markov\ process;$$

where $\theta = (\theta_s)_{s\geq 0}$ is a shift operator on some probability space and ψ_0 is the identity map. Assume further each k-point motion

$$\left((\psi_t(x_1), \cdots, \psi_t(x_k))_{t\geq 0}, (x_1, \cdots, x_k) \in \left(R^d\right)^k\right)$$

of $(\psi_t)_{t\geq 0}$ is a càdlàg Markov process of the semigroup $\left\{V_t^k\right\}_{t\geq 0}$. Clearly, each k-point motion of the flow is exchangeable in the sense that for any permutation τ of $\{1, \cdots, k\}$,

$$V_t^k \left[f \circ \widetilde{\tau}\right] \left(x_{\tau^{-1}(1)}, \cdots, x_{\tau^{-1}(k)}\right) = V_t^k[f](x_1, \cdots, x_k),\ \forall (x_1, \cdots, x_k) \in \left(R^d\right)^k,$$
$$\forall f \in \mathcal{B}_b\left(\left(R^d\right)^k\right),\ \forall t \geq 0,\ where\ f \circ \widetilde{\tau}\ (x_1, \cdots, x_k) = f(x_{\tau(1)}, \cdots, x_{\tau(k)});$$

and satisfies the consistency that each r $(\leq k)$-component of k-point motion is just the r-point motion, and any two particles in R^d must stay together whenever they meet. For such stochastic flows $(\psi_t)_{t\geq 0}$, refer to [14], [16], [17].

Given a consistent family of all exchangeable k-point motions in R^d with the property that any two particles must stay together whenever they meet, assume each k-point motion is of a Feller semigroup $\left\{V_t^k\right\}_{t\geq 0}$ on $C_0\left(\left(R^d\right)^k\right)$, namely,

$$V_t^k f \in C_0\left(\left(R^d\right)^k\right),\ \forall t \geq 0,\ \lim_{r\downarrow 0}\left\|V_r^k f - f\right\| = 0,\ \forall f \in C_0\left(\left(R^d\right)^k\right),$$

where $C_0\left(\left(R^d\right)^k\right)$ is the set of all continuous functions on $\left(R^d\right)^k$ vanishing at infinity; by [16], there is a unique (in law) stochastic flow $(\psi_t)_{t\geq 0}$ of measurable maps associated to the consistent exchangeable family and this correspondence is one-to-one (the conclusion holds on any locally compact separable metric space).

For any initial measure μ on R^d, its transportation under the flow $(\psi_t)_{t\geq 0}$ is given by $((\psi_t)_*\mu)_{t\geq 0}$, where $(\psi_t)_*\mu$ denotes the image measure of μ under ψ_t. For the time being, assume $\mu \in \mathcal{P}\left(R^d\right)$. Then $\left(((\psi_t)_*\mu)_{t\geq 0}, \mu \in \mathcal{P}\left(R^d\right)\right)$, which we call a measure-valued flow, is a càdlàg Markov process of the semigroup $\{T_t\}_{t\geq 0}$ determined by

$$T_t F_{f,k}(\mu) = F_{V_t^k f,k}(\mu),\ \forall \mu \in \mathcal{P}\left(R^d\right),\ \forall F_{f,k} \in \mathcal{B}_p\left(\mathcal{P}\left(R^d\right)\right). \tag{1.1}$$

See [18], [16]. Due to the exchangeability and consistency of any n-point motion, (1.1) does not depend on the expression of $F = F_{f,k}$.

Interests for studying $\left(((\psi_t)_*\mu)_{t\geq 0}, \mu \in \mathcal{P}\left(R^d\right)\right)$ are as follows.

From

$$\langle(\psi_t)_*\mu, f\rangle = \langle\mu, f \circ \psi_t\rangle,\ \forall f \in \mathcal{B}_b\left(R^d\right),\ \forall t \geq 0,\ \forall \mu \in \mathcal{P}\left(R^d\right);$$

we obtain that $\left(((\psi_t)_*\mu)_{t\geq 0}, \mu \in \mathcal{P}\left(R^d\right)\right)$ is a dual of the flow and hence of its own interests (refer to [14] P135-147 for some interests of the process). Moreover, there is a probabilistic notion, decay of correlations, expressing sensitivity of the dynamics, which is of importance in the characterization of complex systems ([20]); here sensitiveness means that orbits forget their initial state as time increases to ∞, which may be expressed by that

$$C_t^\mu(f, g) = \int_{R^d} f(z)(g \circ \psi_t)(z)\mu(dz) - \int_{R^d} f\ d\mu \int_{R^d} g\ d\mu$$

should converges rapidly to zero as $t \to \infty$, for any f, g in some continuous function space $\mathcal{F}$, and $\mu \in \mathcal{P}\left(R^d\right)$ (random or non-random) is a Sinai-Ruelle-Bowen (SRB) measure (note even for stochastic dynamical systems, SRB-measure may be deterministic, see [2]). Assume $\mu_f(dx) = f(x)\mu(dx) \in \mathcal{P}\left(R^d\right)$. Then by assumption, $\int_{R^d} f \; d\mu = 1$ and

$$C_t^\mu(f,g) = \int_{R^d} g(z)(\psi_t)_*\mu_f(dz) - \int_{R^d} g \; d\mu,$$

and the measure-valued flow $((\psi_t)_*\mu_f)_{t\geq 0}$ comes into picture.

Remove the restriction $\mu \in \mathcal{P}\left(R^d\right)$ and suppose μ is σ-finite. When R^d is endowed with a Riemannian structure, for a stochastic flow $(\psi_t)_{t\geq 0}$ of homeomorphisms (measurable maps) on R^d, its incompressibility is defined by

$$(\psi_t)_*\mu = \mu, \; \forall t \geq 0, \; a.s.,$$

where μ is the volume measure of the manifold R^d. Note incompressibility is important for vorticity and turbulence from a view point of physics ([7], [19]); and stochastic flows are usually viewed as turbulence models. Particularly, whether the Lebesgue measure on R^d is preserved by the stochastic flows is of important interests. However, R^d can be endowed with various Riemannian structures and various corresponding volume measures. Hence it is interesting to consider how σ-finite measures are transported under the stochastic flows. On the other hand, for d dimensional Brownian motion, its unique invariant measure (up to a constant) is the Lebesgue measure, in order to consider their entrance laws, stationary distributions and ergodicities, the space $\mathcal{M}\left(R^d\right)$ is small for Markov processes $\left(((\psi_t)_*\mu)_{t\geq 0}, \mu \in \mathcal{M}\left(R^d\right)\right)$ corresponding to stochastic flows for Brownian motions, which also suggests us to study measure-valued flows describing how σ-finite measures evolve under the stochastic flows.

For fixed $r > 0$, let $\phi_r(x) = \left(1+|x|^2\right)^{-r}, \; \forall x \in R^d$; and

$$\Phi\left(R^d\right) = \left\{ f \in C_b\left(R^d\right) \;\middle|\; \sup_{x\in R^d} \left|\frac{f(x)}{\phi_r(x)}\right| < \infty \right\}, \; \|f\|_{\Phi(R^d)} = \left\|\frac{f}{\phi_r}\right\|, \; \forall f \in \Phi\left(R^d\right).$$

Let $\mathcal{M}_r\left(R^d\right)$ be the set of all Radon measures μ on R^d satisfying $\langle \mu, \phi_r\rangle < +\infty$, and endow $\mathcal{M}_r\left(R^d\right)$ with the following τ_r topology:

$$\mu_n \Longrightarrow \mu \text{ if and only if } \lim_{n\to\infty} \langle \mu_n, f\rangle = \langle \mu, f\rangle, \; \forall f \in C_c\left(R^d\right) \cup \{\phi_r\}.$$

Note the Lebesgue measure is in $\mathcal{M}_r\left(R^d\right)$ if and only if $r > \frac{d}{2}$; and $\mathcal{M}_r\left(R^d\right)$ has been used as the state space for $(2, d, \beta)$ superprocesses with $\beta \in (0, 1]$ when $r > \frac{d}{2}$, and for (α, d, β) superprocesses with $\alpha \in (0, 2), \; \beta \in (0, 1]$ when $d + \alpha > r > \frac{d}{2}$ (refer to [4]).

As mentioned before, measure-valued flows describe how measures evolve under "stochastic flows" (see [14], [18], [16] for some existing results). Recently, stochastic differential equation (SDE) with jumps has been received much attention ([3]). In this paper we study SDE with jumps by measure-valued processes, namely, study related measure-valued flow which takes values in $\mathcal{M}_r\left(R^d\right)$.

Clearly, $(\psi_t)_*\mu$ is càdlàg in t for any $\mu \in \mathcal{M}\left(R^d\right)$. But for any $\mu \in \mathcal{M}_r\left(R^d\right) \setminus \mathcal{M}\left(R^d\right)$, due to we need to check

$$(\psi_t)_*\mu \in \mathcal{M}_r\left(R^d\right), \; t \in [0, \infty), \; a.s., \tag{1.2}$$

it is not obvious $(\psi_t)_*\mu$ is càdlàg in τ_r topology in t (note the dominated convergence theorem and (1.2) imply the càdlàg property).

Recall $\left(((\psi_t)_*\mu)_{t\geq 0}, \mu \in \mathcal{M}\left(R^d\right)\right)$ is Markovian because of (1.1) ([18], [16]). Let $\mathbf{F}$ be the space of measurable maps φ on R^d satisfying

$$\varphi_*\mu \in \mathcal{M}_r\left(R^d\right), \; \forall \mu \in \mathcal{M}_r\left(R^d\right),$$

and equip $\mathbf{F}$ with the σ-field generated by the maps $\varphi \to \varphi(x)$ for every $x \in R^d$. Even if $(\psi_t)_{t\geq 0}$ is Markovian, namely, for any bounded measurable function $\mathbf{f}$ on $\mathbf{F}$ and any $t,\ s \geq 0$,

$$E[\mathbf{f}(\psi_{t+s}) \mid \sigma(\psi_v, v \leq s)] = E[\mathbf{f}(\psi_{t+s}) \mid \psi_s], \tag{1.3}$$

because the following formulae may hold:

$$\sigma((\psi_v)_*\mu, v \leq s) \subseteq \sigma(\psi_v, v \leq s),\ \sigma((\psi_v)_*\mu, v \leq s) \neq \sigma(\psi_v, v \leq s),$$
$$\sigma((\psi_s)_*\mu) \subseteq \sigma(\psi_s),\ \sigma((\psi_s)_*\mu) \neq \sigma(\psi_s),\ where\ \mu \in \mathcal{M}_r\left(R^d\right) \setminus \mathcal{M}\left(R^d\right),$$

though $g(\varphi_*\mu):\ \ \varphi \in \mathbf{F} \to R^1$ is a bounded measurable function for any $g \in \mathcal{B}_b\left(\mathcal{M}_r\left(R^d\right)\right)$, one can not obtain directly from (1.3) that if each $(\psi_v)_*\mu$ is $\mathcal{M}_r\left(R^d\right)$-valued, then

$$E[g((\psi_{t+s})_*\mu) \mid \sigma((\psi_v)_*\mu, v \leq s)] = E[g((\psi_{t+s})_*\mu) \mid (\psi_s)_*\mu]. \tag{1.4}$$

In addition, the set of all functions of the forms $F_{f,k}$ on $\mathcal{M}_r\left(R^d\right)$ does not separate points in $\mathcal{M}_r\left(R^d\right)$, (1.4) can not be obtained directly from a formula like (1.1).

Therefore, we first show the $\mathcal{M}_r\left(R^d\right)$-valued measure-valued flows associated to SDE (2.5) with (resp. without) jumps in R^d are càdlàg (resp. continuous), strongly Markovian and weakly Fellerian. Then we extend the existing results on incompressibility in Harris [8] and Kunita [14] Theorem 4.3.2, and prove strong Markov property of the process describing how compact subsets evolve under incompressible "stochastic flows" under a certain condition (note for some diffeomorphism stochastic flow $(\psi_t)_{t\geq 0}$, in [11] an Itô formula is obtained for $(\psi_t(D))_{t\geq 0}$ for any smoothly bounded domain D). For details, see Theorem 2.1 of present paper. In section 3, we give remarks on Theorem 2.1. While in section 4, we present proof of Theorem 2.1.

2. Main result

Let U and U_0 ($\subseteq U$) be two measurable subsets in $R^d \setminus \{0\}$, and $n(du)$ a σ-finite measure on $R^d \setminus \{0\}$, and $h(\cdot,\cdot) = \left(h^1(\cdot,\cdot), \cdots, h^d(\cdot,\cdot)\right)$ an R^d-valued measurable map on $R^d \times U$; and

$$\sigma(\cdot) = \left(\sigma^{pq}(\cdot)\right)_{\substack{1\leq p\leq d\\ 1\leq q\leq l}} \text{ and } b(\cdot) = \left(b^1(\cdot), \cdots, b^d(\cdot)\right)$$

$d \times l$ matrix-valued and R^d-valued continuous maps on R^d respectively. Assume

$$n(U_0) < \infty;\ H(u) := \sup_{x\in R^d} \frac{|h(x,u)|}{\phi_r(x)} < \infty,\ \forall u \in U;\ \int_U H(u)^2\ n(du) < \infty; \tag{2.1}$$

$$\sup_{u\in U\setminus\mathcal{N}} H(u) < \infty \text{ for some } n(du) - \text{zero measurable subset } \mathcal{N} \subseteq U; \tag{2.2}$$

$$|\sigma(x)|^2 + |b(x)|^2 \leq C_1\left(1 + |x|^2\right), \tag{2.3}$$

$$\begin{aligned}&|\sigma(x) - \sigma(y)|^2 + |b(x) - b(y)|^2 + \int_{U\setminus U_0} |h(x,u) - h(y,u)|^2\ n(du)\\ &\quad \leq C_2^{(m)}|x-y|^2,\ \forall |x| \leq m, |y| \leq m,\ \forall m \in (0,\infty);\end{aligned} \tag{2.4}$$

where $C_1,\ C_2^{(m)}$ are constants. Consider the following SDE in R^d which has been studied extensively by [9].

$$\begin{cases} dX_t = & \sigma(X_t)\ dB_t + b(X_t)\ dt + \int_{U_0} h\left(X_{t-},u\right)\ \lambda(dtdu) + \\ & \int_{U\setminus U_0} h(X_{t-},u)\ \eta(dtdu); \\ X_0 = & x \in R^d. \end{cases} \tag{2.5}$$

Where $B = (B_t)_{t\geq 0} = \left(B_t^1, \cdots, B_t^l\right)_{t\geq 0}$ is the standard l-dimensional Brownian motion starting at 0; and $\lambda(dtdu)$ is the counting measure for a stationary Poisson point process $p = (p_t)_{t\geq 0}$ on

$R^d\backslash\{0\}$ which is of the σ-finite Lévy measure $n(du)$ on $R^d\backslash\{0\}$ and independent of $B=(B_t)_{t\geq 0}$, and measure $\eta(dtdu)$ on $[0,\infty)\times R^d\setminus\{0\}$ is given by

$$\eta(dtdu)=\lambda(dtdu)-dt\ n(du).$$

By (2.2) and (2.3), the linear growth condition

$$|\sigma(x)|^2+|b(x)|^2+\int_{U\backslash U_0}|h(x,u)|^2\ n(du)\leq C_3\left(1+|x|^2\right) \tag{2.6}$$

holds for some constant C_3. Remembering (2.4) and (2.6), by the standard SDE theory (c.f. [9]), we see SDE (2.5) has a unique strong solution $(X_t(x))_{t\geq 0}$ for any $x\in R^d$ such that

$$X_t(x) \text{ is measurable in } x\in R^d \text{ for any } t\geq 0,\ a.s.;$$

$X_t(y)$ tends to $X_t(x)$ in probability as $y\to x$ for any $t\geq 0$; and $\left((X_t(x))_{t\geq 0}, x\in R^d\right)$ is a càdlàg (resp. continuous) strong Markov process (resp. when $n=0$) with the following weak generator: For any $f\in C_b^2\left(R^d\right)$ and $x=\left(x^1,\cdots,x^d\right)\in R^d$,

$$\begin{aligned}A_1f(x) &= \frac{1}{2}\sum_{i,j=1}^d a^{ij}(x)\frac{\partial^2 f(x)}{\partial x^i\partial x^j}+\sum_{i=1}^d b^i(x)\frac{\partial f(x)}{\partial x^i}+\\ &\int_{U\backslash U_0}\left[f\left(x+h(x,u)\right)-f(x)-\nabla f(x)\cdot h(x,u)\right]\ n(du)+\\ &\int_{U_0}\left[f\left(x+h(x,u)\right)-f(x)\right]\ n(du);\end{aligned}$$

where for any natural number k, $C_b^2\left(R^k\right)$ is the set of all bounded continuous functions on R^k with bounded continuous derivatives of orders one and two;

$$a(x):=\left(a^{ij}(x)\right)_{1\leq i,j\leq d}=\sigma(x)\sigma^T(x),\ \forall x\in R^d,$$

$\sigma^T(x)$ is the transpose of $\sigma(x)$; ∇ is the gradient operator. Clearly, $\left((X_t(x))_{t\geq 0}, x\in R^d\right)$ is weakly Fellerian in the sense that its semigroup $\{V_t^1\}_{t\geq 0}$ satisfies

$$V_t^1 f\in C_b\left(R^d\right),\ \lim_{s\downarrow 0}V_s^1 f(x)=f(x),\ \forall t\geq 0,\ \forall x\in R^d,\ \forall f\in C_b\left(R^d\right).$$

To state our main Theorem, we need some preliminaries. Given a measure μ on R^d, we say $(X_t)_{t\geq 0}$ is μ-incompressible if

$$(X_t)_*\mu=\mu,\ \forall t\in[0,\infty),\ a.s.;$$

and use $S(\mu)$ to denote the support of measure μ ($S(0)=\emptyset$). **Here we point out,** in [8] and [14], μ-incompressibility means that for any fixed t, $(X_t)_*\mu=\mu,\ a.s.$. Refer to [8] P236, [14] P135 and P139-140 (the proof "(v) implies (i)" of Theorem 4.3.2 in [14]). Let $\Im\left(R^d\right)$ be the set of all closed subsets of R^d endowed with the Hausdorff topology, which is induced by the following metric on $\Im\left(R^d\right)$:

$$\rho(A,B)=\begin{cases}\max\left\{\max\limits_{x\in A}\inf\limits_{y\in B}\frac{|x-y|}{1+|x-y|},\ \max\limits_{y\in B}\inf\limits_{x\in A}\frac{|x-y|}{1+|x-y|}\right\}, & \forall A,B\in\Im\left(R^d\right)\setminus\{\emptyset\},\\ 1, & \forall A\in\Im\left(R^d\right)\setminus\{\emptyset\},\ B=\emptyset,\\ 0, & A=B=\emptyset.\end{cases}$$

For any closed subset $\mathcal{C}$ of R^d, let $\mathcal{K}_{\mathcal{C}}$ be the space of all compact subsets in $\mathcal{C}$. Then $\mathcal{K}_{\mathcal{C}}$ is a closed subset of $\Im\left(R^d\right)$.

