Bounds on the local bases of primitive, non-powerful, minimally strong signed digraphs

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Abstract

In this paper, we study the local bases of primitive, non-powerful, minimally strong signed digraphs of order $n \geq 7$. We obtain the first two or three largest kth local bases, depending on whether n is odd or even, together with complete characterization of the equality cases, for primitive, non-powerful, minimally strong signed digraphs.

Key words: Signed digraph; Local base; Primitive minimally strong digraph; Powerful

AMS Subject Classifications: 05C20, 05C22, 15A48,

1. Introduction

A sign pattern matrix is a matrix each of whose entries is $1, -1$ or 0. For a square sign pattern matrix A, notice that in the computations of (the signs of) the entries of the power A^k, the ambiguous sign may arise when -1 is added to 1. So a new symbol "#" was introduced in [1] to denote the ambiguous sign. In [1], the set $\Gamma = \{0, 1, -1, \#\}$ is defined as the generalized sign set and the addition and multiplication involving the symbol # are defined as follows:

$$(-1) + 1 = 1 + (-1) = \#; \quad a + \# = \# + a = \# \text{ for all } a \in \Gamma;$$

$$0 \cdot \# = \# \cdot 0 = 0; \quad b \cdot \# = \# \cdot b = \# \text{ for all } b \in \Gamma \backslash \{0\}.$$

A matrix with entries in the set Γ is called a generalized sign pattern matrix. In this paper we assume that all the matrix operations considered are operations on matrices over Γ.

We now introduce some graph theoretical concepts.

When we say a digraph, we always permit loops but no multiple arcs. A signed digraph S is a digraph where each arc of S is assigned a sign 1 or -1. A generalized signed digraph S is a digraph where each arc of S is assigned a sign 1, -1 or #. A walk W in a signed digraph is a sequence of arcs $e_1, e_2, \cdots, e_k$ such that the terminal vertex of e_i is the same as the initial vertex of e_{i+1} for $i = 1, 2, \cdots, k-1$. The number k is called the length of the walk W, denoted by $l(W)$. The sign of the walk W (in a signed digraph), denoted by $\text{sgn}(W)$, is defined to be $\prod_{i=1}^{k} \text{sgn}(e_i)$.

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Two walks W_1 and W_2 in a signed digraph are called a pair of $SSSD$ walks, if they have the same initial vertex, same terminal vertex and same length, but they have different signs.

Let $A = (a_{ij})$ be a square sign pattern matrix of order n. The associated digraph $D(A)$ of A (possibly with loops) is defined to be the digraph with vertex set $V = \{1, 2, \cdots, n\}$ and arc set $E = \{(i,j) \mid a_{ij} \neq 0\}$. The associated signed digraph $S(A)$ of A is obtained from $D(A)$ by assigning the sign of a_{ij} to each arc (i,j) in $D(A)$.

A square generalized sign pattern matrix A is called powerful if each power of A contains no # entry. It is easy to see that a sign pattern matrix A is powerful if and only if the associated signed digraph $S(A)$ contains no pairs of $SSSD$ walks.

Definition 1.1 ([2]). Let A be a square generalized sign pattern matrix of order n and $A, A^2, A^3, \cdots$ be the sequence of powers of A. Suppose A^l is the first power that is repeated in the sequence. Namely, suppose l is the least positive integer such that there is a positive integer p such that

$$A^l = A^{l+p}. \tag{1.1}$$

Then l is called the generalized base (or simply base) of A, denoted by $l(A)$. The least positive integer p such that (1.1) holds for $l = l(A)$ is called the generalized period (or simply period) of A, denoted by $p(A)$.

For convenience, we will also define the corresponding concepts for signed digraphs. Let S be a signed digraph of order n. Then there is a sign pattern matrix A of order n such that $S(A) = S$. We say that S is powerful if A is powerful (i.e., S contains no pairs of $SSSD$ walks). Also we define $l(S) = l(A)$ and $p(S) = p(A)$.

A digraph D is called minimally strong provided that D is strong connected (or strong) and each digraph obtained from D by the removal of an arc is not strong.

Let D be a digraph. We denote by $L(D)$ the set of distinct lengths of all cycles of D; and $s(D)$ the length of the shortest cycle of D.

A digraph D is called a primitive digraph, if there is a positive integer k such that for each vertex x and vertex y (not necessarily distinct) in D, there exists a walk of length k from x to y. The least such k is called the primitive exponent of D, denoted by $\exp(D)$. It is well known that D is primitive if and only if D is strong and $\gcd(r_1, r_2, \cdots, r_k) = 1$, where $L(D) = \{r_1, r_2, \cdots, r_k\}$.

A signed digraph S is called primitive if the underlying digraph D is primitive, and in this case we define $\exp(S) = \exp(D)$. Similarly, S is called minimally strong if D is minimally strong.

A square matrix A is reducible if there exists a permutation matrix P such that

$$PAP^T = \begin{pmatrix} B & 0 \\ D & C \end{pmatrix},$$

where B and C are square non-vacuous matrices. The matrix A is irreducible if it is not reducible and is nearly reducible if it is irreducible and each matrix obtained from A by replacing a nonzero entry by 0 is reducible. A square sign pattern matrix A is called primitive if $D(A)$ is primitive and is called nearly reducible if $|A|$ is nearly reducible. Clearly, a sign pattern matrix A is nearly reducible if and only if $D(A)$ is minimally strong.

Let D be a primitive digraph of order n and $x \in V(D)$. The exponent of D at vertex x, denoted by $\exp_D(x)$, is the least positive integer k such that there is a walk of length k from x to each $y \in V(D)$. We choose to order the vertices of D in such a way that $\exp_D(v_{i_1}) \le \exp_D(v_{i_2}) \le \cdots \le \exp_D(v_{i_n})$; then the number $\exp_D(v_{i_k})$ is called the kth local exponent of D, denoted by $\exp_D(k)$. It is well known that $\exp(D) = \exp_D(n)$.

It was shown in [2] that if a signed digraph S is primitive non-powerful, then $l(S)$ is the least positive integer k such that there is a pair of $SSSD$ walks of length k between any two vertices in S.

Definition 1.2 ([3]). Let S be a primitive non-powerful signed digraph of order n. The base of S at a vertex $x \in V(S)$, denoted by $l_S(x)$, is defined to be the least positive integer l such that there is a pair of $SSSD$ walks of length k from x to each $y \in V(S)$ for each integer $k \ge l$. We choose to order the vertices of S in such a way that $l_S(v_{i_1}) \le l_S(v_{i_2}) \le \cdots \le l_S(v_{i_n})$; then we call $l_S(v_{i_k})$ the kth local base of S, denoted by $l_S(k)$.

Clearly, $l(S) = l_S(n)$. Let D be the underlying digraph of S; we define $\exp_S(x) = \exp_D(x)$ and $\exp_S(k) = \exp_D(k)$.