Since $\mu \in \mathcal{M}_r\left(R^d\right) \to \phi_r(x)\mu(dx) \in \mathcal{M}\left(R^d\right)$ is a topological homeomorphism,

$$S: \ \mu \in \mathcal{M}\left(R^d\right) \to S(\mu) \in \Im\left(R^d\right) \ \textit{is measurable} \ ([4] \ \textit{Theorem} \ 9.3.1.2(a)),$$

and $S(\mu) = S(\phi_r(x)\mu(dx))$; $S: \ \mu \in \mathcal{M}_r\left(R^d\right) \to S(\mu) \in \Im\left(R^d\right)$ is measurable.

Let $\mathcal{H}$ be the set of all bounded continuous functions F on $\mathcal{M}_r\left(R^d\right)$ such that

$$F(\mu) = G(\langle \mu, f_1\rangle, \cdots, \langle \mu, f_k\rangle) \ \textit{with} \ G \in C_c^2\left(R^k\right), \ k \geq 1, \ f_i \in C_c^2\left(R^d\right), \ 1 \leq i \leq k,$$
$$\textit{and} \ C_c^2\left(R^k\right) = C_c\left(R^k\right) \cap C_b^2\left(R^k\right).$$

For any $x_i = (x_i^1, \cdots, x_i^d) \in R^d, \ i = 1, 2$, and any $f \in C_b^2\left(\left(R^d\right)^2\right)$, let

$$\begin{aligned} A_2 f(x_1, x_2) = &\frac{1}{2}\sum_{i,j=1}^{2}\sum_{p,q=1}^{d} a^{pq}(x_i, x_j)\frac{\partial^2 f}{\partial x_i^p \partial x_j^q}(x_1, x_2) + \sum_{i=1}^{2}\sum_{p=1}^{d} b^p(x_i)\frac{\partial f}{\partial x_i^p}(x_1, x_2) + \\ &\int_{U\setminus U_0} \left[f\left(x_1 + h(x_1, u), x_2 + h(x_2, u)\right) - f(x_1, x_2) - \right. \\ &\left. \nabla f(x_1, x_2) \cdot (h(x_1, u), h(x_2, u))\right] \ n(du) + \\ &\int_{U_0} \left[f\left(x_1 + h(x_1, u), x_2 + h(x_2, u)\right) - f(x_1, x_2)\right] \ n(du); \end{aligned}$$

where

$$a^{pq}(x, y) = \sum_{k=1}^{l} \sigma^{pk}(x)\sigma^{kq}(y), \ \forall (x, y) \in \left(R^d\right)^2, \ \forall 1 \leq p, q \leq d.$$

Write

$$a(x, y) = \left(a^{pq}(x, y)\right)_{1 \leq p,q \leq d}, \ \forall (x, y) \in \left(R^d\right)^2.$$

Now our main Theorem is stated as follows:

Theorem 2.1. **(i)** $\left(((X_t)_*\mu)_{t\geq 0}, \ \mu \in \mathcal{M}_r\left(R^d\right)\right)$ is the unique (in law) càdlàg (resp. continuous) $\mathcal{M}_r\left(R^d\right)$-valued weakly Fellerian and strongly Markovian process (resp. when $n = 0$) satisfying its weak generator A is determined by that for any $F(\mu) = G(\langle \mu, f_1\rangle, \cdots, \langle \mu, f_k\rangle) \in \mathcal{H}$,

$$\begin{aligned} AF(\mu) \ = \ &\sum_{i=1}^{k} \partial_i G(\langle \mu, f_1\rangle, \cdots, \langle \mu, f_k\rangle)\langle \mu, A_1 f_i\rangle + \\ &\frac{1}{2}\sum_{i,j=1}^{k} \partial_{ij}^2 G(\langle \mu, f_1\rangle, \cdots, \langle \mu, f_k\rangle)\left\langle \mu^2, \nabla f_i(x) a(x, y)\left(\nabla f_j(y)\right)^T\right\rangle + \\ &\int_U \{G(\langle \mu, f_1(x + h(x, u))\rangle, \cdots, \langle \mu, f_k(x + h(x, u))\rangle) - G(\langle \mu, f_1\rangle, \cdots, \langle \mu, f_k\rangle) - \\ &\quad \sum_{i=1}^{k} \partial_i G(\langle \mu, f_1\rangle, \cdots, \langle \mu, f_k\rangle)\langle \mu, f_i(x + h(x, u)) - f_i(x)\rangle\} \ n(du), \end{aligned}$$

and $\left\{\mu \in \mathcal{M}\left(R^d\right) \ \middle| \ \mu\left(R^d\right) = c\right\}$ is an invariant sub state space for any $c \in [0, \infty)$. For the mentioned process, $F((X_t)_*\mu) - F(\mu) - \int_0^t AF((X_s)_*\mu) \ ds, \ t \in [0, \infty)$, is an L^2-martingale for any $F \in \mathcal{H}$ and any $\mu \in \mathcal{M}_r\left(R^d\right)$.

(ii) Given any $\mu \in \mathcal{M}_r\left(R^d\right) \setminus \{0\}$, $(X_t)_{t\geq 0}$ is μ-incompressible if and only if

$$\langle \mu, A_1 f\rangle = 0, \ \left\langle \mu^2, A_2 f^{\otimes 2}\right\rangle = 0, \ \forall f \in C_c^2\left(R^d\right).$$

(iii) If

$$\lim_{\substack{s\downarrow t\\ y\to x}} X_s(y) = X_t(x),\ \lim_{\substack{s\uparrow t\\ y\to x}} X_s(y) = X_{t-}(x),\ \forall (t,x)\in[0,\infty)\times R^d,\ a.s.;$$

and $a.s.$, X_v is a continuous injection for any fixed $v\geq 0$; and $(X_t)_{t\geq 0}$ is μ-incompressible for some $\mu\in\mathcal{M}_r\left(R^d\right)\setminus\{0\}$; then $\left((X_t(D))_{t\geq 0}, D\in\mathcal{K}_{S(\mu)}\right)$ is a càdlàg (resp. continuous) $\mathcal{K}_{S(\mu)}$-valued strong Markov process (resp. when $n=0$).

Remark 2.2. Theorem 2.1 still holds if (2.1) is replaced by

$$n(U)<\infty,\ h\in\mathcal{B}_b\left(R^d\times U\right) \text{ and } h(y,\cdot)\to h(x,\cdot) \text{ in measure } n \text{ as } y\to x \text{ in } R^d. \qquad (2.1)'$$

Though the cocycle property is needed to define flows of maps, it was not shown to hold and not used in proving Theorem 2.1. The reason is that the law of the measure-valued flow does only depend on the laws of all k-point motions (for example, see (1.1)).

Remark 2.3. Note the unique strong solution to SDEs can generate a stochastic flow of measurable maps under a certain condition. For Theorem 2.1, the weak generator A_k of the k-point motion $\left((X_t(x_1),\cdots,X_t(x_k))_{t\geq 0},(x_1,\cdots,x_k)\in\left(R^d\right)^k\right)$ is specified in Lemma 4.1 with

$$a(x,y)=(a^{pq}(x,y))_{1\leq p,q\leq d}=\sigma(x)(\sigma(y))^T,\ \forall (x,y)\in\left(R^d\right)^2.$$

For more general infinitesimal covariance $a(\cdot,\cdot)$ on $\left(R^d\right)^2$ which is of C^2-class and satisfies the linear growth condition, if there is a unique (in law) stochastic flow $(X_t)_{t\geq 0}$ of measurable maps with A_k-process as its k-point motion for any $k\geq 1$, here A_k is of the form specified in Lemma 4.1 for this more general $a(\cdot,\cdot)$; then Theorem 2.1 is still true for this $(X_t)_{t\geq 0}$.

For certain diffeomorphism stochastic flows $(\psi_t)_{t\geq 0}$ on R^d, in [11] an Itô formula is obtained for domain-valued process $(\psi_t(D))_{t\geq 0}$, for any smoothly bounded domain D in R^d. Theorem 2.1 gives strong Markov property of the càdlàg (resp. continuous) process (resp. when $n=0$) describing how compact subsets evolve under incompressible "stochastic flows", such a result does not appear before and its applications are given in section 3.

Recall μ-incompressibility of $(X_t)_{t\geq 0}$ in [8] and [14] means for any fixed t, $(X_t)_*\mu=\mu,\ a.s.$. However, for any $\mu\in\mathcal{M}_r\left(R^d\right)$ with $r>\frac{d}{2}$, the μ-incompressible flows $(X_t)_{t\geq 0}$ in [8] and [14] Theorem 4.3.2 in fact satisfy

$$(X_t)_*\mu=\mu,\ \forall t\in[0,\infty),\ a.s.,\ \textit{because } (X_t)_*\mu \textit{ is continuous in } t\in[0,\infty).$$

Note in [14] Theorem 4.3.2, the infinitesimal covariance $a(\cdot,\cdot)$ is of C^3-class, the equivalences on incompressibility for measures $\Pi(dx)=\pi(x)\,dx$ with that $\pi(x)$ is of C^3-class and strictly positive are obtained. In this paper, even if $\pi(\cdot)$ (not necessarily strictly positive) and $a(\cdot,\cdot)$ are of C^2-class, the corresponding equivalences on incompressibility for $\Pi(dx)=\pi(x)\,dx\in\mathcal{M}_r\left(R^d\right)\setminus\{0\}$ still hold. Refer to Remarks 3.2 and 3.2′ behind. For incompressible flows in [8] and [14] Theorem 4.3.2, the process describing how compact subsets evolve under the flows is a diffusion by Theorem 2.1.

Remark 2.4. There are many papers and results on stochastic flows and associated measure-valued flows which go beyond Kunita's results. In [14] (as well as [16] and this paper), the solutions to the SDEs do not depend on the mass of the flow. In contrast, many authors consider flows of SDEs driven by general Brownian noise (including infinitely many Brownian motions) where the coefficients depend on the total mass of the flow which were first studied by [12]; and accordingly, the associated measure-valued flow is the solution to a quasilinear stochastic partial differential equation (SPDE) (see [12], [15], [5], [6], [13] and references therein).

If one considers infinitely many Brownian motions as driving terms, one obtains the same results (under suitable local Lipschitz and linear growth conditions) including Kunita's flows as a special case. Theorem 2.1 is not in its most possible form. What we emphasize is that measure-valued processes and "flows" can make valuable contributions to each other, and ideas used to study "flows" by measure-valued processes.

3. Remarks on Theorem 2.1

Remark 3.1. Let $n = 0$ and $r > 1$. Then the 2-dimensional Lebesgue measure dx is in $\mathcal{M}_r\left(R^2\right)$. Note the k-point motion of Brownian flows $(Z_t)_{t\geq 0}$ (not necessarily stochastic flows for Brownian motions) in [8] (see [8] (3.4)) satisfies Lemma 4.3 in this paper. Then $\left(((Z_t)_*\mu)_{t\geq 0}, \mu \in \mathcal{M}_r\left(R^2\right)\right)$ is a continuous strongly Markovian and weakly Fellerian process on $\mathcal{M}_r\left(R^2\right)$. So any incompressible flow $(Z_t)_{t\geq 0}$ of [8] in the sense that Z_t preserves the Lebesgue measure for any fixed $t \geq 0$ in fact satisfies

$$(Z_t)_* dx = dx,\ \forall t \geq 0,\ a.s..$$

Recall from [8], $Z_t(x)$ is continuous in $(t, x) \in [0, \infty) \times R^2$ and $Z_t(\cdot)$ is a homeomorphism of R^2 onto itself for all $t \geq 0$. Hence, $((Z_t(D))_{t\geq 0}, D \in \mathcal{K}_{R^d})$ is a $\mathcal{K}_{R^d}$-valued diffusion process for any incompressible Brownian flow $(Z_t)_{t\geq 0}$ in [8].

Remark 3.2. Let $(\varphi_t)_{t\geq 0}$ be the Brownian flow of diffeomorphisms on R^d specified in [14] Theorem 4.3.2. Let $\Pi(dx) = \pi(x)\ dx$ with that $\pi(x)$ is of C^3-class and strictly positive. Then [14] Theorem 4.3.2 (c.f. Harris [8]) states the following statements are equivalent.

(a) $(\varphi_t)_*\Pi = \Pi,\ a.s.$ for any $t \geq 0$.

(b) $\displaystyle\sum_{i=1}^{d} \frac{\partial}{\partial x^i}\left\{\pi(x)a^{ij}(x,y)\right\} = 0,\ \forall 1 \leq j \leq d,\ \forall (x,y) \in \left(R^d\right)^2,$

$div\{\pi(b-c)\}(x) = 0,\ \forall x \in R^d.$

(c) Π and $\Pi \times \Pi$ are invariant measures of $A_1 -$ diffusion process and $A_2 -$ diffusion process respectively.

Where $a(x,y) = \left(a^{ij}(x,y)\right)_{1\leq i,j\leq d}$ and $b(x) = \left(b^1(x), \cdots, b^d(x)\right)$ are the infinitesimal covariance and infinitesimal mean of $(\varphi_t)_{t\geq 0}$ respectively; and

$$c(x) = \left(c^1(x), \cdots, c^d(x)\right),\ c^i(x) = \frac{1}{2}\left\{\sum_{j=1}^{d} \frac{\partial a^{ij}(x,y)}{\partial x^j}\bigg|_{y=x}\right\},\ \forall 1 \leq i \leq d;$$

$$A_k f(x_1, \cdots, x_k) = \frac{1}{2}\sum_{i,j=1}^{k}\sum_{p,q=1}^{d} a^{pq}(x_i, x_j)\frac{\partial^2 f}{\partial x_i^p \partial x_j^q}(x_1, \cdots, x_k) + \sum_{i=1}^{k}\sum_{p=1}^{d} b^p(x_i)\frac{\partial f}{\partial x_i^p}(x_1, \cdots, x_k),$$

$$\forall (x_1, \cdots, x_k) \in \left(R^d\right)^k,\ \forall f \in C_b^2\left(\left(R^d\right)^k\right),\ \forall k \geq 1.$$

Remember $\left(((\varphi_t)_*\mu)_{t\geq 0}, \mu \in \mathcal{M}_r\left(R^d\right)\right)$ is an $\mathcal{M}_r\left(R^d\right)$-valued diffusion process of weak Feller property. **We claim** if $\pi(x)$ (not necessarily strictly positive) is of C^2-class and

$$\Pi(dx) = \pi(x)\ dx \in \mathcal{M}_r\left(R^d\right) \setminus \{0\},$$

then the following statements are equivalent.

$(a)'$ $(\varphi_t)_*\Pi = \Pi,\ \forall t \geq 0,\ a.s..$

$(b)'$ $\displaystyle\sum_{i=1}^{d} \frac{\partial}{\partial x^i}\left\{\pi(x)a^{ij}(x,y)\right\} = 0,\ \forall 1 \leq j \leq d,\ \forall (x,y) \in \left(R^d\right)^2,$

$div\{\pi(b-c)\}(x) = 0,\ \forall x \in R^d.$

$(c)'$ $\langle \Pi, A_1 f\rangle = 0,\ \left\langle \Pi^2, A_2 f^{\otimes 2}\right\rangle = 0,\ \forall f \in C_c^2\left(R^d\right).$

(even if $a(x,y)$ is of $C^2 -$ class, the equivalences still hold)

For such a Π-incompressible flow $(\varphi_t)_{t\geq 0}$, $\left((\varphi_t(D))_{t\geq 0}, D \in \mathcal{K}_{S(\Pi)}\right)$ is a diffusion on $\mathcal{K}_{S(\Pi)}$.

In fact, the proof of Theorem 2.1 shows that $(a)'$ and $(c)'$ are equivalent (c.f. Lemma 4.7). It suffices to prove $(c)'$ and $(b)'$ are equivalent. Integrating by parts, it is easy to see that $\langle \Pi, A_1 f\rangle = 0,\ \forall f \in C_c^2\left(R^d\right)$ is equivalent to

$$\frac{1}{2}\sum_{p,q=1}^{d} \frac{\partial^2[\pi(x)a^{pq}(x,x)]}{\partial x^p \partial x^q} - div(\pi b)(x) = 0.$$

However,

$$\begin{aligned}
&\frac{1}{2}\sum_{p,q=1}^{d} \frac{\partial^2[\pi(x)a^{pq}(x,x)]}{\partial x^p \partial x^q} - div(\pi b)(x) \\
&= \frac{1}{2}\sum_{p=1}^{d}\frac{\partial}{\partial x^p}\sum_{q=1}^{d}\left\{a^{pq}(x,x)\frac{\partial \pi(x)}{\partial x^q} + \left.\frac{\partial a^{pq}(y,x)}{\partial x^q}\right|_{y=x}\pi(x)\right\} + \\
&\quad div(\pi(c-b))(x) \\
&= \frac{1}{2}\sum_{p=1}^{d}\frac{\partial}{\partial x^p}\sum_{q=1}^{d}\left\{a^{qp}(x,x)\frac{\partial \pi(x)}{\partial x^q} + \left.\frac{\partial a^{qp}(x,y)}{\partial x^q}\right|_{y=x}\pi(x)\right\} + \\
&\quad div(\pi(c-b))(x) \\
&\quad \left(\text{since } a^{pq}(x,y) = a^{qp}(y,x),\ \forall 1\leq p,q\leq d,\ \forall (x,y)\in \left(R^d\right)^2\right) \\
&= \frac{1}{2}\sum_{p=1}^{d}\frac{\partial}{\partial x^p}\left\{\sum_{q=1}^{d}\frac{\partial}{\partial x^q}\left.\{\pi(x)a^{qp}(x,y)\}\right|_{y=x}\right\} + div(\pi(c-b))(x).
\end{aligned}$$

So $\langle \Pi, A_1 f\rangle = 0,\ \forall f \in C_c^2\left(R^d\right)$ is equivalent to

$$\frac{1}{2}\sum_{p=1}^{d}\frac{\partial}{\partial x^p}\left\{\sum_{q=1}^{d}\frac{\partial}{\partial x^q}\left.\{\pi(x)a^{qp}(x,y)\}\right|_{y=x}\right\} + div(\pi(c-b))(x) = 0.$$

Note for any $f \in C_b^2\left(R^d\right)$,

$$A_2 f^{\otimes 2}(x_1,x_2) = f(x_1)A_1 f(x_2) + f(x_2)A_1 f(x_1) + \sum_{p,q=1}^{d} a^{pq}(x_1,x_2)\frac{\partial f(x_1)}{\partial x_1^p}\frac{\partial f(x_2)}{\partial x_2^q},$$

$(c)'$ is equivalent to

$$\langle \Pi, A_1 f\rangle = 0,\ \left\langle \Pi^2, \sum_{p,q=1}^{d} a^{pq}(x_1,x_2)\frac{\partial f(x_1)}{\partial x_1^p}\frac{\partial f(x_2)}{\partial x_2^q}\right\rangle = 0,\ \forall f\in C_c^2\left(R^d\right).$$

In the situation of [14] Theorem 4.3.2, $a(x,y)$ is of C^3-class. Let $\left(\left(G^1(x),\cdots,G^d(x)\right)\right)_{x\in R^d}$ be the centered R^d-valued Gaussian random field with the covariance matrix $a(x,y)$. Then by the standard Gaussian random field theory (refer to [1] Theorem 1.4.2 or reproducing kernel Hilbert space method in [1] Chapter 3), one can get each $G^i(x)$ is of C^3-class in x. Therefore, by integration by parts,

$$\left\langle \Pi^2, \sum_{p,q=1}^{d} a^{pq}(x_1,x_2)\frac{\partial f(x_1)}{\partial x_1^p}\frac{\partial f(x_2)}{\partial x_2^q}\right\rangle = 0,\ \forall f\in C_c^2\left(R^d\right)$$

is equivalent to

$$
\begin{aligned}
0 &= \int_{(R^d)^2} f(x_1)f(x_2) \sum_{p,q=1}^{d} \frac{\partial^2}{\partial x_1^p \partial x_2^q} \left[\pi(x_1)\pi(x_2)a^{pq}(x_1,x_2)\right] \ dx_1dx_2 \\
&= \int_{(R^d)^2} f(x_1)f(x_2) \sum_{p,q=1}^{d} \frac{\partial^2}{\partial x_1^p \partial x_2^q} \left[\pi(x_1)\pi(x_2)E\left[G^p(x_1)G^q(x_2)\right]\right] \ dx_1dx_2 \\
&= E\left[\int_{(R^d)^2} f(x_1)f(x_2) \sum_{p,q=1}^{d} \frac{\partial^2}{\partial x_1^p \partial x_2^q} \left[\pi(x_1)\pi(x_2)G^p(x_1)G^q(x_2)\right] \ dx_1dx_2\right] \\
&\qquad \text{(c.f. [1] Theorem 1.4.2)} \\
&= E\left[\left(\int_{R^d} f(x_1) \sum_{p=1}^{d} \frac{\partial}{\partial x_1^p}\left[\pi(x_1)G^p(x_1)\right] \ dx_1\right)^2\right], \ \forall f \in C_c^2\left(R^d\right) \\
\Longleftrightarrow & \sum_{p=1}^{d} \frac{\partial}{\partial x_1^p}\left[\pi(x_1)G^p(x_1)\right] = 0, \ \forall x_1 \in R^d, \ a.s.;
\end{aligned}
$$

which implies that

$$
\begin{aligned}
0 &= E\left[G^q(x_2)\sum_{p=1}^{d} \frac{\partial}{\partial x_1^p}\left[\pi(x_1)G^p(x_1)\right]\right] = E\left[\sum_{p=1}^{d} \frac{\partial}{\partial x_1^p}\left[\pi(x_1)G^p(x_1)G^q(x_2)\right]\right] \\
&= \sum_{p=1}^{d} \frac{\partial}{\partial x_1^p} E\left[\pi(x_1)G^p(x_1)G^q(x_2)\right] \ \text{(c.f. [1] Theorem 1.4.2)} \\
&= \sum_{p=1}^{d} \frac{\partial}{\partial x_1^p}\left[\pi(x_1)a^{pq}(x_1,x_2)\right], \ \forall (x_1,x_2) \in \left(R^d\right)^2.
\end{aligned}
$$

So $(c)'$ implies $(b)'$. From deduction above, one easily see $(b)'$ implies $(c)'$.

Remark 3.2′. Let $n = 0$ in Theorem 2.1. If $a(x,y) = \sigma(x)\sigma(y)^T$, σ is of C_b^2-class and b is of C_b^1-class, then similarly to Remark 3.2 (using $a(x,y) = \sigma(x)\sigma(y)^T$ directly and without using Gaussian random fields), one can check for $\Pi(dx) = \pi(x)\ dx \in \mathcal{M}_r\left(R^d\right) \setminus \{0\}$ with $\pi(x)$ of C^2-class, the following statements are equivalent.

$(a)''$ $(X_t)_*\Pi = \Pi, \ \forall t \geq 0, \ a.s..$

$(b)''$ $\displaystyle\sum_{i=1}^{d} \frac{\partial}{\partial x^i}\left\{\pi(x)a^{ij}(x,y)\right\} = 0, \ \forall 1 \leq j \leq d, \ \forall (x,y) \in \left(R^d\right)^2,$

$div\{\pi(b-c)\}(x) = 0, \ \forall x \in R^d.$

$(c)''$ $\langle \Pi, A_1 f\rangle = 0, \ \left\langle \Pi^2, A_2 f^{\otimes 2}\right\rangle = 0, \ \forall f \in C_c^2\left(R^d\right).$

For such a Π-incompressible flow $(X_t)_{t\geq 0}$, $\left((X_t(D))_{t\geq 0}, D \in \mathcal{K}_{S(\Pi)}\right)$ is a $\mathcal{K}_{S(\Pi)}$-valued diffusion.

Remark 3.3. In the situation of Theorem 2.1, let

$$
\begin{aligned}
&\sigma(x) \equiv 0, \ b(x) \equiv 0; \ U_0 = U, \ 0 < n(U) < \infty; \\
&h(x,u) \equiv e \text{ for some vector } e = (e^1, \cdots, e^d) \in R^d \setminus \{0\}; \\
&0 \leq \pi(x), \ \Pi(dx) = \pi(x)\ dx \in \mathcal{M}_r\left(R^d\right) \setminus \{0\}.
\end{aligned}
$$

Then $\left(((X_t)_*\mu)_{t\geq 0}, \mu \in \mathcal{M}_r\left(R^d\right)\right)$ is an $\mathcal{M}_r\left(R^d\right)$-valued càdlàg strongly Markovian and weakly Fellerian process. **We assert** that

$$
\lim_{\substack{s\downarrow t \\ y\to x}} X_s(y) = X_t(x), \ \lim_{\substack{s\uparrow t \\ y\to x}} X_s(y) = X_{t-}(x), \ \forall (t,x) \in [0,\infty) \times R^d, \ a.s.;
$$

and $a.s.$, X_v is a continuous injection for any fixed $v \geq 0$. Moreover,

$$\langle \Pi, A_1 f \rangle = 0, \ \left\langle \Pi^2, A_2 f^{\otimes 2} \right\rangle = 0, \ \forall f \in C_c^2\left(R^d\right)$$

if and only if

$$\pi(y) = \pi(y+e), \ dy \ - a.s. \ y \in R^d.$$

Hence, $\left((X_t(D))_{t\geq 0}, D \in \mathcal{K}_{S(\Pi)}\right)$ is a càdlàg $\mathcal{K}_{S(\Pi)}$-valued strong Markov process when

$$\pi(y) = \pi(y+e), \ dy \ - a.s. \ y \in R^d.$$

More generally, if

$$\begin{aligned} & h(x,u) = h(u), \ h \in \mathcal{B}_b(U), \ \{h(u) \mid u \in U\} \subseteq \{ke \mid k = 0, \pm 1, \pm 2, \cdots\}, \\ & \pi(y) = \pi(y+e), \ dy \ - a.s. \ y \in R^d, \end{aligned}$$

then $(X_t)_{t\geq 0}$ is Π-incompressible and $\left((X_t(D))_{t\geq 0}, D \in \mathcal{K}_{S(\Pi)}\right)$ is a càdlàg $\mathcal{K}_{S(\Pi)}$-valued strong Markov process.

Indeed, the first part of assertion follows easily from the SDE

$$dX_t(x) = \int_U e \ \lambda(dtdu), \ X_0(x) = x.$$

Since for any $f \in C_c^2\left(R^d\right)$,

$$\begin{aligned} \langle \Pi, A_1 f \rangle &= \int_{R^d} \int_U [f(x+e) - f(x)] \ n(du) \ \pi(x) \ dx \\ &= n(U) \int_{R^d} f(x)[\pi(x-e) - \pi(x)] \ dx, \\ \left\langle \Pi^2, A_2 f^{\otimes 2} \right\rangle &= n(U) \int_{(R^d)^2} f(x_1) f(x_2) [\pi(x_1 - e)\pi(x_2 - e) - \pi(x_1)\pi(x_2)] \ dx_1 dx_2; \end{aligned}$$

one can easily get the rest of the assertion holds true.

Remark 3.3′. In the case of Theorem 2.1, let

$$\begin{aligned} & \sigma(x) \equiv 0, \ b(x) \equiv 0; \ U_0 = \emptyset, \ 0 < n(U) < \infty; \\ & h(x,u) = h(u), \ h \in \mathcal{B}_b(U), \ \{h(u) \mid u \in U\} \subseteq \{ke \mid k = 0, \pm 1, \pm 2, \cdots\} \\ & \quad \text{for some } e = (e^1, \cdots, e^d) \in R^d \setminus \{0\}; \\ & 0 \leq \pi(x) \text{ is of } C^1\text{-class}, \ \pi(y+e) = \pi(y), \ \forall y \in R^d; \\ & \Pi(dx) = \pi(x) \ dx \in \mathcal{M}_r\left(R^d\right) \setminus \{0\}. \end{aligned}$$

Then $\left(((X_t)_*\mu)_{t\geq 0}, \mu \in \mathcal{M}_r\left(R^d\right)\right)$ is a càdlàg strongly Markovian and weakly Fellerian process on $\mathcal{M}_r\left(R^d\right)$. **We claim** that

$$\lim_{\substack{s \downarrow t \\ y \to x}} X_s(y) = X_t(x), \ \lim_{\substack{s \uparrow t \\ y \to x}} X_s(y) = X_{t-}(x), \ \forall (t,x) \in [0,\infty) \times R^d, \ a.s.;$$

and $a.s.$, X_v is a continuous injection for any fixed $v \geq 0$. Further,

$$\langle \Pi, A_1 f \rangle = 0, \ \left\langle \Pi^2, A_2 f^{\otimes 2} \right\rangle = 0, \ \forall f \in C_c^2\left(R^d\right)$$

is equivalent to

$$\nabla \pi(x) \cdot \int_U h(u) \ n(du) = 0, \ \forall x \in R^d.$$

So under $\nabla\pi(x)\cdot\int_U h(u)\ n(du) = 0,\ \forall x\in R^d,\ \left((X_t(D))_{t\geq 0}, D\in\mathcal{K}_{S(\Pi)}\right)$ is a càdlàg $\mathcal{K}_{S(\Pi)}$-valued strong Markov process.

In fact, from

$$dX_t(x) = \int_U h(u)\ \lambda(dtdu) + \int_U h(u)\ n(du)\ dt, X_0(x) = x,$$

one can see the first part of the claim is true. However, for any $f\in C_c^2\left(R^d\right)$, it is easy to check

$$\begin{aligned}
\langle\Pi, A_1 f\rangle &= \int_U\int_{R^d} f(x)[\nabla\pi(x)\cdot\ h(u)]\ dxn(du) \\
&= \int_{R^d} f(x)\left[\nabla\pi(x)\cdot\ \int_U h(u)\ n(du)\right] dx, \\
\left\langle\Pi^2, A_2 f^{\otimes 2}\right\rangle &= \int_U\int_{(R^d)^2} f(x_1)f(x_2)\{\pi(x_1)\ \nabla\pi(x_2)\cdot\ h(u) + \pi(x_2)\nabla\pi(x_1)\cdot\ h(u)\} \\
&\qquad dx_1dx_2n(du) \\
&= \int_{(R^d)^2} f(x_1)f(x_2)\left\{\pi(x_1)\ \nabla\pi(x_2)\cdot\int_U h(u)\ n(du)+\right. \\
&\qquad \left.\pi(x_2)\nabla\pi(x_1)\cdot\int_U h(u)\ n(du)\right\} dx_1dx_2;
\end{aligned}$$

which implies the second part of the claim holds.