In [3], L. Wang et al. obtained sharp bounds of local bases for primitive non-powerful signed digraphs. In [4], B. Liu and L. You gave sharp upper bounds of the base for primitive nearly reducible sign pattern matrices. Define:

$$m_1(n,k) = \begin{cases} 2n^2 - 8n + 9 + k, & \text{if } 1 \le k \le n-2, \\ 2n^2 - 8n + 8 + k, & \text{if } n-1 \le k \le n; \end{cases}$$

$$m_2(n,k) = \begin{cases} 2n^2 - 10n + 13 + k, & \text{if } 1 \le k \le n-3, \\ 2n^2 - 10n + 12 + k, & \text{if } n-2 \le k \le n; \end{cases}$$

and

$$m_3(n,k) = \begin{cases} 2n^2 - 12n + 20 + k, & \text{if } 1 \le k \le n-4, \\ 2n^2 - 12n + 19 + k, & \text{if } n-3 \le k \le n-2, \\ 2n^2 - 12n + 18 + k, & \text{if } n-1 \le k \le n. \end{cases}$$

In the remainder of this paper, let $D_{n,s}$ $(n \ge 4, 2 \le s \le n-1)$ and H_n $(n \ge 6)$ be the digraphs of order n given in Fig. 1 and $H_n^{(i)}$ $(i = 1,2,3,4,5)$ be the primitive, minimally strong digraph of order $n \ge 6$ given in Fig. 3, respectively. In this paper, we study the local bases of primitive, non-powerful, minimally strong signed digraphs and obtain the following:

Main Theorem. Let S be a primitive, non-powerful, minimally strong signed digraph of order $n \ge 7$. Then

(1) $l_S(k) \le m_1(n,k)$ for $1 \le k \le n$,

with equality if and only if the underlying digraph is isomorphic to $D_{n,n-2}$.

(2) For each integer l with $m_2(n,k) < l < m_1(n,k)$ or $m_3(n,k) < l < m_2(n,k)$, there is no primitive, non-powerful, minimally strong signed digraph of order n with $l_S(k) = l$ for $1 \le k \le n$.

(3) $l_S(k) = m_2(n,k)$ for $1 \le k \le n$ if and only if n is even and the underlying digraph is isomorphic to $D_{n,n-3}$; and there is no primitive, non-powerful, minimally strong signed digraph of order n with $l_S(k) = m_2(n,k)$ if n is odd.

(4) $l_S(k) = m_3(n,k)$ for $k = 1, 2, \cdots, n-4, n-2, n$ if and only if the underlying digraph is isomorphic to H_n; $l_S(n-1) = m_3(n,n-1)$ if and only if the underlying digraph is isomorphic to H_n or $H_n^{(1)}$ whose two cycles of length $n-2$ have the same sign in S; and $l_S(n-3) = m_3(n,n-3)$ if and only if the underlying digraph is isomorphic to H_n or $H_n^{(i)}$ $(i = 1, 2)$ whose two cycles of length $n-2$ have the same sign in S.

Theorem 4.1 in [4] is exactly the case $l_S(n) = l(S)$ in Main Theorem.

2. Some preliminaries

In this section, we introduce some definitions, notations and properties which we need to use in the next sections.

Lemma 2.1 ([2]). If S is a primitive signed digraph, then S is non-powerful if and only if S contains a pair of cycles C_1 and C_2 (say, with lengths p_1 and p_2, respectively) satisfying one of the following two conditions:

(A1) p_1 is odd and p_2 is even and $\text{sgn}(C_2) = -1$;

(A2) Both p_1 and p_2 are odd and $\text{sgn}(C_1) = -\text{sgn}(C_2)$.

A pair of cycles C_1 and C_2 satisfying (A_1) or (A_2) is a "distinguished cycle pair". It is easy to see that if C_1 and C_2 are a distinguished cycle pair with lengths p_1 and p_2, respectively, then the closed walks $W_1 = p_2C_1$ (walk around C_1 p_2 times) and $W_2 = p_1C_2$ have the same length p_1p_2 and different signs:

$$(\text{sgn}(C_1))^{p_2} = -(\text{sgn}(C_2))^{p_1} \tag{2.1}$$

If t is a nonnegative integer, we denote by $R_t(x)$ the set of vertices of digraph D that can be reached by a walk of length t that begins at vertex x.

Lemma 2.2. Let D be a primitive digraph and x, y be two different vertices in D with $R_t(x) = \{y\}$. Then $\exp_D(x) = \exp_D(y) + t$.

Proof. Since $R_t(x) = \{y\}$, it is obvious that $\exp_D(x) \le \exp_D(y) + t$. If $t \ge \exp_D(x)$, then by the definition of $\exp_D(x)$, we have $R_t(x) = V(D) \ne \{y\}$, which is a contradiction. Hence $t < \exp_D(x)$. Since there is a walk of length $\exp_D(x)$ from x to each $v \in V(D)$, and $R_t(x) = \{y\}$; it is clear that there is a walk of length $\exp_D(x) - t$ from y to each $v \in V(D)$. Therefore $\exp_D(y) \le \exp_D(x) - t$. Hence $\exp_D(x) = \exp_D(y) + t$. □

Lemma 2.3 ([5]). Let D be a primitive digraph of order n. Then

$$\exp_D(k+1) \le \exp_D(k) + 1 \quad \text{for} \quad 1 \le k \le n-1.$$

Let $a_1, a_2, \cdots, a_k$ be positive integers. Define the Frobenius set $S(a_1, a_2, \cdots, a_k)$ as:

$$S(a_1, a_2, \cdots, a_k) = \{r_1 a_1 + \cdots + r_k a_k \mid r_1, \cdots, r_k \text{ are nonnegative integers}\}.$$

It is well known that if $\gcd(a_1, a_2, \cdots, a_k) = 1$, then $S(a_1, a_2, \cdots, a_k)$ contains all the sufficiently large positive integers. In this case we define the Frobenius number $\phi(a_1, a_2, \cdots, a_k)$ to be the least integer ϕ such that $m \in S(a_1, a_2, \cdots, a_k)$ for all integers $m \geq \phi$. Clearly, $\phi(a_1, a_2, \cdots, a_k) - 1$ is not in $S(a_1, a_2, \cdots, a_k)$. It is well known that if a, b are coprime positive integers, then $\phi(a, b) = (a-1)(b-1)$.

Also, by using the formula for the Frobenius number of arithmetical progressions ([6]), we have

$$\phi(n-4, n-3, n-2) \leq \lfloor \frac{n-4}{2} \rfloor (n-4). \tag{2.2}$$

Let $R = \{l_1, l_2, \cdots, l_k\}$ be a set of cycle lengths in a primitive digraph D such that $\gcd(l_1, l_2, \cdots, l_k) = 1$. For any $x, y \in V(D)$, the relative distance $d_R(x, y)$ from x to y is defined to be the length of the shortest walk from x to y which meets at least one cycle of each length l_i for $i = 1, 2, \cdots, k$. Let $\phi_R = \phi(l_1, l_2, \cdots, l_k)$ be the Frobenius number, $d_R = \max\limits_{x,y \in V(D)} d_R(x, y)$. We have the following known upper bounds ([7]):

$$\exp_D(x) \leq \phi_R + \max_{y \in V(D)} d_R(x, y); \tag{2.3}$$

$$\exp(D) \leq \phi_R + d_R. \tag{2.4}$$

An ordered pair of vertices x, y in a digraph D is said to have the unique walk property if every walk from x to y of length at least $d_{L(D)}(x, y)$ consists of some walk π of length $d_{L(D)}(x, y)$ form x to y augmented by a number of cycles each of which has a vertex in common with π.

Lemma 2.4 ([8]). Let D be a primitive digraph with $d_{L(D)}(x, y) = d_{L(D)}$. If the ordered pair of vertices x, y has the unique walk property, then

$$\exp(D) = \phi_{L(D)} + d_{L(D)}.$$

Lemma 2.5 ([4]). Let $R = \{l_1, l_2, \cdots, l_k\}$ be a set of cycle lengths in a primitive digraph D of order n with $\frac{n}{2} < l_1 < l_2 < \cdots < l_k$ and $\gcd(l_1, l_2, \cdots, l_k) = 1$. Then for each vertex x and each vertex y in D, we have

$$d_R(x, y) \leq n - 1 + \max\{l_{i+1} - l_i \mid i \in \{1, 2, \cdots, k-1\}\}.$$

Lemma 2.6 ([9]). Let D be a primitive digraph of order n and $L(D) = \{p, q\}$ with $3 \leq p < q$, $p + q > n$. Then $\exp(D) \leq n + p(q-2)$.

Lemma 2.7 ([10]). Let D be a primitive, minimally strong digraph of order n. Then the length of the longest cycle of D does not exceed $n - 1$.

Lemma 2.8 ([4]). Let D be a primitive, minimally strong digraph of order n with a cycle of length $n-1$. Then there only exists a unique cycle of length l $(1 < l < n-1)$ satisfying $\gcd(n-1, l) = 1$ in D.

Lemma 2.9 ([11]). Let D be a primitive, minimally strong digraph of order n, and $s(D) = s$. Then

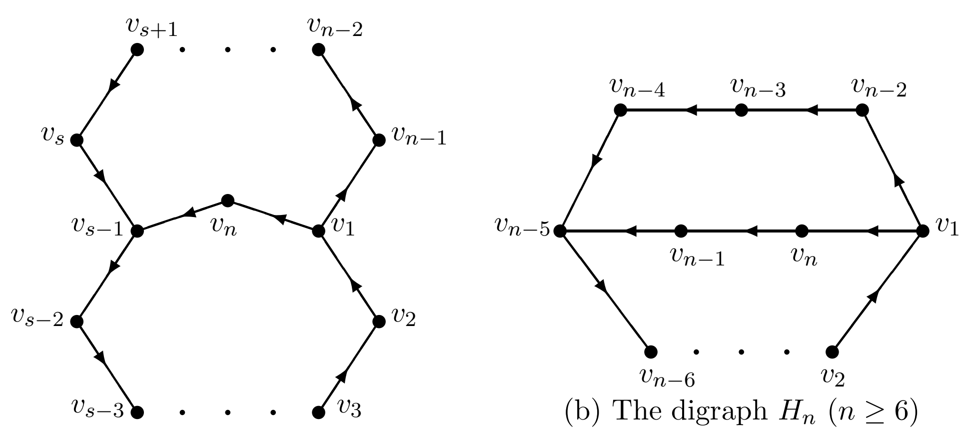
$$\exp_D(k) \le \begin{cases} k + 1 + s(n-3), & \text{if } 1 \le k \le s, \\ k + s(n-3), & \text{if } s+1 \le k \le n, \end{cases}$$

with equality if and only if D is isomorphic to $D_{n,s}$. If $\gcd(s, n-1) \ne 1$, then

$$\exp_D(k) < \begin{cases} k + 1 + s(n-3), & \text{if } 1 \le k \le s, \\ k + s(n-3), & \text{if } s+1 \le k \le n. \end{cases}$$

And if $\gcd(s, n-1) = 1$, then $D_{n,s}$ is a primitive, minimally strong digraph of order n with

$$\exp_{D_{n,s}}(k) = \begin{cases} k + 1 + s(n-3), & \text{if } 1 \le k \le s, \\ k + s(n-3), & \text{if } s+1 \le k \le n. \end{cases}$$



(a) The digraph $D_{n,s}$ $(n \ge 4, 2 \le s \le n-1)$

(b) The digraph H_n $(n \ge 6)$

Fig. 1. The digraph $D_{n,s}$ and the digraph H_n

Lemma 2.10 ([3]). Let S be a primitive, non-powerful signed digraph of order n. Then

$$l_S(k+1) \le l_S(k) + 1 \quad \text{for} \quad 1 \le k \le n-1.$$

Lemma 2.11. Let S be a primitive, non-powerful signed digraph and x, y be two different vertices in S with $R_t(x) = \{y\}$. If all the walks of length t from x to y have the same sign, then $l_S(x) = l_S(y) + t$.

Proof. Let v be any given vertex in S. By the definition of local base, there is a pair of $SSSD$ walks W_1 and W_2 (Q_1 and Q_2, respectively) from y (x, respectively) to v with length $l_S(y)$ ($l_S(x)$, respectively). Since $R_t(x) = \{y\}$, it is clear that there is a pair of $SSSD$ walks from x to v with length $l_S(y) + t$. So $l_S(x) \le l_S(y) + t$. For $i = 1, 2$, let Q_i' be the subwalk of

Q_i from y to v with length $l_S(x) - t > 0$. (If $t \geq l_S(x)$, then $R_t(x) = V(S) \neq \{y\}$, which is a contradiction). Since all the walks of length t from x to y have the same sign, Q'_1 and Q'_2 are also a pair of $SSSD$ walks. So $l_S(y) \leq l_S(x) - t$. Hence $l_S(x) = l_S(y) + t$. $\square$

Let S be a primitive, non-powerful signed digraph. For any $x \in V(S)$, let $r(x)$ be the least positive integer k such that there is a pair of $SSSD$ walks of length k from x to x. It is clear that $r(x) \leq l_S(x)$. From Lemma 2.6 in [3], we know that if there is a pair of $SSSD$ walks with length r from x to x, then $l_S(x) \leq \exp_S(x) + r$. So the following Lemma 2.12 holds.

Lemma 2.12. Let S be a primitive, non-powerful signed digraph and $x \in V(S)$. Then $l_S(x) \leq \exp_S(x) + r(x)$.

3. Some special cases

In this section, we consider those primitive, non-powerful, minimally strong signed digraphs whose underlying digraphs are $D_{n,s}$, H_n and $H_n^{(i)} (i = 1, 2, 3, 4, 5)$.

In the remainder of this paper, let $D_{n,t,s}$ $(n \geq 4, 1 \leq t \leq n - s, 2 \leq s \leq n - 1)$ be the digraph given in Fig. 2. Then we have $D_{n,s} = D_{n,1,s}$ and $H_n = D_{n,2,n-3}$. So we first consider the primitive, non-powerful signed digraph whose underlying digraph is $D_{n,t,s}$.

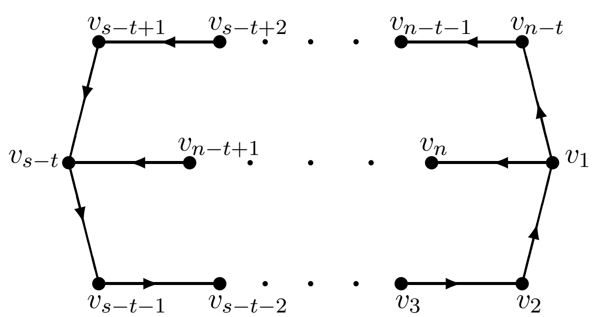


Fig. 2. The digraph $D_{n,t,s}$ $(n \geq 4, 1 \leq t \leq n - s, 2 \leq s \leq n - 1)$.