Remark 3.4. Fix $r > \frac{d}{2}$. Take $\psi:\ R^d\to R^d$ as follows:

Write $x = \left(x^1,\cdots,x^d\right)\in R^d$ into the polar coordinate form

$$x^k = \rho\left(\prod_{i=1}^{k-1}\sin\theta_i\right)\cos\theta_k,\ \forall k\leq d-1,\ x^d = \rho\left(\prod_{i=1}^{d-1}\sin\theta_i\right),$$

where $(\rho,\theta_1,\cdots,\theta_{d-1})\in[0,\infty)\times[0,2\pi)^{d-1}$. Choose

$$\varphi\in C_b^1([0,\infty)),\ \sup_{\rho\geq 0}\left\{\left(1+\rho^2\right)^r|\varphi(\rho)|\right\}<\infty.$$

Define

$$\begin{aligned}
&\psi(x) = \left(\psi^1(x),\cdots,\psi^d(x)\right)\in R^d,\ \forall x\in R^d; \\
&\psi^k(x) = \rho\left(\prod_{i=1}^{k-1}\sin(\theta_i+\varphi(\rho))\right)\cos(\theta_k+\varphi(\rho)) - x^k,\ \forall k\leq d-1, \\
&\psi^d(x) = \rho\left(\prod_{i=1}^{d-1}\sin(\theta_i+\varphi(\rho))\right) - x^d.
\end{aligned}$$

In the setting of Theorem 2.1, let $\sigma(x)\equiv 0,\ b(x)\equiv 0,\ n(U)<\infty$; and

$$h(x,u) = \psi(x),\ \forall(x,u)\in R^d\times U.$$

Then $\left(((X_t)_*\mu)_{t\geq 0}, \mu\in\mathcal{M}_r\left(R^d\right)\right)$ is a càdlàg strongly Markovian and weakly Fellerian process on $\mathcal{M}_r\left(R^d\right)$. Use polar coordinate and the integration by parts formulae, similarly to Remark 3.3 and 3.3′, one can check $(X_t)_{t\geq 0}$ is dx-incompressible if and only if

$$div\ \psi(x) = 0, \forall x\in R^d \text{ or } n(U\setminus U_0) = 0.$$

Under additional suitable conditions, $(X_t)_{t\geq 0}$ can be a homeomorphic stochastic flows, and $((X_t(D))_{t\geq 0}, D \in \mathcal{K}_{R^d})$ is a càdlàg $\mathcal{K}_{R^d}$-valued strong Markov process for dx-incompressible flow $(X_t)_{t\geq 0}$. For example, if further

$$U_0 = \{|x| \leq 1\} \subseteq U,\ \varphi \in C_b^2([0,\infty)),$$

then $(X_t)_{t\geq 0}$ is a stochastic flow of locally C^1-diffeomorphisms (see [21] section 4).

Remark 3.5. From remarks above, diffusion-jump type examples can be constructed, we leave this to the interested readers.

In Theorem 2.1, jumps are bounded. Note α-stable process is of big jumps for $\alpha \in (0,2)$. Choose $r \in (\frac{d}{2}, \frac{d}{2} + \frac{\alpha}{2})$, then ([4])

$$|\Delta_\alpha \phi_r| \leq constant\ \phi_r,\ \left|\Delta_\alpha \phi_r^2\right| \leq constant\ \phi_r^2.$$

One can check (see Lemma 4.3) the semigroup $\left\{V_{t,\alpha}^1\right\}_{t\geq 0}$ generated by Δ_α must satisfies

$$\sup_{t\leq T} \left\| \left[V_{t,\alpha}^1 \phi_r\right] / \phi_r \right\| < \infty,\ \sup_{t\leq T} \left\| \left\{V_{t,\alpha}^1 \left[\phi_r^2\right]\right\} / \left[\phi_r^2\right] \right\| < \infty,\ \forall T \geq 0.$$

For the 2-point motion of any stochastic flows for α-stable processes, write $\left\{V_{t,\alpha}^2\right\}_{t\geq 0}$ for its semigroup. Then

$$\sup_{t\leq T} \left\| \left\{V_{t,\alpha}^2 \phi_r^{\otimes 2}\right\} / \phi_r^{\otimes 2} \right\| < \infty,\ \forall T \geq 0.$$

Indeed, by the Cauchy-Schwartz inequality, for any $t \leq T$,

$$\begin{aligned} V_{t,\alpha}^2 \phi_r^{\otimes 2}(x_1, x_2) &\leq \left\{V_{t,\alpha}^1 \left[\phi_r^2\right](x_1)\right\}^{\frac{1}{2}} \left\{V_{t,\alpha}^1 \left[\phi_r^2\right](x_2)\right\}^{\frac{1}{2}} \\ &\leq \left\{C_T \phi_r^2(x_1)\right\}^{\frac{1}{2}} \left\{C_T \phi_r^2(x_2)\right\}^{\frac{1}{2}} \\ &= C_T\ \phi_r(x_1)\phi_r(x_2),\ \text{for some constant } C_T \text{ depending on } T. \end{aligned}$$

So the method for proving Theorem 2.1 can be applied to this setting except that the corresponding $\mathcal{M}_r\left(R^d\right)$-valued process is càdlàg (new method is needed to verify this).

4. Proof of Theorem 2.1

Note (2.5) satisfies local Lipschitz and linear growth assumptions. Then the system of k SDEs (2.5) as an Itô SDE in $\left(R^d\right)^k$ with initial points $(x_1, \cdots, x_k)$ has a unique strong Markov solution which is weakly Fellerian. Combining with the Itô's formula, we obtain the following

Lemma 4.1. For any natural number k, the k-point weakly Fellerian process

$$\left((X_t(x_1), \cdots, X_t(x_k))_{t\geq 0}, (x_1, \cdots, x_k) \in \left(R^d\right)^k\right)$$

is the unique strong Markov processes with the semigroup $\left\{V_t^k\right\}_{t\geq 0}$ corresponding to the following weak generator

$$\begin{aligned} &A_k f(x_1, \cdots, x_k) \\ &= \frac{1}{2} \sum_{i,j=1}^k \sum_{p,q=1}^d a^{pq}(x_i, x_j) \frac{\partial^2 f}{\partial x_i^p \partial x_j^q}(x_1, \cdots, x_k) + \sum_{i=1}^k \sum_{p=1}^d b^p(x_i) \frac{\partial f}{\partial x_i^p}(x_1, \cdots, x_k) + \\ &\quad \int_{U\setminus U_0} \left[f\left(x_1 + h(x_1, u), \cdots, x_k + h(x_k, u)\right) - f(x_1, \cdots, x_k) - \right. \end{aligned}$$

$$\nabla f(x_1, \cdots, x_k) \cdot (h(x_1, u), \cdots, h(x_k, u))] \ n(du) +$$
$$\int_{U_0} [f(x_1 + h(x_1, u), \cdots, x_k + h(x_k, u)) - f(x_1, \cdots, x_k)] \ n(du),$$
$$\forall x_i = \left(x_i^1, \cdots, x_i^d\right), \ 1 \le i \le k, \ \forall f \in C_b^2\left(\left(R^d\right)^k\right).$$

Lemma 4.2. Endow $\mathcal{M}\left(R^d\right)$ with the weak topology. Then $\left(((X_t)_*\mu)_{t\ge 0}, \mu \in \mathcal{M}\left(R^d\right)\right)$ is a càdlàg (resp. continuous) $\mathcal{M}\left(R^d\right)$-valued weakly Fellerian processes (resp. when $n = 0$) such that
$$F((X_t)_*\mu) - F(\mu) - \int_0^t AF((X_s)_*\mu) \ ds, \ t \in [0, \infty),$$
is a martingale for any $F \in \mathcal{H}$.

Proof. Since for any $t \ge 0$, $(X_t)_*0 = 0$ and
$$(X_t)_*\mu = \mu\left(R^d\right) \ (X_t)_* \frac{\mu}{\mu(R^d)}, \ \mu \in \mathcal{M}\left(R^d\right) \setminus \{0\},$$
it suffices to prove the lemma for initial measures $\mu \in \mathcal{P}\left(R^d\right)$. It is known ([16], [18]) that $\left(((X_t)_*\mu)_{t\ge 0}, \mu \in \mathcal{P}\left(R^d\right)\right)$ is a $\mathcal{P}\left(R^d\right)$-valued càdlàg (resp. continuous) weakly Fellerian processes (resp. when $n = 0$) with the following semigroup $\left\{T_t^{\mathcal{P}}\right\}_{t\ge 0}$ determined by
$$T_t^{\mathcal{P}} F_{f,k}(\mu) = F_{V_t^k f,k}(\mu), \ \forall \mu \in \mathcal{P}\left(R^d\right), \ \forall f \in \mathcal{B}_b\left(\left(R^d\right)^k\right), \ \forall t \ge 0, \ \forall k \ge 1.$$

Now we are in the position to verify that $F((X_t)_*\mu) - F(\mu) - \int_0^t AF((X_s)_*\mu) \ ds, \ t \in [0, \infty)$, is a martingale for any $\mu \in \mathcal{P}\left(R^d\right)$ and
$$F(\nu) = G(\langle \nu, f_1\rangle, \cdots, \langle \nu, f_k\rangle) \in \mathcal{H}.$$

In fact, by the Itô's formula, for any $f \in C_b^2\left(R^d\right)$,
$$\begin{aligned} f(X_t(x)) \quad &= \quad f(x) + \int_0^t A_1 f(X_s(x)) \ ds + \int_0^t \nabla f(X_s(x)) \cdot \ \sigma(X_s(x)) \ dB_s + \\ &\int_0^t \int_U \{f(X_{s-}(x) + h(X_{s-}(x), u)) - f(X_{s-}(x))\} \ \eta(dsdu). \end{aligned}$$
Write $Y_t = (X_t)_*\mu, \ t \ge 0$. Then $\langle Y_t, f_i\rangle$ is a semimartingale. By the Itô's formula, we get
$$\begin{aligned} &G(\langle Y_t, f_1\rangle, \ \cdots, \langle Y_t, f_k\rangle) \\ &= G(\langle \mu, f_1\rangle, \ \cdots, \langle \mu, f_k\rangle) + \sum_{i=1}^k \int_0^t \partial_i G(\langle Y_{s-}, f_1\rangle, \ \cdots, \langle Y_{s-}, f_k\rangle) \ d\langle Y_s, f_i\rangle + \\ &\frac{1}{2} \sum_{i,j=1}^k \int_0^t \partial_{ij}^2 G(\langle Y_{s-}, f_1\rangle, \ \cdots, \langle Y_{s-}, f_k\rangle) \ d\left\langle \langle Y_\cdot, f_i\rangle^c, \langle Y_\cdot, f_j\rangle^c\right\rangle_s + \\ &\sum_{s\le t} \{\Delta G(\langle Y_s, f_1\rangle, \ \cdots, \langle Y_s, f_k\rangle) - \\ &\quad \nabla G(\langle Y_{s-}, f_1\rangle, \ \cdots, \langle Y_{s-}, f_k\rangle) \cdot \ \Delta(\langle Y_s, f_1\rangle, \ \cdots, \langle Y_s, f_k\rangle)\} \\ &:= G(\langle \mu, f_1\rangle, \ \cdots, \langle \mu, f_k\rangle) + K_1(t) + K_2(t) + K_3(t); \end{aligned}$$

where $\langle Y_\cdot, f_i\rangle^c$ is the continuous martingale part of semimartingale $\langle Y_\cdot, f_i\rangle$; and for a càdlàg map g from $[0,\infty)$ into some R^p, $\Delta g(t) = g(t) - g(t-),\ \forall t \geq 0$.

However, due to $\langle Y_s, 1\rangle = \langle \mu, 1\rangle = 1,\ \forall s \in [0,\infty)$,

$$\begin{aligned}
K_1(t) &= \sum_{i=1}^{k} \left\langle \mu, \int_0^t \partial_i G(\langle Y_{s-}, f_1\rangle, \cdots, \langle Y_{s-}, f_k\rangle) A_1 f_i(X_s(x))\ ds \right\rangle + \\
&\quad \sum_{i=1}^{k} \left\langle \mu, \int_0^t \partial_i G(\langle Y_{s-}, f_1\rangle, \cdots, \langle Y_{s-}, f_k\rangle)\ \nabla f_i(X_s(x)) \cdot \sigma(X_s(x)) dB_s \right\rangle + \\
&\quad \sum_{i=1}^{k} \int_{R^d} \int_0^t \int_U \partial_i G(\langle Y_{s-}, f_1\rangle, \cdots, \langle Y_{s-}, f_k\rangle) \\
&\qquad\qquad [f_i(X_{s-}(x) + h(X_{s-}(x), u)) - f_i(X_{s-}(x))]\ \eta(dsdu)\mu(dx) \\
&:= K_{11}(t) + K_{12}(t) + K_{13}(t) \\
&= \sum_{i=1}^{k} \int_0^t \partial_i G(\langle Y_s, f_1\rangle, \cdots, \langle Y_s, f_k\rangle) \langle Y_s, A_1 f_i\rangle\ ds + K_{12}(t) + K_{13}(t) \\
&\qquad \big(\text{where } (K_{12}(t))_{t\geq 0} \text{ and } (K_{13}(t))_{t \geq 0} \text{ are two } L^2\text{-martingales}\big)\,; \\
K_2(t) &= \left\langle \mu^2, \frac{1}{2} \sum_{i,j=1}^{k} \int_0^t \partial_{ij}^2 G(\langle Y_{s-}, f_1\rangle, \cdots, \langle Y_{s-}, f_k\rangle) \right. \\
&\quad \left. \left\langle \sum_{p=1}^{d} \partial_p f_i(X_\cdot(x)) \sum_{q=1}^{l} \sigma^{pq}(X_\cdot(x)) dB^q_\cdot, \sum_{p=1}^{d} \partial_p f_j(X_\cdot(y)) \sum_{q=1}^{l} \sigma^{pq}(X_\cdot(y)) dB^q_\cdot \right\rangle_s \right\rangle \\
&= \frac{1}{2} \sum_{i,j=1}^{k} \int_0^t \partial_{ij}^2 G(\langle Y_s, f_1\rangle, \cdots, \langle Y_s, f_k\rangle) \left\langle Y_s^2, \nabla f_i(x) a(x,y) (\nabla f_j)^T(y) \right\rangle\ ds; \\
K_3(t) &= \int_0^t \int_U \left\{ \Delta G(\langle Y_s, f_1\rangle, \cdots, \langle Y_s, f_k\rangle) - \sum_{i=1}^{k} \partial_i G(\langle Y_{s-}, f_1\rangle, \cdots, \langle Y_{s-}, f_k\rangle) \Delta\langle Y_s, f_i\rangle \right\} \\
&\qquad \lambda(dsdu) \\
&= \int_0^t \int_U \{ G(\langle \mu, f_1(X_{s-}(x) + h(X_{s-}(x), u))\rangle, \cdots, \langle \mu, f_k(X_{s-}(x) + h(X_{s-}(x), u))\rangle) \\
&\qquad - G(\langle \mu, f_1(X_{s-}(x))\rangle, \cdots, \langle \mu, f_k(X_{s-}(x))\rangle) - \\
&\qquad \sum_{i=1}^{k} \{ \partial_i G(\langle Y_{s-}, f_1\rangle, \cdots, \langle Y_{s-}, f_k\rangle) \\
&\qquad\qquad \langle \mu, f_i(X_{s-}(x) + h(X_{s-}(x), u)) - f_i(X_{s-}(x))\rangle \} \}\ \lambda(dsdu) \\
&= \int_0^t \int_U \{ G(\langle Y_{s-}, f_1(x + h(x,u))\rangle, \cdots, \langle Y_{s-}, f_k(x + h(x,u))\rangle) \\
&\qquad - G(\langle Y_{s-}, f_1(x)\rangle, \cdots, \langle Y_{s-}, f_k(x)\rangle) - \\
&\qquad \sum_{i=1}^{k} \{ \partial_i G(\langle Y_{s-}, f_1\rangle, \cdots, \langle Y_{s-}, f_k\rangle) \\
&\qquad\qquad \langle Y_{s-}, f_i(x + h(x,u)) - f_i(x)\rangle \} \}\ \eta(dsdu) + \\
&\quad \int_0^t \int_U \{ G(\langle Y_{s-}, f_1(x + h(x,u))\rangle, \cdots, \langle Y_{s-}, f_k(x + h(x,u))\rangle) \\
&\qquad - G(\langle Y_{s-}, f_1(x)\rangle, \cdots, \langle Y_{s-}, f_k(x)\rangle) - \\
&\qquad \sum_{i=1}^{k} \{ \partial_i G(\langle Y_{s-}, f_1\rangle, \cdots, \langle Y_{s-}, f_k\rangle)
\end{aligned}$$

$$
\begin{aligned}
& \qquad\qquad \langle Y_{s-}, f_i(x+h(x,u)) - f_i(x)\rangle\}\}\ dsn(du) \\
:= \quad & K_{31}(t) + K_{32}(t) \\
& \text{(where } (K_{31}(t))_{t\geq 0} \text{ is an } L^2\text{-martingale)} \\
= \quad & K_{31}(t) + \int_0^t \int_U \{G(\langle Y_s, f_1(x+h(x,u))\rangle, \cdots, \langle Y_s, f_k(x+h(x,u))\rangle) - \\
& \qquad G(\langle Y_s, f_1(x)\rangle, \cdots, \langle Y_s, f_k(x)\rangle) - \\
& \qquad \sum_{i=1}^k \{\partial_i G(\langle Y_s, f_1\rangle, \cdots, \langle Y_s, f_k\rangle) \\
& \qquad\quad \langle Y_s, f_i(x+h(x,u)) - f_i(x)\rangle\}\}\ dsn(du).
\end{aligned}
$$

Now it is easy to show that

$$
F((X_t)_*\mu) - F(\mu) - \int_0^t AF((X_s)_*\mu)\ ds = F(Y_s) - F(\mu) - \int_0^t AF(Y_s)\ ds,\ t \in [0,\infty),
$$

is an L^2-martingale, and

$$
\lim_{t\to 0+} \frac{1}{t}\left\{E[G(\langle Y_t, f_1\rangle,\ \cdots, \langle Y_t, f_k\rangle)] - G(\langle \mu, f_1\rangle,\ \cdots, \langle \mu, f_k\rangle)\right\} = AF(\mu).
$$