Theorem 3.1. Let S be a primitive, non-powerful signed digraph of order $n \geq 4$ with $D_{n,t,s}$ as its underlying digraph. Then

(1)

$$\exp_S(k) = \begin{cases} s(n-t-2)+t+k, & \text{if } 1 \leq k \leq s-t+1, \\ s(n-t-2)+t+k-1, & \text{if } s-t+2 \leq k \leq s-t+3, \\ s(n-t-2)+t+k-2, & \text{if } s-t+4 \leq k \leq s-t+5, \\ \quad\vdots & \quad\quad\vdots \\ s(n-t-2)+k+1, & \text{if } s+t-2 \leq k \leq s+t-1, \\ s(n-t-2)+k, & \text{if } s+t \leq k \leq n. \end{cases} \quad (3.1)$$

(2) $l_S(k) = \exp_S(k) + (n-t)s$, i.e.,

$$l_S(k) = \begin{cases} 2s(n-t-1)+t+k, & \text{if } 1 \le k \le s-t+1, \\ 2s(n-t-1)+t+k-1, & \text{if } s-t+2 \le k \le s-t+3, \\ 2s(n-t-1)+t+k-2, & \text{if } s-t+4 \le k \le s-t+5, \\ \vdots & \vdots \\ 2s(n-t-1)+k+1, & \text{if } s+t-2 \le k \le s+t-1, \\ 2s(n-t-1)+k, & \text{if } s+t \le k \le n. \end{cases} \tag{3.2}$$

Proof. Since S is primitive, and $L(S) = \{n-t, s\}$, we know that $\gcd(n-t, s) = 1$ and $t < n-s$. Let C_{n-t} and C_s be the cycles of lengths $n-t$ and s in S.

(1) Note that $d_{L(S)} = d_{L(S)}(v_{n-t}, v_{s-t+1}) = n-t+n-s-1 = 2n-s-t-1$, and the vertices v_{n-t}, v_{s-t+1} have the unique walk property. By Lemma 2.4 and (2.3), we have

$$\exp(S) = \exp_S(v_{n-t}) = \phi(n-t, s) + d_{L(S)} = (n-t-1)(s-1) + 2n-s-t-1 = (n-t-2)s + n.$$

Since $|R_1(v_i)| = 1$ for $2 \le i \le n$, it follows from Lemma 2.2 that $\exp_S(v_{n-t+1}) = \exp_S(v_{s-t}) + 1$ and $\exp_S(v_i) = \exp_S(v_{i-1}) + 1$ for $i = 2, \cdots, n-t, n-t+2, \cdots, n$. Hence we have $\exp_S(v_i) = (n-t-2)s + t + i$ for $1 \le i \le n-t$ and $\exp_S(v_{n-t+j}) = \exp_S(v_{s-t+j})$ for $1 \le j \le t$.

So by directly computing, we can obtain formula (3.1). In particular, $\exp_S(v_1) = \exp_S(1)$.

(2) First we show that $l_S(v_1) = \exp_S(v_1) + (n-t)s = 2s(n-t-1) + t + 1$. Since S is non-powerful and C_{n-t} and C_s are the only two cycles of S, C_{n-t} and C_s must be a distinguished cycle pair by Lemma 2.1. So sC_{n-t} and $(n-t)C_s$ have different signs by (2.1). Because v_1 is a common vertex of C_{n-t} and C_s, we have $r(v_1) \le (n-t)s$. Hence $l_S(v_1) \le \exp_S(v_1) + (n-t)s$ by Lemma 2.12.

Next we show that there is no pair of $SSSD$ walks of length $k = 2s(n-t-1)+t$ from v_1 to v_{s-t+1}. Suppose that W_1, W_2 are two walks of length k from v_1 to v_{s-t+1}. Then each W_i $(i = 1, 2)$ is a "union" of the path P from v_1 to v_{s-t+1} with length $n-s$ and cycles, that is, $W_i = P + a_iC_{n-t} + b_iC_s$, $a_i \ge 0, b_i \ge 0, (i = 1, 2)$. Thus we have

$$k = l(W_i) = n - s + a_i(n-t) + b_is, \quad a_i \ge 0, b_i \ge 0, (i = 1, 2).$$

So $(a_2 - a_1)(n-t) = (b_1 - b_2)s$. Write $b_1 - b_2 = (n-t)x$; then $a_2 - a_1 = sx$. We claim that $x = 0$.

If $x \ge 1$, then $a_2 \ge s$; so $k = n-s+a_2(n-t)+b_2s = n-s+(a_2-s)(n-t)+s(n-t)+b_2s$, which implies that $\phi(n-t, s) - 1 = (n-t-1)(s-1) - 1 = k-n+s-(n-t)s = (a_2-s)(n-t)+b_2s \in S(n-t, s)$, contradicting the definition of $\phi(n-t, s)$. Similarly we can get a contradiction if $x \le -1$. Thus we have $x = 0$. So $a_1 = a_2, b_1 = b_2$ and thus $\text{sgn}(W_1) = \text{sgn}(W_2)$. This argument shows that $l_S(v_1) \ge k+1 = \exp_S(v_1) + (n-t)s$. Hence $l_S(v_1) = \exp_S(v_1) + (n-t)s$.

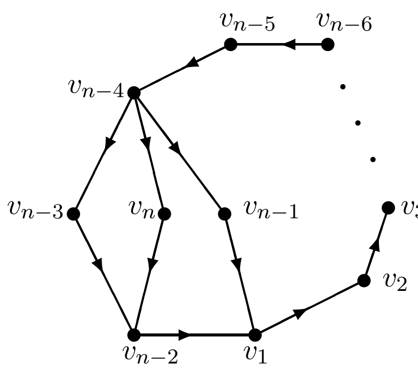
Again since $|R_1(v_i)| = 1$ for $2 \le i \le n$, it follows from Lemma 2.11 that $l_S(v_{n-t+1}) = l_S(v_{s-t}) + 1$ and $l_S(v_i) = l_S(v_{i-1}) + 1$ for $i = 2, \cdots, n-t, n-t+2, \cdots, n$. So it is not difficult to see that $l_S(v_i) = \exp_S(v_i) + (n-t)s$ for $1 \le i \le n$. Furthermore, $l_S(k) = \exp_S(k) + (n-t)s$ for $1 \le k \le n$. Hence by (3.1), we can obtain formula (3.2). $\square$

Since $D_{n,s} = D_{n,1,s}$, it is easy to check that the following Corollary 3.1 holds by Theorem 3.1.

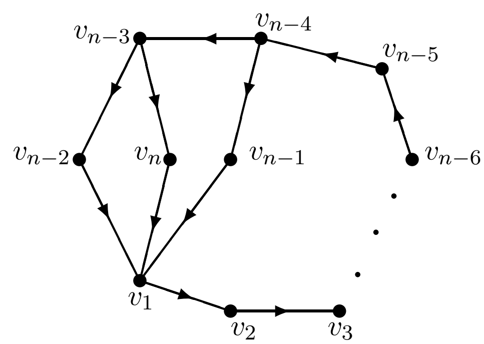
Corollary 3.1. Let S be a primitive, non-powerful signed digraph of order $n \geq 4$ with $D_{n,s}$ as its underlying digraph. Then

$$l_S(k) = \begin{cases} 2s(n-2)+k+1, & \text{if } 1 \leq k \leq s, \\ 2s(n-2)+k, & \text{if } s+1 \leq k \leq n. \end{cases} \tag{3.3}$$

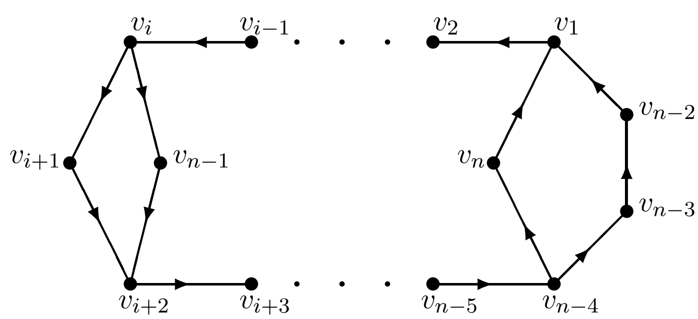
Note that the digraph $D_{n,n-2}$ is primitive and $D_{n,n-3}$ is primitive if and only if n is even. So the following Corollaries 3.2 and 3.3 hold by Corollary 3.1.