□

Lemma 4.3. There is a sequence $\left\{C_4^{(k)}\right\}_{k\geq 1}$ of positive constants such that $\forall t \geq 0, \forall k \geq 1$,

$$
\begin{aligned}
&\left|A_k \phi_r^{\otimes k}\right| \leq C_4^{(k)} \phi_r^{\otimes k},\ \left|A_1\left[\phi_r^2\right]\right| \leq C_4^{(1)} \phi_r^2; \\
&V_t^k \phi_r^{\otimes k} \leq e^{tC_4^{(k)}} \phi_r^{\otimes k},\ V_t^1\left[\phi_r^2\right] \leq e^{tC_4^{(1)}} \phi_r^2.
\end{aligned}
$$

Moreover, $\forall \mu_i \in \mathcal{M}_r\left(R^d\right),\ \forall 1 \leq i \leq k,\ \forall k \geq 1,\ \forall t \geq 0,$

$$
\begin{aligned}
&\left\langle \prod_{i=1}^k \mu_i, V_t^k \phi_r^{\otimes k}\right\rangle \leq e^{tC_4^{(k)}} \prod_{i=1}^k \langle \mu_i, \phi_r\rangle < \infty, \\
&\left\langle \mu_1, V_t^1\left[\phi_r^2\right]\right\rangle \leq e^{tC_4^{(1)}} \left\langle \mu_1, \phi_r^2\right\rangle < \infty.
\end{aligned}
$$

Proof. Step 1. By (2.3), for any $1 \leq i \neq j \leq k$ and any $1 \leq p, q \leq d$,

$$
\begin{aligned}
&\left| a^{pq}(x_i, x_j) \frac{\partial^2 \phi_r^{\otimes k}}{\partial x_i^p \partial x_j^q}(x_1, \cdots, x_k)\right| \\
&= \left| \left(\sum_{m=1}^l \sigma^{pm}(x_i)\sigma^{qm}(x_j)\right) \frac{\partial^2 \phi_r^{\otimes k}}{\partial x_i^p \partial x_j^q}(x_1, \cdots, x_k)\right| \\
&\leq \left(\sum_{m=1}^l (\sigma^{pm}(x_i))^2\right)^{\frac{1}{2}} \left(\sum_{m=1}^l (\sigma^{qm}(x_j))^2\right)^{\frac{1}{2}} \left| \frac{\partial^2[\phi_r(x_i)\phi_r(x_j)]}{\partial x_i^p \partial x_j^q} \prod_{1\leq s\neq i,j\leq k} \phi_r(x_s)\right| \\
&\leq C_1 \sqrt{(1+|x_i|^2)(1+|x_j|^2)} \left| \frac{\partial \phi_r(x_i)}{\partial x_i^p} \frac{\partial \phi_r(x_j)}{\partial x_j^q}\right| \prod_{1\leq s\neq i,j\leq k} \phi_r(x_s) \\
&= r^2 C_1 \sqrt{(1+|x_i|^2)(1+|x_j|^2)}\ \ \phi_r(x_i)\phi_r(x_j) \left| \frac{2x_i^p}{1+|x_i|^2} \frac{2x_j^q}{1+|x_j|^2}\right| \prod_{1\leq s\neq i,j\leq k} \phi_r(x_s)
\end{aligned}
$$

$$
\begin{aligned}
&\leq r^2 C_1 \left(\sup_{(x,y)\in (R^d)^2} \frac{4|x^p y^q|}{\sqrt{(1+|x|^2)(1+|y|^2)}} \right) \phi_r^{\otimes k}(x_1,\cdots,x_k) \\
&= r^2 C_1 \left(\sup_{(x,y)\in (R^d)^2} \frac{4|x^1 y^1|}{\sqrt{(1+|x|^2)(1+|y|^2)}} \right) \phi_r^{\otimes k}(x_1,\cdots,x_k) \\
&= \widetilde{C}_1 \ \phi_r^{\otimes k}(x_1,\cdots,x_k), \ \forall (x_1,\cdots,x_k) \in \left(R^d\right)^k.
\end{aligned}
$$

Similarly, for any $1 \leq i \leq k$ and any $1 \leq p \neq q \leq d$,

$$
\begin{aligned}
&\left| a^{pq}(x_i,x_i) \frac{\partial^2 \phi_r^{\otimes k}}{\partial x_i^p \partial x_i^q}(x_1,\cdots,x_k) \right| \leq C_1 \ (1+|x_i|^2) \left(\prod_{1\leq s\neq i\leq k} \phi_r(x_s) \right) \left| \frac{\partial^2 \phi_r(x_i)}{\partial x_i^p \partial x_i^q} \right| \\
&= r(r+1)C_1 \ \left(\prod_{s=1}^k \phi_r(x_s) \right) \frac{4|x_i^p x_i^q|}{1+|x_i|^2} \leq r(r+1)C_1 \ \left(\sup_{x\in R^d} \frac{4|x^1 x^2|}{1+|x|^2} \right) \phi_r^{\otimes k}(x_1,\cdots,x_k) \\
&:= \widetilde{C}_2 \ \phi_r^{\otimes k}(x_1,\cdots,x_k), \ \forall (x_1,\cdots,x_k) \in \left(R^d\right)^k;
\end{aligned}
$$

and for any $1 \leq i \leq k$ and any $1 \leq p \leq d$,

$$
\begin{aligned}
&\left| a^{pp}(x_i,x_i) \frac{\partial^2 \phi_r^{\otimes k}}{\partial^2 x_i^p}(x_1,\cdots,x_k) \right| \leq C_1 \ (1+|x_i|^2) \left(\prod_{1\leq s\neq i\leq k} \phi_r(x_s) \right) \left| \frac{\partial^2 \phi_r(x_i)}{\partial^2 x_i^p} \right| \\
&= C_1 \ \left(\prod_{s=1}^k \phi_r(x_s) \right) \left| 2r - \frac{4r(r+1)\,(x_i^p)^2}{1+|x_i|^2} \right| \leq (2r+4r(r+1))C_1 \phi_r^{\otimes k}(x_1,\cdots,x_k) \\
&:= \widetilde{C}_3 \ \phi_r^{\otimes k}(x_1,\cdots,x_k), \ \forall (x_1,\cdots,x_k) \in \left(R^d\right)^k; \\
&\left| b^p(x_i) \frac{\partial \phi_r^{\otimes k}}{\partial x_i^p}(x_1,\cdots,x_k) \right| \leq \sqrt{C_1 \ (1+|x_i|^2)} \ \left(\prod_{1\leq s\neq i\leq k} \phi_r(x_s) \right) \ \left| \frac{\partial \phi_r(x_i)}{\partial x_i^p} \right| \\
&= \sqrt{C_1} \ \frac{2|x_i^p|}{\sqrt{1+|x_i|^2}} \ \phi_r^{\otimes k}(x_1,\cdots,x_k) \leq \sqrt{C_1} \ \left(\sup_{x\in R^d} \frac{2|x^1|}{\sqrt{1+|x|^2}} \right) \ \phi_r^{\otimes k}(x_1,\cdots,x_k) \\
&:= \widetilde{C}_4 \ \phi_r^{\otimes k}(x_1,\cdots,x_k), \ \forall (x_1,\cdots,x_k) \in \left(R^d\right)^k.
\end{aligned}
$$

Therefore, for some positive constant $C_{4,1}^{(k)}$ depending on k,

$$
\begin{aligned}
&\left| \frac{1}{2} \sum_{i,j=1}^k \sum_{p,q=1}^d a^{pq}(x_i,x_j) \frac{\partial^2 \phi_r^{\otimes k}}{\partial x_i^p \partial x_j^q}(x_1,\cdots,x_k) + \sum_{i=1}^k \sum_{p=1}^d b^p(x_i) \frac{\partial \phi_r^{\otimes k}}{\partial x_i^p}(x_1,\cdots,x_k) \right| \\
&\leq C_{4,1}^{(k)} \ \phi_r^{\otimes k}(x_1,\cdots,x_k), \ \forall (x_1,\cdots,x_k) \in \left(R^d\right)^k.
\end{aligned}
$$

Step 1′. Similarly to Step 1, one can verify that for some positive constant $\widetilde{C}_{4,1}^{(1)}$,

$$
\left| \frac{1}{2} \sum_{p,q=1}^d a^{pq}(x) \frac{\partial^2 \left[\phi_r^2\right]}{\partial x^p \partial x^q}(x) + \sum_{p=1}^d b^p(x) \frac{\partial \left[\phi_r^2\right]}{\partial x^p}(x) \right| \leq \widetilde{C}_{4,1}^{(1)} \ \phi_r^2(x), \ \forall x \in R^d.
$$

Step 2. For any $(x_1,\cdots,x_k) \in \left(R^d\right)^k$ and any $u \in U \setminus U_0$, by the remainder formula on the Taylor expansion,

$$
\phi_r^{\otimes k}(x_1+h(x_1,u),\cdots,x_k+h(x_k,u)) - \phi_r^{\otimes k}(x_1,\cdots,x_k) -
$$

$$\nabla\phi_r^{\otimes k}(x_1,\cdots,x_k)\cdot\ (h(x_1,u),\cdots,h(x_k,u))$$
$$=\sum_{i,j=1}^{k}\sum_{p,q=1}^{d}h^p(x_i,u)\left(\frac{\partial^2\phi_r^{\otimes k}}{\partial y_i^p\partial y_j^q}(y_1,\cdots,y_k)\bigg|_{(y_1,\cdots,y_k)=(\xi_1,\cdots,\xi_k)}\right)h^q(x_j,u)$$

for some $(\xi_1,\cdots,\xi_k)\in\left(R^d\right)^k$ satisfying

$$|(\xi_1,\cdots,\xi_k)-(x_1,\cdots,x_k)|\leq\sqrt{\sum_{i=1}^{k}|h(x_i,u)|^2}$$
$$\leq H(u)\leq\sup_{v\in U\setminus\mathcal{N}}H(v)=\widetilde{C}_5<\infty,\ n(du)\ -a.e..$$
$$\text{(where we have used (2.1) and (2.2))}$$

However, it is easy to check that for some positive constant $\widetilde{C}_6$ not depending on k,

$$\left|\frac{\partial^2\phi_r^{\otimes k}(y_1,\cdots,y_k)}{\partial y_i^p\partial y_j^q}\right|\leq\widetilde{C}_6\ \phi_r^{\otimes k}(y_1,\cdots,y_k),$$
$$\forall(y_1,\cdots,y_k)\in\left(R^d\right)^k,\ \forall 1\leq p,q\leq d,\ \forall 1\leq i,j\leq k.$$

Hence,

$$\left|\frac{\partial^2\phi_r^{\otimes k}(y_1,\cdots,y_k)}{\partial y_i^p\partial y_j^q}\bigg|_{(y_1,\cdots,y_k)=(\xi_1,\cdots,\xi_k)}\right|\leq\widetilde{C}_6\ \phi_r^{\otimes k}(\xi_1,\cdots,\xi_k)$$
$$\leq\widetilde{C}_6\phi_r^{\otimes k}(x_1,\cdots,x_k)\left(\sup_{|\vec{z}-\vec{w}|\leq\widetilde{C}_5}\frac{\phi_r^{\otimes k}(z_1,\cdots,z_k)}{\phi_r^{\otimes k}(w_1,\cdots,w_k)}\right)$$
$$:=\widetilde{C}_7(k)\ \phi_r^{\otimes k}(x_1,\cdots,x_k)<\infty,$$

where $\vec{z}=(z_1,\cdots,z_k)\in\left(R^d\right)^k,\ \vec{w}=(w_1,\cdots,w_k)\in\left(R^d\right)^k$. Therefore,

$$\int_{U\setminus U_0}\left|\phi_r^{\otimes k}(x_1+h(x_1,u),\cdots,x_k+h(x_k,u))-\phi_r^{\otimes k}(x_1,\cdots,x_k)-\right.$$
$$\left.\nabla\phi_r^{\otimes k}(x_1,\cdots,x_k)\cdot\ (h(x_1,u),\cdots,h(x_k,u))\right|\ n(du)$$
$$\leq\widetilde{C}_7(k)\sum_{i,j=1}^{k}\sum_{p,q=1}^{d}\int_{U\setminus U_0}|h^p(x_i,u)||h^q(x_j,u)|\ \phi_r^{\otimes k}(x_1,\cdots,x_k)\ n(du)$$
$$\leq\widetilde{C}_7(k)\sum_{i,j=1}^{k}\sum_{p,q=1}^{d}\int_{U\setminus U_0}H(u)^2\ \phi_r^{\otimes k}(x_1,\cdots,x_k)\ n(du)$$
$$=k^2d^2\widetilde{C}_7(k)\int_{U\setminus U_0}H(u)^2\ n(du)\phi_r^{\otimes k}(x_1,\cdots,x_k)$$
$$:=C_{4,2}^{(k)}\ \phi_r^{\otimes k}(x_1,\cdots,x_k),\quad\forall(x_1,\cdots,x_k)\in\left(R^d\right)^k.$$

Remembering (2.1) and (2.2), similarly, we can show that for some positive constant $C_{4,3}^{(k)}$ depending on k,

$$\int_{U_0}\left|\phi_r^{\otimes k}(x_1+h(x_1,u),\cdots,x_k+h(x_k,u))-\phi_r^{\otimes k}(x_1,\cdots,x_k)\right|\ n(du)$$
$$\leq C_{4,3}^{(k)}\ \phi_r^{\otimes k}(x_1,\cdots,x_k),\quad\forall(x_1,\cdots,x_k)\in\left(R^d\right)^k.$$

Step 2′. Similarly to Step 2, there are positive constants $\widetilde{C}_{4,2}^{(1)}$ and $\widetilde{C}_{4,3}^{(1)}$ such that

$$\begin{aligned}&\int_{U\setminus U_0} \left|\phi_r^2(x+h(x,u)) - \phi_r^2(x) - \nabla\phi_r^2(x)\cdot\ h(x,u)\right|\ n(du) \le \widetilde{C}_{4,2}^{(1)}\ \phi_r^2(x),\\&\int_{U_0} \left|\phi_r^2(x+h(x,u)) - \phi_r^2(x)\right|\ n(du) \le \widetilde{C}_{4,3}^{(1)}\ \phi_r^2(x),\\&\qquad\qquad \forall x\in R^d.\end{aligned}$$

Step 3. By Steps 1-2 and Steps $1'-2'$, it is not difficult to see that there is a sequence $\left\{C_4^{(k)}\right\}_{k\ge1}$ of positive constants such that

$$\left|A_k\phi_r^{\otimes k}\right| \le C_4^{(k)}\phi_r^{\otimes k},\ \forall k\ge1;\ \left|A_1\left[\phi_r^2\right]\right| \le C_4^{(1)}\phi_r^2.$$

So

$$V_t^k\phi_r^{\otimes k} = \phi_r^{\otimes k} + \int_0^t V_sA_k\phi_r^{\otimes k}\ ds \le \phi_r^{\otimes k} + C_4^{(k)}\int_0^t V_s^k\phi_r^{\otimes k}\ ds,\ \forall t\ge0,$$

and by the Gronwall inequality,

$$V_t^k\phi_r^{\otimes k} \le e^{tC_4^{(k)}}\phi_r^{\otimes k},\ \forall t\ge0.$$

Similarly, we can prove

$$V_t^1\left[\phi_r^2\right] \le e^{tC_4^{(1)}}\phi_r^2,\ \forall t\ge0.$$

Therefore, $\forall\mu_i\in\mathcal{M}_r\left(R^d\right),\ \forall 1\le i\le k,\ \forall k\ge1,\ \forall t\ge0,$

$$\begin{aligned}\left\langle\prod_{i=1}^k\mu_i, V_t^k\phi_r^{\otimes k}\right\rangle &\le e^{tC_4^{(k)}}\prod_{i=1}^k\langle\mu_i,\phi_r\rangle < \infty,\\ \left\langle\mu_1, V_t^1\left[\phi_r^2\right]\right\rangle &\le e^{tC_4^{(1)}}\left\langle\mu_1,\phi_r^2\right\rangle < \infty.\end{aligned}$$

□

Lemma 4.4. For any fixed $t\in[0,\infty)$ and $\mu\in\mathcal{M}_r\left(R^d\right)$, $(X_t)_*\mu$ is $\mathcal{M}_r\left(R^d\right)$-valued. Define a family $\{T_t\}_{t\ge0}$ of operators from $\left(C_b\left(\mathcal{M}_r\left(R^d\right)\right),\|\cdot\|\right)$ into $\left(\mathcal{B}_b\left(\mathcal{M}_r\left(R^d\right)\right),\|\cdot\|\right)$ as follows:

$$T_tF(\mu) = E[F((X_t)_*\mu)],\ \forall F\in C_b\left(\mathcal{M}_r\left(R^d\right)\right),\ \forall\mu\in\mathcal{M}_r\left(R^d\right);\ \forall t\ge0.$$

Then $\{T_t\}_{t\ge0}$ is a weakly Fellerian semigroup on $C_b\left(\mathcal{M}_r\left(R^d\right)\right)$.

Proof. Step 1. For any fixed $t\in[0,\infty)$ and $\mu\in\mathcal{M}_r\left(R^d\right)$, by Lemma 4.3,

$$\begin{aligned}E[\langle(X_t)_*\mu,\phi_r\rangle] &= E\left[\int_{R^d}\phi_r(X_t(x))\ \mu(dx)\right] = \int_{R^d}E[\phi_r(X_t(x))]\ \mu(dx)\\&= \int_{R^d}V_t^1\phi_r(x)\mu(dx) \le e^{tC_4^{(1)}}\int_{R^d}\phi_r(x)\ \mu(dx) < +\infty,\end{aligned}$$

which implies $(X_t)_*\mu$ is $\mathcal{M}_r\left(R^d\right)$-valued.

Step 2. Endow $\mathcal{M}\left(R^d\right)$ with the weak topology. Since $\mu\in\mathcal{M}_r\left(R^d\right)\to\phi_r(x)\mu(dx)\in\mathcal{M}\left(R^d\right)$ is a topological homeomorphism, it is not difficult to see that

$$\mu_n\Longrightarrow\mu \text{ in } \mathcal{M}_r\left(R^d\right) \text{ implies } \lim_{n\to\infty}\langle\mu_n,f\rangle = \langle\mu,f\rangle,\ \forall f\in\Phi\left(R^d\right).$$

For any $t \in [0, \infty)$ and any $f \in \Phi\left(R^d\right)$, remembering Lemmas 4.3 and 4.1, we have

$$V_t^2 f^{\otimes 2} \in C_b^2\left(\left(R^d\right)^2\right), \ \left|\frac{V_t^2 f^{\otimes 2}}{\phi_r^{\otimes 2}}\right| \leq \|f\|_{\Phi(R^d)}^2 \ e^{tC_4^{(2)}} < \infty. \tag{4.1}$$

So

$$\begin{aligned}
&E\left[|\langle (X_t)_*\mu - (X_t)_*\nu, f\rangle|^2\right] \\
&= E\left[\left\langle ((X_t)_*\mu)^2, f^{\otimes 2}\right\rangle + \left\langle ((X_t)_*\nu)^2, f^{\otimes 2}\right\rangle\right] - 2E\left[\left\langle (X_t)_*\mu \times (X_t)_*\nu, f^{\otimes 2}\right\rangle\right] \\
&= \left\langle \mu^2, V_t^2 f^{\otimes 2}\right\rangle + \left\langle \nu^2, V_t^2 f^{\otimes 2}\right\rangle - 2\left\langle \mu \times \nu, V_t^2 f^{\otimes 2}\right\rangle \\
&\to 0, \ \text{as } \mu \to \nu \text{ in } \mathcal{M}_r\left(R^d\right),
\end{aligned}$$

which implies that $(X_t)_*\mu$ is continuous in $\mu \in \mathcal{M}_r\left(R^d\right)$ in probability. Therefore, for any $F \in C_b\left(\mathcal{M}_r\left(R^d\right)\right)$, $T_t F(\mu)$ is continuous in $\mu \in \mathcal{M}_r\left(R^d\right)$.

Step 3. For any $s, t \in [0, \infty)$ and any $F \in C_b\left(\mathcal{M}_r\left(R^d\right)\right)$,

$$\begin{aligned}
&T_t T_s F(\mu) = \lim_{k\to\infty} T_t T_s F(\mu_k) \\
&= \lim_{k\to\infty} E[T_s F((X_t)_*\mu_k)] = \lim_{k\to\infty} E[(T_s F)|_{\mathcal{M}(R^d)}((X_t)_*\mu_k)] \\
&\quad \left(\text{since } (X_t)_*\mu_k \text{ is } \mathcal{M}\left(R^d\right) - \text{valued}\right) \\
&= \lim_{k\to\infty} E\left[T_s^{\mathcal{M}} F((X_t)_*\mu_k)\right] = \lim_{k\to\infty} T_t^{\mathcal{M}} T_s^{\mathcal{M}} F(\mu_k) \\
&\quad \left(\text{since } (T_s F)|_{\mathcal{M}(R^d)} = T_s^{\mathcal{M}}\left(F|_{\mathcal{M}(R^d)}\right)\right) \\
&= \lim_{k\to\infty} T_{t+s}^{\mathcal{M}} F(\mu_k) = \lim_{k\to\infty} E[F((X_{t+s})_*\mu_k)] = E[F((X_{t+s})_*\mu)] = T_{t+s}F(\mu),
\end{aligned}$$

where $(T_s F)|_{\mathcal{M}(R^d)}$ and $F|_{\mathcal{M}(R^d)}$ are the restrictions of $T_s F$ and F to $\mathcal{M}\left(R^d\right)$ respectively, $\left\{T_v^{\mathcal{M}}\right\}_{v\geq 0}$ is the semigroup of $\mathcal{M}\left(R^d\right)$-valued process in Lemma 4.2, $\mu_k(dx) = I_{\{|x|\leq k\}}\mu(dx)$.

Step 4. For any $f \in \Phi\left(R^d\right)$, Lemma 4.3 implies that $\left|V_t^1 f\right| \leq e^{tC_4^{(1)}} \ \phi_r$, note (4.1) and

$$\lim_{t\downarrow 0} V_t^2 f^{\otimes 2}(x,y) = f^{\otimes 2}(x,y), \ \lim_{t\downarrow 0} V_t^1 f(x) = f(x), \ \forall (x,y) \in \left(R^d\right)^2,$$

by the dominated convergence theorem,

$$\begin{aligned}
&E\left[|\langle (X_t)_*\mu - \mu, f\rangle|^2\right] \\
&= E\left[\left\langle ((X_t)_*\mu)^2, f^{\otimes 2}\right\rangle + \left\langle \mu^2, f^{\otimes 2}\right\rangle\right] - 2E\left[\left\langle (X_t)_*\mu \times \mu, f^{\otimes 2}\right\rangle\right] \\
&= \left\langle \mu^2, V_t^2 f^{\otimes 2}\right\rangle + \left\langle \mu^2, f^{\otimes 2}\right\rangle - 2\left\langle \mu, V_t^1 f\right\rangle \ \langle \mu, f\rangle \to 0, \ \text{as } t \to 0+;
\end{aligned}$$

which says $(X_t)_*\mu \to \mu \in \mathcal{M}_r\left(R^d\right)$ in probability as $t \to 0+$. Now for any $F \in C_b\left(\mathcal{M}_r\left(R^d\right)\right)$,

$$\lim_{t\to 0+} T_t F(\mu) = \lim_{t\to 0+} E[F((X_t)_*\mu)] = F(\mu), \ \forall \mu \in \mathcal{M}_r\left(R^d\right).$$

□

Lemma 4.5. For any $\mu \in \mathcal{M}_r\left(R^d\right)$, $((X_t)_*\mu)_{t\geq 0}$ is càdlàg (resp. continuous when $n = 0$) on $\mathcal{M}_r\left(R^d\right)$; and

$$F((X_t)_*\mu) - F(\mu) - \int_0^t AF((X_s)_*\mu) \ ds, \ t \in [0, \infty),$$

is an L^2-martingale for any $F \in \mathcal{H}$. Further, the processes $\left(((X_t)_*\mu)_{t\geq 0}, \mu \in \mathcal{M}_r\left(R^d\right)\right)$ is strongly Markovian.

Proof. Since the second part of the Lemma is a consequence of Lemma 4.4 and the first part of the Lemma, it suffices to prove the first part of the Lemma.

Step 1. Given any $\mu \in \mathcal{M}_r\left(R^d\right)$, recall

$$\mu_k(dx) = I_{\{|x|\le k\}}\mu(dx) \in \mathcal{M}_r\left(R^d\right) \text{ for any } \ k > 0.$$

Clearly, for any fixed $t \in [0,\infty)$, as $k \uparrow \infty$, $(X_t)_*\mu_k \to (X_t)_*\mu \in \mathcal{M}_r\left(R^d\right)$, *a.s.*. Note each $((X_t)_*\mu_k)_{t\ge 0}$ is càdlàg (resp. continuous) on both $\mathcal{M}\left(R^d\right)$ and $\mathcal{M}_r\left(R^d\right)$ (resp. when $n=0$), and $\{((X_t)_*\mu_k)_{t\ge0}\}_{k\ge1}$ converges weakly to $((X_t)_*\mu)_{t\ge0}$ in finite dimensional distributions. In order to prove $((X_t)_*\mu)_{t\ge0}$ is càdlàg (resp. continuous) on $\mathcal{M}_r\left(R^d\right)$ (resp. when $n=0$), it suffices to show that for any $f \in C_b^2\left(R^d\right) \cap \Phi\left(R^d\right)$ and any $T \in [0,\infty)$,

$$\lim_{\substack{k\to\infty\\ m\to\infty}} \sup_{t\le T} |\langle (X_t)_*\mu_k, f\rangle - \langle (X_t)_*\mu_m, f\rangle| = 0, \ a.s..$$

In fact, since

$$\begin{aligned}
\sup_{t\le T} |\langle (X_t)_*\mu_k, f\rangle - \langle (X_t)_*\mu_m, f\rangle| &= \sup_{t\le T}\left| \int_{\{k\vee m\ge|x|>k\wedge m\}} f(X_t(x))\ \mu(dx)\right| \\
&\le \|f\|_{\Phi(R^d)} \sup_{t\le T}\int_{\{k\vee m\ge|x|>k\wedge m\}} \phi_r(X_t(x))\ \mu(dx) \\
&\le \|f\|_{\Phi(R^d)} \int_{\{k\vee m\ge|x|>k\wedge m\}} \sup_{t\le T}\phi_r(X_t(x))\ \mu(dx),
\end{aligned}$$

it only needs to check

$$E\left[\int_{R^d} \sup_{t\le T}\phi_r(X_t(x))\ \mu(dx)\right] = \int_{R^d} E\left[\sup_{t\le T}\phi_r(X_t(x))\right]\mu(dx) < \infty.$$

However, by the Itô's formula,

$$\begin{aligned}
\phi_r(X_t(x)) &= \phi_r(x) + \int_0^t A_1\phi_r(X_s(x))\ ds + \int_0^t \nabla\phi_r(X_s(x))\cdot\ \sigma(X_s(x))dB_s + \\
&\int_0^t\int_U [\phi_r(X_{s-}(x) + h(X_{s-}(x),u)) - \phi_r(X_{s-}(x)) - \\
&\qquad \nabla\phi_r(X_{s-}(x))\cdot\ h(X_{s-}(x),u)]\ \eta(dsdu) + \\
&\int_0^t\int_U \nabla\phi_r(X_{s-}(x))\cdot\ h(X_{s-}(x),u)\ \eta(dsdu) \\
&= \phi_r(x) + \int_0^t A_1\phi_r(X_s(x))\ ds + \int_0^t \nabla\phi_r(X_s(x))\cdot\ \sigma(X_s(x))dB_s + \\
&\int_0^t\int_U [\phi_r(X_{s-}(x) + h(X_{s-}(x),u)) - \phi_r(X_{s-}(x))]\ \eta(dsdu).
\end{aligned}$$

Hence,

$$\begin{aligned}
\sup_{s\le t}\phi_r(X_s(x)) &\le \phi_r(x) + C_4^{(1)}\int_0^t \sup_{v\le s}\phi_r(X_v(x))\ ds + \\
&\sup_{0\le s\le t}\left|\int_0^s\int_U [\phi_r(X_{v-}(x) + h(X_{v-}(x),u)) - \phi_r(X_{v-}(x))]\ \eta(dvdu)\right| + \\
&\sup_{0\le s\le t}\left|\int_0^s \nabla\phi_r(X_v(x))\cdot\ \sigma(X_v(x))dB_v\right|;
\end{aligned}$$

$$
\begin{aligned}
E\left[\sup_{s\le t}\phi_r(X_s(x))\right] &\le \phi_r(x) + C_4^{(1)}\int_0^t E\left[\sup_{v\le s}\phi_r(X_v(x))\right]\ ds + \\
&\left\{E\left[\sup_{0\le s\le t}\left|\int_0^s\int_U \left[\phi_r(X_{v-}(x)+h(X_{v-}(x),u)) - \phi_r(X_{v-}(x))\right]\ \eta(dvdu)\right|^2\right]\right\}^{\frac{1}{2}} + \\
&\left\{E\left[\sup_{0\le s\le t}\left|\int_0^s \nabla\phi_r(X_v(x))\cdot\ \sigma(X_v(x))dB_v\right|^2\right]\right\}^{\frac{1}{2}} \\
&\le \phi_r(x) + C_4^{(1)}\int_0^t E\left[\sup_{v\le s}\phi_r(X_v(x))\right]\ ds + \\
&2\left\{E\left[\left|\int_0^t\int_U \left[\phi_r(X_{v-}(x)+h(X_{v-}(x),u)) - \phi_r(X_{v-}(x))\right]\ \eta(dvdu)\right|^2\right]\right\}^{\frac{1}{2}} + \\
&2\left\{E\left[\left|\int_0^t \nabla\phi_r(X_v(x))\cdot\ \sigma(X_v(x))dB_v\right|^2\right]\right\}^{\frac{1}{2}} \\
&\qquad \text{(By the Doob's maximal inequality of martingales)} \\
&= \phi_r(x) + C_4^{(1)}\int_0^t E\left[\sup_{v\le s}\phi_r(X_v(x))\right]\ ds + \\
&2\left\{E\left[\int_0^t\int_U \left|\phi_r(X_{v-}(x)+h(X_{v-}(x),u)) - \phi_r(X_{v-}(x))\right|^2\ dvn(du)\right]\right\}^{\frac{1}{2}} + \\
&2\left\{E\left[\int_0^t \sum_{i,j=1}^d a^{ij}(X_v(x))\partial_i\phi_r(X_v(x))\partial_j\phi_r(X_v(x))\ dv\right]\right\}^{\frac{1}{2}}
\end{aligned}
$$

Since there are positive constants $C_5,\ C_6$ such that $\forall z, y \in R^d$,

$$
\begin{aligned}
&|\phi_r(z+y) - \phi_r(z)| \le C_5\ |y| \quad \text{(c.f. Step 2 of proof of Lemma 4.3)}, \\
&\sum_{i,j=1}^d \left|a^{ij}(y)\partial_i\phi_r(y)\partial_j\phi_r(y)\right| \le C_6\ \phi_r(y)^2 \quad \text{(c.f. Step 1 of proof of Lemma 4.3)};
\end{aligned}
$$

by Lemma 4.3, we have

$$
\begin{aligned}
E&\left[\int_0^t\int_U \left|\phi_r(X_{v-}(x)+h(X_{v-}(x),u)) - \phi_r(X_{v-}(x))\right|^2\ dvn(du)\right] \\
&\le (C_5)^2 E\left[\int_0^t\int_U h(X_{v-}(x),u)^2\ dv\ n(du)\right] \\
&\le (C_5)^2 E\left[\int_0^t\int_U H(u)^2\phi_r(X_{v-}(x))^2\ dv\ n(du)\right] \\
&= (C_5)^2\int_U H(u)^2\ n(du)\ \int_0^t E\left[\phi_r(X_{v-}(x))^2\right] dv \\
&\le t\ e^{tC_4^{(1)}}(C_5)^2\int_U H(u)^2\ n(du)\ \phi_r(x)^2; \\
E&\left[\int_0^t \sum_{i,j=1}^d a^{ij}(X_v(x))\partial_i\phi_r(X_v(x))\partial_j\phi_r(X_v(x))\ dv\right] \\
&\le C_6\int_0^t E\left[\phi_r(X_v(x))^2\right]\ dv \le t\ e^{tC_4^{(1)}} C_6\ \phi_r(x)^2.
\end{aligned}
$$

Therefore, for any $t \le T$,

$$E\left[\sup_{s\le t}\phi_r(X_s(x))\right] \le \phi_r(x) + C_4^{(1)}\int_0^t E\left[\sup_{v\le s}\phi_r(X_v(x))\right]\ ds +$$
$$C_5\left\{\int_U H(u)^2\ n(du)\right\}^{\frac{1}{2}}\ \sqrt{C(T)}\ \phi_r(x) + \sqrt{C_6 C(T)}\ \phi_r(x),$$

where $C(T) = Te^{TC_4^{(1)}}$; and by the Gronwall inequality, there is a positive constant C_7 satisfying

$$E\left[\sup_{s\le t}\phi_r(X_s(x))\right] \le C_7\ \phi_r(x),\ \forall t \le T.$$

So

$$E\left[\int_{R^d}\sup_{t\le T}\phi_r(X_t(x))\ \mu(dx)\right] = \int_{R^d} E\left[\sup_{t\le T}\phi_r(X_t(x))\right]\mu(dx) < \infty.$$