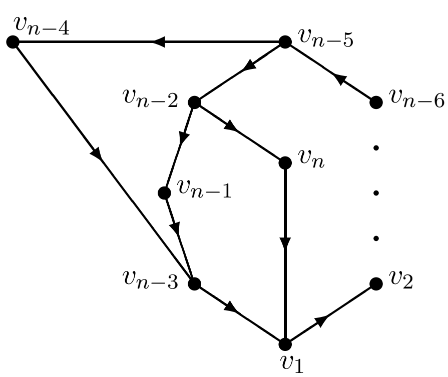
(a) The digraph $H_n^{(1)}$



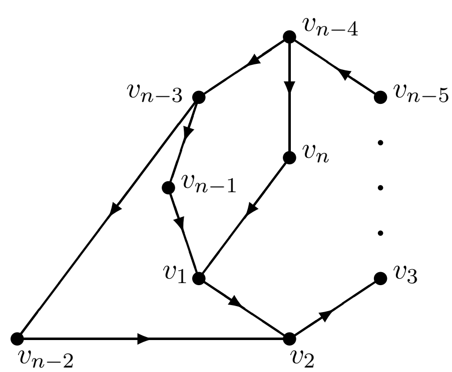
(b) The digraph $H_n^{(2)}$



(c) The digraph $H_n^{(3)}(i = 1, 2, \cdots, n-6)$



(d) The digraph $H_n^{(4)}$



(e) The digraph $H_n^{(5)}$

Fig. 3. The digraph $H_n^{(i)}(i = 1, 2, 3, 4, 5)$.

Corollary 3.2. Let S_1 be a primitive, non-powerful signed digraph of order $n \geq 4$ with

$D_{n,n-2}$ as its underlying digraph. Then

$$l_{S_1}(k) = m_1(n,k) \quad \text{for} \quad 1 \le k \le n.$$

Corollary 3.3. Let $n \ge 6, n \equiv 0 \pmod 2$. Let S_2 be a primitive, non-powerful signed digraph with $D_{n,n-3}$ as its underlying digraph. Then

$$l_{S_2}(k) = m_2(n,k) \quad \text{for} \quad 1 \le k \le n.$$

It is clear that H_n $(n \ge 6)$ is primitive. Since $H_n = D_{n,2,n-3}$, the following Corollary 3.4 holds by Theorem 3.1.

Corollary 3.4. Let S_3 be a primitive, non-powerful signed digraph of order $n \ge 6$ with H_n as its underlying digraph. Then

$$l_{S_3}(k) = m_3(n,k) \quad \text{for} \quad 1 \le k \le n.$$

Let D be a primitive, minimally strong digraph of order $n \ge 6$ with $L(D) = \{n-2, n-3\}$. Then according to the results in [10], we know that D is isomorphic to H_n or $H_n^{(i)}$ for some $i \in \{1,2,3,4,5\}$, and we have:

$$\exp(H_n^{(i)}) = \exp_{H_n^{(i)}}(v_{n-3}) = n^2 - 6n + 11 \quad \text{for} \quad i = 1,2,3; \tag{3.4}$$

$$\exp(H_n^{(i)}) = \exp_{H_n^{(i)}}(v_{n-1}) = n^2 - 6n + 10 \quad \text{for} \quad i = 4,5. \tag{3.5}$$

In the following, we consider the primitive, non-powerful signed digraph with $H_n^{(i)}$ $(i = 1,2,3,4,5)$ as its underlying digraph respectively.

Lemma 3.1. Let $S^{(1)}$ be a primitive, non-powerful signed digraph of order $n \ge 6$ with $H_n^{(1)}$ as its underlying digraph. Then

(1)

$$\exp_{S^{(1)}}(k) = \begin{cases} n^2 - 7n + 13 + k, & \text{if} \ 1 \le k \le n-3, \\ n^2 - 7n + 12 + k, & \text{if} \ n-2 \le k \le n-1, \\ n^2 - 7n + 11 + k, & \text{if} \ k = n. \end{cases} \tag{3.6}$$

(2) If the (only) two cycles of length $n-2$ of $S^{(1)}$ have different signs, then

$$l_{S^{(1)}}(k) \le \exp_{S^{(1)}}(k) + n - 2 \qquad \text{for} \ 1 \le k \le n. \tag{3.7}$$

(3) If the (only) two cycles of length $n-2$ of $S^{(1)}$ have the same sign, then $l_{S^{(1)}}(k) = \exp_{S^{(1)}}(k) + (n-2)(n-3)$, i.e.,

$$l_{S^{(1)}}(k) = \begin{cases} 2n^2 - 12n + 19 + k, & \text{if} \ 1 \le k \le n-3, \\ 2n^2 - 12n + 18 + k, & \text{if} \ n-2 \le k \le n-1, \\ 2n^2 - 12n + 17 + k, & \text{if} \ k = n. \end{cases} \tag{3.8}$$

In particular, $l_{S^{(1)}}(k) = m_3(n,k)$ for $k = n-3, n-1$ and $l_{S^{(1)}}(k) < m_3(n,k)$ for $k = 1, 2, \cdots, n-4, n-2, n$.

Proof. (1) From (3.4), we have $\exp_{S^{(1)}}(v_{n-3}) = n^2 - 6n + 11$. Note that v_n is a copy of v_{n-3} with respect to adjacency, so $\exp_{S^{(1)}}(v_n) = \exp_{S^{(1)}}(v_{n-3})$. Since $|R_1(v_j)| = 1$ for $j = 1, 2, \cdots, n-5, n-3, n-2, n-1, n$, it follows from Lemma 2.2 that $\exp_{S^{(1)}}(v_{n-2}) = \exp_{S^{(1)}}(v_{n-3}) - 1$; $\exp_{S^{(1)}}(v_1) = \exp_{S^{(1)}}(v_{n-2}) - 1$; $\exp_{S^{(1)}}(v_j) = \exp_{S^{(1)}}(v_{j-1}) - 1$ for $j = 2, 3, \cdots, n-4$ and $\exp_{S^{(1)}}(v_{n-1}) = \exp_{S^{(1)}}(v_1) + 1$. So by directly computing, we can obtain (3.6). In Particular, $\exp_{S^{(1)}}(v_{n-4}) = \exp_{S^{(1)}}(1)$.