Step 2. For $\{\mu_k\}_{k\ge1}$ specified in Step 1, for any $F(\nu) = G(\langle\nu, f_1\rangle, \cdots, \langle\nu, f_m\rangle) \in \mathcal{H}$, the proof of Lemma 4.2 gives that

$$\begin{aligned}&F((X_t)_*\mu_k) - F(\mu_k) - \int_0^t AF((X_s)_*\mu_k)\ ds\\ &:= K_{12}(t,\mu_k) + K_{13}(t,\mu_k) + K_{31}(t,\mu_k),\ \forall t \in [0,\infty),\end{aligned}$$

with $(K_{12}(t,\mu_k))_{t\ge0}$, $(K_{13}(t,\mu_k))_{t\ge0}$ and $(K_{31}(t,\mu_k))_{t\ge0}$ are three L^2-martingales given by

$$\begin{aligned}
K_{12}(t,\mu_k) &= \sum_{i=1}^m \left\langle \mu_k, \int_0^t \partial_i G(\langle Y_{s-}^k, f_1\rangle, \cdots, \langle Y_{s-}^k, f_m\rangle)\ \nabla f_i(X_s(x))\cdot\sigma(X_s(x))dB_s\right\rangle,\\
K_{13}(t,\mu_k) &= \sum_{i=1}^m \int_{R^d}\int_0^t\int_U \partial_i G(\langle Y_{s-}^k, f_1\rangle, \cdots, \langle Y_{s-}^k, f_m\rangle)\\
&\qquad [f_i(X_{s-}(x) + h(X_{s-}(x),u)) - f_i(X_{s-}(x))]\ \eta(dsdu)\mu_k(dx),\\
K_{31}(t,\mu_k) &= \int_0^t\int_U \{G(\langle Y_{s-}^k, f_1(x+h(x,u))\rangle, \cdots, \langle Y_{s-}^k, f_m(x+h(x,u))\rangle)\\
&\quad -G(\langle Y_{s-}^k, f_1(x)\rangle, \cdots, \langle Y_{s-}^k, f_m(x)\rangle) -\\
&\quad \sum_{i=1}^m \big\{\partial_i G(\langle Y_{s-}^k, f_1\rangle, \cdots, \langle Y_{s-}^k, f_m\rangle)\\
&\qquad \big\langle Y_{s-}^k, f_i(x+h(x,u)) - f_i(x)\big\rangle\big\}\big\}\ \eta(dsdu);
\end{aligned}$$

where $Y_s^k = (X_s)_*\mu_k,\ \forall s \in [0,\infty)$. Write

$$C_8 = \sup_{\substack{(x,y)\in(R^d)^2\\ 1\le i\le m}}\left\{\frac{\left|\nabla f_i(x)\sigma(x)\sigma^T(y)(\nabla f_i(y))^T\right|}{\phi_r(x)\phi_r(y)}\right\} < \infty.$$

Then by the Doob's maximal inequality and Lemma 4.3, for any $k \ge 1$,

$$\begin{aligned}
&E\left[\sup_{s\le t}\left\{K_{12}(s,\mu_k)^2\right\}\right] \le 4E\left[K_{12}(t,\mu_k)^2\right]\\
&\le 4m^2\sum_{i=1}^m E\left[\left\langle \mu_k^2, \|\partial_i G\|^2\int_0^t \left|\nabla f_i(X_s(x))\sigma(X_s(x))\sigma^T(X_s(y))(\nabla f_i(X_s(y)))^T\right|\ ds\right\rangle\right]\\
&\le 4m^3 C_8\left(\max_{1\le i\le m}\|\partial_i G\|^2\right)\int_0^t \left\langle \mu_k^2, E[\phi_r(X_s(x))\phi_r(X_s(y))]\right\rangle\ ds
\end{aligned}$$

$$= 4m^3 C_8 \left(\max_{1\le i\le m} \|\partial_i G\|^2 \right) \int_0^t \left\langle \mu_k^2, V_s^2 \left[\phi_r^{\otimes 2}\right] \right\rangle \, ds$$
$$\le 4m^3 C_8 \left(\max_{1\le i\le m} \|\partial_i G\|^2 \right) \; t e^{tC_4^{(2)}} \left\langle \mu^2, \phi_r^{\otimes 2} \right\rangle < \infty, \; \forall t \in [0,\infty).$$

By the remainder formula of the Taylor expansion,

$$\max_{1\le i\le m} |f_i(x+h(x,u)) - f_i(x)| \le C_9 \; |h(x,u)|, \; \forall x \in R^d, \text{ for some constant } C_9.$$

Using again the Doob's maximal inequality and Lemma 4.3, we have

$$E\left[\sup_{s\le t} \left\{K_{13}(s,\mu_k)^2\right\}\right] \le 4E\left[K_{13}(t,\mu_k)^2\right]$$
$$\le 4m^2 \sum_{i=1}^m E\left\{\left[\int_{R^d}\int_0^t\int_U \partial_i G(\langle Y_{s-}^k, f_1\rangle, \cdots, \langle Y_{s-}^k, f_m\rangle) \right.\right.$$
$$\left.\left. [f_i(X_{s-}(x)+h(X_{s-}(x),u)) - f_i(X_{s-}(x))] \;\; \eta(dsdu)\mu_k(dx)\right]^2\right\}$$
$$= 4m^2 \sum_{i=1}^m E\left[\int_{(R^d)^2}\int_0^t\int_U \partial_i G(\langle Y_{s-}^k, f_1\rangle, \cdots, \langle Y_{s-}^k, f_m\rangle)\right.$$
$$[f_i(X_{s-}(x)+h(X_{s-}(x),u)) - f_i(X_{s-}(x))] \;\; \eta(dsdu)$$
$$\int_0^t\int_U \partial_i G(\langle Y_{s-}^k, f_1\rangle, \cdots, \langle Y_{s-}^k, f_m\rangle)$$
$$\left.[f_i(X_{s-}(y)+h(X_{s-}(y),u)) - f_i(X_{s-}(y))] \; \eta(dsdu) \;\; \mu_k^2(dxdy)\right]$$
$$= 4m^2 \sum_{i=1}^m E\int_{(R^d)^2}\int_0^t\int_U \partial_i G(\langle Y_{s-}^k, f_1\rangle, \cdots, \langle Y_{s-}^k, f_m\rangle)^2$$
$$[f_i(X_{s-}(x)+h(X_{s-}(x),u)) - f_i(X_{s-}(x))]$$
$$[f_i(X_{s-}(y)+h(X_{s-}(y),u)) - f_i(X_{s-}(y))] \; dsn(du) \;\; \mu_k^2(dxdy)$$
$$\le 4m^3 \left(\max_{1\le i\le m} \|\partial_i G\|^2 \right) C_9^2$$
$$E\left\langle \mu_k^2, \int_0^t\int_U |h(X_{s-}(x),u)| \; |h(X_{s-}(y),u)| \; dsn(du) \right\rangle$$
$$\le 4m^3 \left(\max_{1\le i\le m} \|\partial_i G\|^2 \right) C_9^2$$
$$E\left\langle \mu_k^2, \int_0^t\int_U \phi_r(X_{s-}(x))\phi_r(X_{s-}(y))H(u)^2 dsn(du) \right\rangle$$
$$= 4m^3 \left(\max_{1\le i\le m} \|\partial_i G\|^2 \right) C_9^2 \int_U H(u)^2 \; n(du) \int_0^t \left\langle \mu_k^2, E[\phi_r(X_s(x))\phi_r(X_s(y))] \right\rangle \; ds$$
$$\le 4m^3 \left(\max_{1\le i\le m} \|\partial_i G\|^2 \right) C_9^2 \int_U H(u)^2 \; n(du) \; t e^{tC_4^{(2)}} \left\langle \mu^2, \phi_r^{\otimes 2} \right\rangle < \infty, \quad \forall k \ge 1, \; \forall t \ge 0.$$

Similarly to $\{K_{13}(\cdot,\mu_k)\}_{k\ge 1}$, one can check that

$$\sup_{k\ge 1} E\left[\sup_{s\le t} \left\{K_{31}(s,\mu_k)^2\right\}\right] < +\infty, \; \forall t \in [0,\infty).$$

Now combining with Step 1, let $k \to \infty$, we can obtain that

$$F((X_t)_*\mu) - F(\mu) - \int_0^t AF((X_s)_*\mu) \; ds, \; t \in [0,\infty),$$

is an L^2-martingale. $\square$

Lemma 4.6. If $Y=(Y_t)_{t\geq 0}$ is a càdlàg (resp. continuous) $\mathcal{M}_r\left(R^d\right)$-valued weakly Fellerian process (resp. when $n=0$) with the weak generator $(A,\mathcal{H})$ such that

$$\left\{\mu\in\mathcal{M}\left(R^d\right)\ \middle|\ \mu\left(R^d\right)=c\right\}$$

is its invariant sub state space for any $c\in[0,\infty)$, then for any $\mu\in\mathcal{M}_r\left(R^d\right)$, Y $(Y_0=\mu)$ and $((X_t)_*\mu)_{t\geq 0}$ have the same law.

Proof. Step 1. Let $\left\{\widetilde{T}_t\right\}_{t\geq 0}$ be the semigroup of Y and $\widetilde{P}_t(\mu,d\nu)$ (resp. $P_t(\mu,d\nu)$) the kernel probability measure associated to $\widetilde{T}_t$ (resp. T_t) for any $t\geq 0$. Then by assumption, $\left\{\widetilde{T}_t\right\}_{t\geq 0}$ is a weakly Fellerian semigroup on $C_b\left(\mathcal{M}_r\left(R^d\right)\right)$, and $\widetilde{P}_t(\mu,d\nu)$ and $P_t(\mu,d\nu)$ are continuous in $\mu\in\mathcal{M}_r\left(R^d\right)$ for any $t\geq 0$.

Note $\mathcal{M}\left(R^d\right)$ is dense in $\mathcal{M}_r\left(R^d\right)$. So $\left\{\widetilde{P}_t(\mu,d\nu)\right\}_{\mu\in\mathcal{M}_r(R^d)}$ and $\{P_t(\mu,d\nu)\}_{\mu\in\mathcal{M}_r(R^d)}$ are uniquely determined by $\left\{\widetilde{P}_t(\mu,d\nu)\right\}_{\mu\in\mathcal{M}(R^d)}$ and $\{P_t(\mu,d\nu)\}_{\mu\in\mathcal{M}(R^d)}$ respectively, for any $t\geq 0$. Recall

$$\Xi(c):=\left\{\mu\in\mathcal{M}_r\left(R^d\right)\ \middle|\ \mu\left(R^d\right)=c\right\}$$

is an invariant sub state space of both Y and $\left(((X_t)_*\mu)_{t\geq 0},\mu\in\mathcal{M}_r\left(R^d\right)\right)$ for any $c\geq 0$. In order to prove Lemma 4.6, it suffices to verify that for any $\mu\in\mathcal{M}\left(R^d\right)$, Y $(Y_0=\mu)$ and $((X_t)_*\mu)_{t\geq 0}$ have the same law on $D_{\Xi(\mu(R^d))}$.

Step 2. Given any $\mu\in\mathcal{M}\left(R^d\right)$, $k\geq 1$, and $f_i\in C_c^2\left(R^d\right)$, $1\leq i\leq k$. Write $\delta:=\mu\left(R^d\right)$. Note (2.1)-(2.2), one can check that

$$A_1 f_i\in C_c\left(R^d\right),\ \forall 1\leq i\leq k.$$

Recall $\sup\limits_{u\in U\setminus\mathcal{N}} H(u)=\widetilde{C}_5<\infty$. Let

$$m=\delta\left\{\sum_{i=1}^k\left(\|f_i\|+\widetilde{C}_5\right)^2\right\}^{\frac{1}{2}}+1.$$

Take $G:=G^{(m)}\in C_c^2\left(R^k\right)$ satisfying

$$G(x)=\begin{cases}\prod\limits_{i=1}^k x^i, & \text{if } |x|\leq m+1,\\ 0, & \text{if } |x|\geq m+2.\end{cases}$$

Let

$$F(\nu)=G(\langle\nu,f_1\rangle,\cdots,\langle\nu,f_k\rangle)\in\mathcal{H}.$$

Then for any $\nu\in\Xi(\delta)$,

$$\begin{aligned}&|(\langle\nu,f_1\rangle,\cdots,\langle\nu,f_k\rangle)|<m,\\ &|(\langle\nu,f_1(x+h(x,u))\rangle,\cdots,\langle\nu,f_k(x+h(x,u))\rangle)|<m,\ n(du)\ -a.e..\end{aligned}$$

Therefore, for any $\nu\in\Xi(\delta)$, by a straightforward calculation,

$$AF(\nu)\ =\ \sum_{i=1}^k\left(\prod_{1\leq j\neq i\leq k}\langle\nu,f_j\rangle\right)\langle\nu,A_1f_i\rangle+$$

$$
\begin{aligned}
&\frac{1}{2}\sum_{1\le i\neq j\le k}\left(\prod_{1\le s\neq i,j\le k}\langle\nu,f_s\rangle\right)\left\langle\nu^2,\nabla f_i(x)a(x,y)(\nabla f_j(y))^T\right\rangle+\\
&\int_U\left\{\prod_{i=1}^k\langle\nu,f_i(x_i+h(x_i,u))\rangle-\prod_{i=1}^k\langle\nu,f_i\rangle-\right.\\
&\qquad\left.\sum_{i=1}^k\left(\prod_{1\le j\neq i\le k}\langle\nu,f_j\rangle\right)\langle\nu,f_i(x_i+h(x_i,u))-f_i(x_i)\rangle\right\}\ n(du)\\
=\ &\sum_{i=1}^k\left(\prod_{1\le j\neq i\le k}\langle\nu,f_j\rangle\right)\left\langle\nu,\frac{1}{2}\sum_{p,q=1}^d a^{pq}(x_i)\frac{\partial^2 f_i(x_i)}{\partial x_i^p\partial x_i^q}+\sum_{p=1}^d b^p(x_i)\frac{\partial f_i(x_i)}{\partial x_i^p}\right\rangle+\\
&\frac{1}{2}\sum_{1\le i\neq j\le k}\left(\prod_{1\le s\neq i,j\le k}\langle\nu,f_s\rangle\right)\left\langle\nu^2,\nabla f_i(x)a(x,y)(\nabla f_j(y))^T\right\rangle+\\
&\int_{U\setminus U_0}\left\{\prod_{i=1}^k\langle\nu,f_i(x_i+h(x_i,u))\rangle-\prod_{i=1}^k\langle\nu,f_i\rangle-\right.\\
&\qquad\left.\sum_{i=1}^k\left(\prod_{1\le j\neq i\le k}\langle\nu,f_j\rangle\right)\langle\nu,\nabla f_i(x_i)\cdot\ h(x_i,u)\rangle\right\}\ n(du)+\\
&\int_{U_0}\left\{\prod_{i=1}^k\langle\nu,f_i(x_i+h(x_i,u))\rangle-\prod_{i=1}^k\langle\nu,f_i\rangle\right\}\ n(du).
\end{aligned}
$$

While for any $(x_1,\cdots,x_k)\in\left(R^d\right)^k$,

$$
\begin{aligned}
&A_k\left[\otimes_{i=1}^k f_i\right](x_1,\cdots,x_k)\\
&=\sum_{i=1}^k\left(\prod_{1\le j\neq i\le k}f_j(x_j)\right)\left\{\frac{1}{2}\sum_{p,q=1}^d a^{pq}(x_i)\frac{\partial^2 f_i(x_i)}{\partial x_i^p\partial x_i^q}+\sum_{p=1}^d b^p(x_i)\frac{\partial f_i(x_i)}{\partial x_i^p}\right\}+\\
&\quad\frac{1}{2}\sum_{1\le i\neq j\le k}\left(\prod_{1\le s\neq i,j\le k}f_s(x_s)\right)\sum_{p,q=1}^d a^{pq}(x_i,x_j)\frac{\partial f_i(x_i)}{\partial x_i^p}\frac{\partial f_j(x_j)}{\partial x_j^q}+\\
&\quad\int_{U\setminus U_0}\left\{\prod_{i=1}^k f_i(x_i+h(x_i,u))-\prod_{i=1}^k f_i(x_i)-\right.\\
&\qquad\left.\sum_{i=1}^k\sum_{p=1}^d\left(\prod_{1\le j\neq i\le k}f_j(x_j)\right)\frac{\partial f_i(x_i)}{\partial x_i^p}h^p(x_i,u)\right\}\ n(du)+\\
&\quad\int_{U_0}\left\{\prod_{i=1}^k f_i(x_i+h(x_i,u))-\prod_{i=1}^k f_i(x_i)\right\}\ n(du),
\end{aligned}
$$

where $\otimes_{i=1}^k f_i(x_1,\cdots,x_k)=\prod\limits_{i=1}^k f_i(x_i),\ \forall(x_1,\cdots,x_k)\in\left(R^d\right)^k$. Therefore,

$$
AF(\nu)=\left\langle\nu^k,A_k\left[\otimes_{i=1}^k f_i\right]\right\rangle,\ \forall\nu\in\Xi(\delta).
$$

Step 3. Given $\mu\in\Xi(\delta)$. Write

$$
\widetilde{\mathcal{H}}_1=\left\{F_{\otimes_{i=1}^k f_i,k}(\nu)\ \middle|\ \nu\in\Xi(\delta),\ \forall f_i\in C_c^2\left(R^d\right),\ \forall 1\le i\le k,\ \forall k\ge 1\right\}.
$$

Let $\mathcal{H}_1$ be the algebra generated by $\widetilde{\mathcal{H}}_1$. Then $\widetilde{\mathcal{H}}_1$ and $\mathcal{H}_1$ separates points in $\mathcal{P}(\Xi(\delta))$ ([18]), and $(A, \mathcal{H}_1)$ is uniquely determined by $\left(A, \widetilde{\mathcal{H}}_1\right)$.