(2) If the two cycles of length $n-2$ of $S^{(1)}$ have different signs, then it is easy to see that $r(v_j) \le n-2$ for $j = 1, 2, \cdots, n-4, n-2$. So $l_{S^{(1)}}(v_j) \le \exp_{S^{(1)}}(v_j) + n - 2$ for $j = 1, 2, \cdots, n-4, n-2$ by Lemma 2.12. Since $R_1(v_j) = \{v_{n-2}\}$ for $j = n-3, n$ and $R_1(v_{n-1}) = \{v_1\}$, we have $l_{S^{(1)}}(v_j) = l_{S^{(1)}}(v_{n-2}) + 1 \le \exp_{S^{(1)}}(v_{n-2}) + (n-2) + 1 = \exp_{S^{(1)}}(v_j) + n - 2$ for $j = n-3, n$ and $l_{S^{(1)}}(v_{n-1}) = l_{S^{(1)}}(v_1) + 1 \le \exp_{S^{(1)}}(v_1) + (n-2) + 1 = \exp_{S^{(1)}}(v_{n-1}) + n - 2$. Hence the formula (3.7) holds.

(3) If the two cycles of length $n-2$ of $S^{(1)}$ have the same sign, then by Lemma 2.1, each cycle of length $n-2$ and the cycle of length $n-3$ will form a distinguished cycle pair. Since v_{n-4} is a common vertex of one of the distinguished cycle pairs of $S^{(1)}$, we have $r(v_{n-4}) \le (n-2)(n-3)$. Hence $l_{S^{(1)}}(v_{n-4}) \le \exp_{S^{(1)}}(v_{n-4}) + (n-2)(n-3) = \exp_{S^{(1)}}(1) + (n-2)(n-3) = 2n^2 - 12n + 20$ by Lemma 2.12.

Now we show that there is no pair of $SSSD$ walks of length $k = 2n^2 - 12n + 19$ from v_{n-4} to v_{n-2}. Suppose that W_1, W_2 are two walks of length k from v_{n-4} to v_{n-2}. Then each W_i $(i = 1, 2)$ is a "union" of path $P_1 = (v_{n-4}, v_{n-3}, v_{n-2})$ or $P_2 = (v_{n-4}, v_n, v_{n-2})$ and cycles. Since the two cycles of length $n-2$ of $S^{(1)}$ have the same sign, then $\text{sgn}(P_1) = \text{sgn}(P_2)$ and thus we have

$$k = l(W_i) = 2 + a_i(n-3) + b_i(n-2), \quad a_i \ge 0, b_i \ge 0, (i = 1, 2).$$

So $(a_2 - a_1)(n-3) = (b_1 - b_2)(n-2)$. Write $b_1 - b_2 = (n-3)x$; then $a_2 - a_1 = (n-2)x$. We claim that $x = 0$.

If $x \ge 1$, then $a_2 \ge n-2$; so $k = 2 + [a_2 - (n-2)](n-3) + (n-2)(n-3) + b_2(n-2)$, which implies that $\phi(n-2, n-3) - 1 = (n-3)(n-4) - 1 = k - (n^2 - 5n + 8) = [a_2 - (n-2)](n-3) + b_2(n-2) \in S(n-2, n-3)$, contradicting the definition of $\phi(n-2, n-3)$. Similarly we can get a contradiction if $x \le -1$. Thus we have $x = 0$. So $a_1 = a_2, b_1 = b_2$ and thus $\text{sgn}(W_1) = \text{sgn}(W_2)$. This argument shows that $l_{S^{(1)}}(v_{n-4}) \ge k + 1 = \exp_{S^{(1)}}(v_{n-4}) + (n-2)(n-3)$. Hence $l_{S^{(1)}}(v_{n-4}) = \exp_{S^{(1)}}(v_{n-4}) + (n-2)(n-3)$.

Again since $|R_1(v_j)| = 1$ for $j = 1, 2, \cdots, n-5, n-3, n-2, n-1, n$, it follows from Lemma 2.11 that $l_{S^{(1)}}(v_j) = l_{S^{(1)}}(v_{j+1}) + 1$ for $j = 1, 2, \cdots, n-5, n-3$; $l_{S^{(1)}}(v_{n-2}) = l_{S^{(1)}}(v_{n-1}) = l_{S^{(1)}}(v_1) + 1$ and $l_{S^{(1)}}(v_n) = l_{S^{(1)}}(v_{n-2}) + 1$. So it is not difficult to check that $l_{S^{(1)}}(v_i) = \exp_{S^{(1)}}(v_i) + (n-2)(n-3)$ for $1 \le i \le n$. Furthermore, $l_{S^{(1)}}(k) = \exp_{S^{(1)}}(k) + (n-2)(n-3)$ for $1 \le k \le n$. Hence by (3.6), we can obtain formula (3.8). By the definition of $m_3(n,k)$, $l_{S^{(1)}}(k) \le m_3(n,k)$, with equality if and only if $k = n-3$ or $n-1$. $\Box$

Lemma 3.2. Let $S^{(2)}$ be a primitive, non-powerful signed digraph of order $n \geq 6$ with $H_n^{(2)}$ as its underlying digraph. Then

(1)

$$\exp_{S^{(2)}}(k) = \begin{cases} n^2 - 7n + 13 + k, & \text{if } 1 \leq k \leq n-3, \\ n^2 - 7n + 12 + k, & \text{if } k = n-2, \\ n^2 - 7n + 11 + k, & \text{if } n-1 \leq k \leq n. \end{cases} \tag{3.9}$$

(2) If the (only) two cycles of length $n-2$ of $S^{(2)}$ have different signs, then

$$l_{S^{(2)}}(k) \leq \exp_{S^{(2)}}(k) + n - 2 \qquad \text{for } 1 \leq k \leq n. \tag{3.10}$$

(3) If the (only) two cycles of length $n-2$ of $S^{(2)}$ have the same sign, then $l_{S^{(2)}}(k) = \exp_{S^{(2)}}(k) + (n-2)(n-3)$, i.e.,

$$l_{S^{(2)}}(k) = \begin{cases} 2n^2 - 12n + 19 + k, & \text{if } 1 \leq k \leq n-3, \\ 2n^2 - 12n + 18 + k, & \text{if } k = n-2, \\ 2n^2 - 12n + 17 + k, & \text{if } n-1 \leq k \leq n. \end{cases} \tag{3.11}$$

In particular, $l_{S^{(2)}}(k) = m_3(n,k)$ for $k = n-3$ and $l_{S^{(2)}}(k) < m_3(n,k)$ for $k = 1, 2, \cdots, n-4, n-2, n-1, n$.

Proof. Note that $R_2(v_{n-3}) = \{v_1\}$. If the two cycles of length $n-2$ of $S^{(2)}$ have different signs, then $r(v_j) \leq n-2$ for $j = 1, 2, \cdots, n-3$. Also if the two cycles of length $n-2$ of $S^{(2)}$ have the same sign, the only two walks of length 2 from v_{n-3} to v_1 have the same sign too. So we can prove this lemma by using a method similar to the proof of Lemma 3.1. $\Box$

Lemma 3.3. Let $S^{(3)}$ be a primitive, non-powerful signed digraph of order $n \geq 7$ with $H_n^{(3)}$ as its underlying digraph. Then

(1)

$$\exp_{S^{(3)}}(k) = \begin{cases} n^2 - 7n + 13 + k, & \text{if } 1 \leq k \leq n-4-i, \\ n^2 - 7n + 12 + k, & \text{if } n-3-i \leq k \leq n-2, \\ n^2 - 7n + 11 + k, & \text{if } n-1 \leq k \leq n. \end{cases} \tag{3.12}$$

(2) If the (only) two cycles of length $n-2$ of $S^{(3)}$ have different signs, then

$$l_{S^{(3)}}(k) \leq \exp_{S^{(3)}}(k) + n - 2 \qquad \text{for } 1 \leq k \leq n. \tag{3.13}$$

(3) If the (only) two cycles of length $n-2$ of $S^{(3)}$ have the same sign, then $l_{S^{(3)}}(k) = \exp_{S^{(3)}}(k) + (n-2)(n-3)$, i.e.,

$$l_{S^{(3)}}(k) = \begin{cases} 2n^2 - 12n + 19 + k, & \text{if } 1 \leq k \leq n-4-i, \\ 2n^2 - 12n + 18 + k, & \text{if } n-3-i \leq k \leq n-2, \\ 2n^2 - 12n + 17 + k, & \text{if } n-1 \leq k \leq n. \end{cases} \tag{3.14}$$

Furthermore, we have $l_{S^{(3)}}(k) < m_3(n,k)$ for $1 \leq k \leq n$.

Proof. Note that $R_2(v_i) = \{v_{i+2}\}$. If the two cycles of length $n-2$ of $S^{(3)}$ have different signs, then $r(v_j) \le n-2$ for $j = 1, 2, \cdots, i, i+2, i+3, \cdots, n-2$. So similar to the proof of (1) and (2) in Lemma 3.1, we can obtain (3.12) and (3.13).

If the only two cycles of length $n-2$ of $S^{(3)}$ have the same sign, then the only two cycles of length $n-3$ of $S^{(3)}$ must have the same sign too. So by Lemma 2.1, each cycle of length $n-2$ and each cycle of length $n-3$ will form a distinguished cycle pair; and note that the only two walks of length 2 from v_i to v_{i+2} have the same sign, using the method similar to (3) in Lemma 3.1, we can obtain (3.14). Since $1 \le i \le n-6$, we have $l_{S^{(3)}}(k) < m_3(n,k)$ for $1 \le k \le n$. □

Lemma 3.4. Let $S^{(i)}$ be a primitive, non-powerful signed digraph of order $n \ge 7$ with $H_n^{(i)}$ $(i = 4, 5)$ as its underlying digraph. Then

(1)

$$\exp_{S^{(i)}}(k) = \begin{cases} n^2 - 7n + 12 + k, & \text{if } 1 \le k \le n-3, \\ n^2 - 7n + 11 + k, & \text{if } n-2 \le k \le n-1, \\ n^2 - 7n + 10 + k, & \text{if } k = n. \end{cases} \tag{3.15}$$

(2) If the (only) two cycles of length $n-3$ of $S^{(i)}$ have different signs, then

$$l_{S^{(i)}}(k) \le \exp_{S^{(i)}}(k) + n - 2 \qquad \text{for } 1 \le k \le n. \tag{3.16}$$

(3) If the (only) two cycles of length $n-3$ of $S^{(i)}$ have the same sign, then $l_{S^{(i)}}(k) = \exp_{S^{(i)}}(k) + (n-2)(n-3)$, i.e.,

$$l_{S^{(i)}}(k) = \begin{cases} 2n^2 - 12n + 18 + k, & \text{if } 1 \le k \le n-3, \\ 2n^2 - 12n + 17 + k, & \text{if } n-2 \le k \le n-1, \\ 2n^2 - 12n + 16 + k, & \text{if } k = n. \end{cases} \tag{3.17}$$

Furthermore, we have $l_{S^{(i)}}(k) < m_3(n,k)$ for $1 \le k \le n$.

Proof. We only show the case $i = 4$; and the proof for the case $i = 5$ is similar to $i = 4$.

(1) From (3.5), we have $\exp_{S^{(4)}}(v_{n-1}) = n^2 - 6n + 10$. Since $|R_1(v_j)| = 1$ for $j = 1, 2, \cdots, n-6, n-4, n-3, n-1, n$, by Lemma 2.2, we know that $\exp_{S^{(4)}}(v_{n-3}) = \exp_{S^{(4)}}(v_{n-1}) - 1 = n^2 - 6n + 9$; $\exp_{S^{(4)}}(v_1) = \exp_{S^{(4)}}(v_{n-3}) - 1 = n^2 - 6n + 8$; $\exp_{S^{(4)}}(v_{n-4}) = \exp_{S^{(4)}}(v_{n-3}) + 1 = n^2 - 6n + 10$; $\exp_{S^{(4)}}(v_n) = \exp_{S^{(4)}}(v_1) + 1 = n^2 - 6n + 9$ and $\exp_{S^{(4)}}(v_j) = \exp_{S^{(4)}}(v_{j-1}) - 1$ for $j = 2, 3, \cdots, n-5$, or equivalently, $\exp_{S^{(4)}}(v_j) = n^2 - 6n + 9 - j$ for $j = 1, 2, \cdots, n-5$.

Now we show that $\exp_{S^{(4)}}(v_{n-2}) = n^2 - 7n + 13$. Since $R_1(v_{n-5}) \supset \{v_{n-2}\}$, it is clear that $\exp_{S^{(4)}}(v_{n-5}) \le \exp_{S^{(4)}}(v_{n-2}) + 1$. Hence $\exp_{S^{(4)}}(v_{n-2}) \ge \exp_{S^{(4)}}(v_{n-5}) - 1 = n^2 - 7n + 13$. For nonnegative integer i, let $A_i = R_{i(n-3)+1}(v_{n-2})$. Suppose $|\bigcup_{j=0}^{i-1} A_j| < n$ and $|A_i \backslash \bigcup_{j=0}^{i-1} A_j| = 0$. Then $|A_m \backslash \bigcup_{j=0}^{i-1} A_j| = 0$ for all $m \ge i$, and so $|\bigcup_{j=0}^{\infty} A_j| < n$, which implies $H_n^{(4)}$ is imprimitive, a contradiction. Therefore

$$|A_i \backslash \bigcup_{j=0}^{i-1} A_j| \ge 1, \qquad \text{provided} \quad |\bigcup_{j=0}^{i-1} A_j| < n.$$

Since $A_0 = \{v_n, v_{n-1}\}$ and $A_1 = \{v_n, v_{n-1}, v_{n-2}, v_{n-3}, v_{n-4}\}$, we have $|A_{n-4}| = n$ and so $\exp_{S^{(4)}}(v_{n-2}) \le (n-4)(n-3) + 1 = n^2 - 7n + 13$. Hence $\exp_{S^{(4)}}(v_{n-2}) = n^2 - 7n + 13$.

So by ordering the above local exponents, we can obtain (3.15).

(2) If the two cycles of length $n-3$ of $S^{(4)}$ have different signs, then it is easy to see that $r(v_j) \leq n-3$ for $j = 1, 2, \cdots, n-5$. So $l_{S^{(4)}}(v_j) \leq \exp_{S^{(4)}}(v_j) + n - 3$ for $j = 1, 2, \cdots, n-5$ by Lemma 2.12. Since $R_1(v_j) = \{v_1\}$ for $j = n, n-3$ and $R_1(v_j) = \{v_{n-3}\}$ for $j = n-1, n-4$, by Lemma 2.11, we know that $l_{S^{(4)}}(v_j) = l_{S^{(4)}}(v_1) + 1 \leq \exp_{S^{(4)}}(v_1) + (n-3) + 1 = \exp_{S^{(4)}}(v_j) + n - 3$ for $j = n, n-3$ and $l_{S^{(4)}}(v_j) = l_{S^{(4)}}(v_{n-3}) + 1 \leq \exp_{S^{(4)}}(v_{n-3}) + (n-3) + 1 = \exp_{S^{(4)}}(v_j) + n - 3$ for $j = n-1, n-4$.