Note each $\left(A_k, C_c^2\left(\left(R^d\right)^k\right)\right)$-martingale problem is well-posed. Remembering Step 2, similarly to the standard dual method for proving that the martingale problem of Fleming-Viot processes is well-posed ([4]), one can check that $(A, \mathcal{H}_1)$-martingale problem on $D_{\Xi(\delta)}$ is well-posed. While the laws of $((X_t)_*\mu)_{t\geq 0}$ and Y $(Y_0 = \mu)$ on $D_{\Xi(\delta)}$ are martingale solutions to $(A, \mathcal{H}_1)$-martingale problem on $D_{\Xi(\delta)}$ for initial point μ. Therefore, $((X_t)_*\mu)_{t\geq 0}$ and Y $(Y_0 = \mu)$ have the same law on $D_{\Xi(\delta)}$. □

Lemma 4.7. Given $\mu \in \mathcal{M}_r(R^d) \setminus \{0\}$, $(X_t)_{t\geq 0}$ is μ-incompressible if and only if

$$\langle \mu, A_1 f\rangle = 0,\ \left\langle \mu^2, A_2 f^{\otimes 2}\right\rangle = 0,\ \forall f \in C_c^2\left(R^d\right).$$

Proof. Recall Lemma 4.3. $(X_t)_{t\geq 0}$ is μ-incompressible if and only if $(X_t)_*\mu = \mu$, $\forall t \in [0,\infty)$, $a.s.$; which is equivalent to that

$$E\left[\left|\langle (X_t)_*\mu, f\rangle - \langle \mu, f\rangle\right|^2\right] = 0,\ \forall f \in C_c^2\left(R^d\right),\ \forall t \in [0,\infty), \tag{4.2}$$

because $((X_t)_*\mu)_{t\geq 0}$ is càdlàg in $t \in [0,\infty)$. Note that for any $f \in C_c^2\left(R^d\right)$,

$$\begin{aligned}
&E[\langle (X_t)_*\mu, f\rangle] = \left\langle \mu, V_t^1 f\right\rangle,\ E\left[\langle (X_t)_*\mu, f\rangle^2\right] = \left\langle \mu^2, V_t^2 f^{\otimes 2}\right\rangle;\\
&E\left[\left|\langle (X_t)_*\mu, f\rangle - \langle \mu, f\rangle\right|^2\right] = E\left[\langle (X_t)_*\mu, f\rangle^2\right] + \langle \mu, f\rangle^2 - 2\langle \mu, f\rangle E[\langle (X_t)_*\mu, f\rangle]\\
&= E\left[\left\langle ((X_t)_*\mu)^2, f^{\otimes 2}\right\rangle\right] + \langle \mu, f\rangle^2 - 2\langle \mu, f\rangle E[\langle (X_t)_*\mu, f\rangle]\\
&= \left\langle \mu^2, V_t^2 f^{\otimes 2}\right\rangle + \langle \mu, f\rangle^2 - 2\langle \mu, f\rangle \left\langle \mu, V_t^1 f\right\rangle.
\end{aligned}$$

It is not difficult to obtain that (4.2) is equivalent to that

$$\left\langle \mu^2, V_t^2 f^{\otimes 2}\right\rangle = \left\langle \mu^2, f^{\otimes 2}\right\rangle,\ \left\langle \mu, V_t^1 f\right\rangle = \langle \mu, f\rangle,\ \forall f \in C_c^2\left(R^d\right),\ \forall t \geq 0. \tag{4.3}$$

While for any $f \in C_c^2\left(R^d\right)$, note (2.1)-(2.2), one can check

$$A_2 f^{\otimes 2} \in C_c\left(\left(R^d\right)^2\right),\ A_1 f \in C_c\left(R^d\right).$$

Hence, for any $f \in C_c^2\left(R^d\right)$, remembering

$$V_t^2 f^{\otimes 2} = f^{\otimes 2} + \int_0^t V_s^2 A_2 f^{\otimes 2}\, ds,\ t \geq 0,$$

by Lemma 4.3,

$$\begin{aligned}
\frac{1}{t}\left|V_t^2 f^{\otimes 2} - f^{\otimes 2}\right| &= \frac{1}{t}\left|\int_0^t V_s^2 A_2 f^{\otimes 2}\, ds\right| \leq \frac{1}{t}\int_0^t V_s^2\left[\left|A_2 f^{\otimes 2}\right|\right]\, ds\\
&\leq \frac{1}{t}\int_0^t V_s^2 \phi_r^{\otimes 2}\, ds\ \left\|\frac{A_2 f^{\otimes 2}}{\phi_r^{\otimes 2}}\right\| \leq \frac{1}{t}\int_0^t e^{sC_4^{(2)}}\phi_r^{\otimes 2}\, ds\ \left\|\frac{A_2 f^{\otimes 2}}{\phi_r^{\otimes 2}}\right\|\\
&\leq e^{C_4^{(2)}}\left\|\frac{A_2 f^{\otimes 2}}{\phi_r^{\otimes 2}}\right\|\ \phi_r^{\otimes 2}\quad (\text{when } t \leq 1).
\end{aligned}$$

Therefore, by the dominated convergence theorem and Lemma 4.3, for any $f \in C_c^2\left(R^d\right)$,

$$\lim_{t\downarrow 0}\frac{1}{t}\left\langle \mu^2, V_t^2 f^{\otimes 2} - f^{\otimes 2}\right\rangle = \left\langle \mu^2, \lim_{t\downarrow 0}\frac{1}{t}\left\{V_t^2 f^{\otimes 2} - f^{\otimes 2}\right\}\right\rangle = \left\langle \mu^2, A_2 f^{\otimes 2}\right\rangle.$$

Similarly,

$$\lim_{t\downarrow 0}\frac{1}{t}\left\langle \mu, V_t^1 f - f\right\rangle = \left\langle \mu, \lim_{t\downarrow 0}\frac{1}{t}\left\{V_t^1 f - f\right\}\right\rangle = \left\langle \mu, A_1 f\right\rangle,\ \forall f\in C_c^2\left(R^d\right).$$

Thus, it is easy to see that (4.3) is equivalent to

$$\left\langle \mu, A_1 f\right\rangle = 0,\ \left\langle \mu^2, A_2 f^{\otimes 2}\right\rangle = 0,\ \forall f\in C_c^2\left(R^d\right).$$

□

Lemma 4.8. Assume

$$\lim_{\substack{s\downarrow t\\ y\to x}} X_s(y) = X_t(x),\ \lim_{\substack{s\uparrow t\\ y\to x}} X_s(y) = X_{t-}(x),\ \forall (t,x)\in[0,\infty)\times R^d,\ a.s.;$$

and $a.s.$, X_v is a continuous injection for any fixed $v\geq 0$; and $(X_t)_{t\geq 0}$ is μ-incompressible for some $\mu\in\mathcal{M}_r\left(R^d\right)\setminus\{0\}$. Then $\left((X_t(D))_{t\geq 0}, D\in\mathcal{K}_{S(\mu)}\right)$ is a càdlàg (resp. continuous) $\mathcal{K}_{S(\mu)}$-valued strong Markov process (resp. when $n=0$).

Proof. Step 1. Define the following map:

$$\begin{aligned}\mathcal{I}\quad &:\quad \mathcal{K}_{S(\mu)}\to\mathcal{M}_r\left(R^d\right),\\ &\qquad D\to\mu^D,\end{aligned}$$

where $\mu^D(dx) = I_D(x)\ \mu(dx)$. Then $\mathcal{I}$ is continuous and $\mathcal{I}\left(\mathcal{K}_{S(\mu)}\right)$ is a measurable subset of $\mathcal{M}_r\left(R^d\right)$, and $\mathcal{I}:\mathcal{K}_{S(\mu)}\to\mathcal{I}\left(\mathcal{K}_{S(\mu)}\right)$ is a measurable homeomorphism.

Indeed, for any $f\in\Phi\left(R^d\right)$, $\left\langle\mu^D, f\right\rangle = \int_D f(x)\ \mu(dx) = \int_{R^d} f(x) I_D(x)\ \mu(dx)$ is continuous in $D\in\mathcal{K}_{S(\mu)}$, which implies the continuity of $\mathcal{I}$. In order to prove $\mathcal{I}\left(\mathcal{K}_{S(\mu)}\right)$ is a measurable subset of $\mathcal{M}_r\left(R^d\right)$, note

$$\mathcal{K}_{S(\mu)} = \bigcup_{k=1}^{\infty}\left\{D\ \middle|\ D\subseteq S(\mu)\cap\left\{x\in R^d\ \mid |x|\leq k\right\},\ D\text{ is compact in } R^d\right\} := \bigcup_{k=1}^{\infty}\mathcal{K}_{S(\mu)}^{(k)},$$

and $\left\{D\ \middle|\ D\subseteq\left\{x\in R^d\ \mid |x|\leq k\right\},\ D\text{ is compact in } R^d\right\}$ is separable and compact with respect to the Hausdorff topology, we get that $\mathcal{K}_{S(\mu)}^{(k)}$ is also separable and compact with respect to the Hausdorff topology. Combining with the continuity of $\mathcal{I}$, we have $\mathcal{I}\left(\mathcal{K}_{S(\mu)}^{(k)}\right)$ is closed in $\mathcal{M}_r\left(R^d\right)$. Thus

$$\mathcal{I}\left(\mathcal{K}_{S(\mu)}\right) = \bigcup_{k=1}^{\infty}\mathcal{I}\left(\mathcal{K}_{S(\mu)}^{(k)}\right)$$

is a measurable subset of $\mathcal{M}_r\left(R^d\right)$.

Clearly, $\mathcal{I}:\mathcal{K}_{S(\mu)}\to\mathcal{I}\left(\mathcal{K}_{S(\mu)}\right)$ is bijective and onto, and its inverse map is the support map

$$S:\nu\in\mathcal{I}\left(\mathcal{K}_{S(\mu)}\right)\to S(\nu)\in\mathcal{K}_{S(\mu)}$$

that is a measurable map. So $\mathcal{I}:\mathcal{K}_{S(\mu)}\to\mathcal{I}\left(\mathcal{K}_{S(\mu)}\right)$ is a measurable homeomorphism.

Step 2. Clearly,

$$S\left(\mu^D\right) = D,\ \forall D\in\mathcal{K}_{S(\mu)}.$$

For any continuous injection $\psi:\ R^d\to R^d$ that preserves μ,

$$\mu^{\psi(D)} = \psi_*\left(\mu^D\right)\ \text{ and } \psi(D) = S\left(\psi_*\left(\mu^D\right)\right),\ \forall D\in S(\mu).$$

In fact, for any $f \in \Phi\left(R^d\right)$,

$$\begin{aligned} &\int_{R^d} f(x)\ \psi_*\left(\mu^D\right)(dx) = \int_D f(\psi(y))\ \mu(dy), \\ &\int_{R^d} f(x)\ \mu^{\psi(D)}(dx) = \int_{\psi(D)} f(x)\ \mu(dx) = \int_{\psi(D)} f(x)\ \psi_*\mu(dx) \\ &\qquad = \int_{\psi^{-1}(\psi(D))} f(\psi(y))\ \mu(dy) = \int_D f(\psi(y))\ \mu(dy). \end{aligned}$$

Therefore,

$$\mu^{\psi(D)} = \psi_*\left(\mu^D\right).$$

Step 3. For any $D \in \mathcal{K}_{S(\mu)}$, $(X_t(D))_{t\geq 0}$ is càdlàg (resp. continuous when $n = 0$) in $t \in [0, \infty)$.

Indeed, for any $0 \leq s < t < \infty$,

$$\begin{aligned} &\sup\left(\sup_{x\in D}\inf_{y\in D}\frac{|X_t(x)-X_s(y)|}{1+|X_t(x)-X_s(y)|},\ \sup_{y\in D}\inf_{x\in D}\frac{|X_s(y)-X_t(x)|}{1+|X_t(x)-X_s(y)|}\right) \\ &\quad\leq \sup\left(\sup_{x\in D}\frac{|X_t(x)-X_s(x)|}{1+|X_t(x)-X_s(x)|},\ \sup_{y\in D}\frac{|X_s(y)-X_t(y)|}{1+|X_t(y)-X_s(y)|}\right) \\ &\quad\leq \sup_{x\in D}|X_t(x)-X_s(x)|; \\ &\sup\left(\sup_{x\in D}\inf_{y\in D}\frac{|X_{t-}(x)-X_s(y)|}{1+|X_{t-}(x)-X_s(y)|},\ \sup_{y\in D}\inf_{x\in D}\frac{|X_s(y)-X_{t-}(x)|}{1+|X_s(y)-X_{t-}(x)|}\right) \\ &\quad\leq \sup_{x\in D}|X_{t-}(x)-X_s(x)|. \end{aligned}$$

Since $\lim\limits_{\substack{t\downarrow s\\ y\to x}} X_t(y) = X_s(x)$, $\lim\limits_{\substack{s\uparrow t\\ y\to x}} X_s(y) = X_{t-}(x)$ (resp. $X_t(x)$ when $n = 0$), and D is compact, we have

$$\begin{aligned} &\sup\left(\sup_{x\in D}\inf_{y\in D}\frac{|X_t(x)-X_s(y)|}{1+|X_t(x)-X_s(y)|},\ \sup_{y\in D}\inf_{x\in D}\frac{|X_s(y)-X_t(x)|}{1+|X_t(x)-X_s(y)|}\right) \\ &\quad\leq \lim_{t\downarrow s}\sup_{x\in D}|X_t(x)-X_s(x)| = 0; \\ &\sup\left(\sup_{x\in D}\inf_{y\in D}\frac{|X_{t-}(x)-X_s(y)|}{1+|X_{t-}(x)-X_s(y)|},\ \sup_{y\in D}\inf_{x\in D}\frac{|X_s(y)-X_{t-}(x)|}{1+|X_s(y)-X_{t-}(x)|}\right) \\ &\quad\leq \lim_{s\uparrow t}\sup_{x\in D}|X_-t(x)-X_s(x)| = 0 \\ &\qquad(\text{note } X_{t-}(x) = X_t(x) \text{ when } \ n = 0); \end{aligned}$$

which verifies $(X_t(D))_{t\geq 0}$ is càdlàg (resp. continuous when $n = 0$) in $t \in [0, \infty)$.

Step 4. Clearly, $\mathcal{K}_{S(\mu)}$ is the sate space of process $\left((X_t(D))_{t\geq 0}, D \in \mathcal{K}_{S(\mu)}\right)$. While by Step 2, $\mathcal{I}\left(\mathcal{K}_{S(\mu)}\right)$ is the state space of process $\left(\left((X_t)_*\left(\mu^D\right)\right)_{t\geq 0}, \mu^D \in \mathcal{I}\left(\mathcal{K}_{S(\mu)}\right)\right)$. Since by Steps 1-3, $\left((X_t(D))_{t\geq 0}, D \in \mathcal{K}_{S(\mu)}\right)$ is the image process of

$$\left(\left((X_t)_*\left(\mu^D\right)\right)_{t\geq 0}, \mu^D \in \mathcal{I}\left(\mathcal{K}_{S(\mu)}\right)\right)$$

under the measurable homeomorphism map $\mathcal{I}^{-1} : \mathcal{I}\left(\mathcal{K}_{S(\mu)}\right) \to \mathcal{K}_{S(\mu)}$; and $\mathcal{I}\left(\mathcal{K}_{S(\mu)}\right)$ is an invariant sub state space of strong Markov process

$$\left(((X_t)_*\nu)_{t\geq 0}, \nu \in \mathcal{M}_r\left(R^d\right)\right).$$

Hence, $\left((X_t(D))_{t\geq 0}, D \in \mathcal{K}_{S(\mu)}\right)$ is a $\mathcal{K}_{S(\mu)}$-valued strong Markov process. □

Proof of Theorem 2.1. Theorem 2.1 follows from Lemmas 4.4-4.8 immediately (Similarly, one can prove Theorem 2.1 under (2.1)′, (2.2)-(2.4)). □

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