For v_{n-2}, because $R_1(v_{n-2}) \supseteq \{v_n\}$, we have $l_{S^{(4)}}(v_{n-2}) \leq l_{S^{(4)}}(v_n) + 1 \leq \exp_{S^{(4)}}(v_n) + (n-3) + 1 = n^2 - 6n + 9 + n - 2$.

Now by computing, we can obtain that

$$l_{S^{(4)}}(k) \leq \begin{cases} n^2 - 6n + 10 + k, & \text{if } 1 \leq k \leq n-4, \\ n^2 - 6n + 9 + k, & \text{if } n-3 \leq k \leq n-2, \\ n^2 - 6n + 8 + k, & \text{if } k = n-1, \\ n^2 - 6n + 7 + k, & \text{if } k = n. \end{cases}$$

Hence $l_{S^{(4)}}(k) \leq \exp_{S^{(4)}}(k) + n - 2$ for $1 \leq k \leq n$.

(3) In this case, by using the method similar to (3) in Lemma 3.1, we can show that there is no pair of $SSSD$ walks of length $k = \exp_{S^{(4)}}(v_j) + (n-2)(n-3) - 1$ from v_j to v_{n-1} for $j = n-5, n-2$. And furthermore, we can obtain (3.17) and $l_{S^{(4)}}(k) < m_3(n,k)$ for $1 \leq k \leq n$. $\square$

4. Proof of Main Theorem

Proof of Main Theorem. Let D be the underlying digraph of S and $s = s(D)$. By Lemma 2.7, we know that there is no cycle with length n in D. Suppose D contains a cycle of length $n-1$. Then by Lemma 2.8, D consists of two cycles of length $n-1$ and l, where $1 < l < n-1$ and $\gcd(n-1, l) = 1$. Thus $l = s$ and D is isomorphic to $D_{n,s}$. If $s = n-2$, then $l_S(k) = m_1(n,k)$ for $1 \leq k \leq n$ by Corollary 3.2. If $s = n-3$, then n is even since $D_{n,3}$ is primitive; and $l_S(k) = m_2(n,k)$ for $1 \leq k \leq n$ by Corollary 3.3. If $s \leq n-4$, then by Corollary 3.1, $l_S(k) \leq 2(n-4)(n-2) + k + 1 = 2n^2 - 12n + 17 + k < m_3(n,k)$ for $1 \leq k \leq n$.

Suppose $L(D) = \{n-2, n-3\}$. Then D is isomorphic to H_n or $H_n^{(i)}$ for some $i \in \{1, 2, 3, 4, 5\}$. By Corollary 3.4 and Lemmas 3.1-3.4, we have $l_S(k) \leq m_3(n,k)$ for $1 \leq k \leq n$. If D is isomorphic to H_n, then $l_S(k) = m_3(n,k)$ for $1 \leq k \leq n$ by Corollary 3.4. If D is isomorphic to $H_n^{(1)}$, then by Lemma 3.1, $l_S(k) = m_3(n,k)$ if and only if the two cycles of length $n-2$ in S have the same sign and $k = n-3, n-1$. If D is isomorphic to $H_n^{(2)}$, then by Lemma 3.2, $l_S(k) = m_3(n,k)$ if and only if the two cycles of length $n-2$ in S have the same sign and $k = n-3$. If D is isomorphic to $H_n^{(i)}$ $(i = 3, 4, 5)$, then by Lemmas 3.3 and 3.4, we have $l_S(k) < m_3(n,k)$ for $1 \leq k \leq n$.

Note that $m_3(n,k) < m_2(n,k) < m_1(n,k)$ $(n \geq 7)$. So it is easy to see that in order to obtain the four parts of this theorem, we only need to show that $l_S(k) < m_3(n,k)$ for $1 \leq k \leq n$ if D contains no cycle of length $n-1$ and $L(D) \neq \{n-2, n-3\}$.

In the following, we assume that D contains no cycle of length $n-1$ and $L(D) \neq \{n-2, n-3\}$. We will show that $l_S(k) < m_3(n,k)$ for $1 \leq k \leq n$. By Lemma 2.10 we know that $l_S(k) \leq l_S(1) + k - 1$ for $1 \leq k \leq n$. Hence by the definition of $m_3(n,k)$, it suffices to show that $l_S(1) < m_3(n,1) - 2 = 2n^2 - 12n + 19$.

Since S is primitive non-powerful, there is a distinguished cycle pair C_1 and C_2 (with lengths, say, p_1 and p_2 respectively) by Lemma 2.1, where p_1C_2 and p_2C_1 have different signs by (2.1). Let $p_1 \leq p_2$.

Case 1. C_1 and C_2 have no common vertices.

Then $p_1 + p_2 \leq n$, and so $p_1 \leq \frac{n}{2}$. Let Q_1 be a shortest path with length q_1 from C_1 to C_2. Let $\{v_1\} = V(Q_1) \bigcap V(C_1), \{v_2\} = V(Q_1) \bigcap V(C_2)$, let Q_2 be a shortest path of length q_2 from v_2 to v_1. Then $q_1 \leq n - p_1 - p_2 + 1$ and $q_2 \leq n - 1$. Clearly, $p_2C_1 + Q_1 + Q_2$ and $p_1C_2 + Q_1 + Q_2$ are a pair of $SSSD$ walks of length $p_1p_2 + q_1 + q_2$ from vertex v_1 to v_1, hence $r(v_1) \leq p_1p_2 + q_1 + q_2$. From the proof of Case 1 in [3, Theorem 3.3], we know that $\exp_D(v_1) \leq p_1(n-2) + 1$.

Subcase 1.1. $n \geq 8$. Since $p_1p_2 + q_1 + q_2 \leq p_1p_2 - p_1 - p_2 + 2n = (p_1 - 1)(p_2 - 1) + 2n - 1 \leq (\frac{p_1 + p_2 - 2}{2})^2 + 2n - 1 \leq (\frac{n-2}{2})^2 + 2n - 1 = \frac{n^2}{4} + n$, by Lemma 2.12, $l_S(v_1) \leq \exp_D(v_1) + r(v_1) \leq \frac{n}{2}(n-2) + 1 + \frac{n^2}{4} + n = \frac{3n^2}{4} + 1$. Thus $l_S(1) \leq l_S(v_1) \leq \frac{3n^2}{4} + 1 < 2n^2 - 12n + 19$ for $n \geq 8$.

Subcase 1.2. $n = 7$. Let $V(D) = \{v_1, v_2, \cdots, v_7\}$.

Subcase 1.2.1. $p_1 = 2$. Then $p_2 \leq 5$ and $\exp_D(v_1) \leq p_1(n-2) + 1 = 2 \times 5 + 1 = 11$. Now, $r(v_1) \leq p_1p_2 + q_1 + q_2 \leq p_1p_2 - p_1 - p_2 + 2n = (p_1 - 1)(p_2 - 1) + 2n - 1 \leq 1 \times 4 + 13 = 17$. By Lemma 2.12, $l_S(v_1) \leq 11 + 17 = 28$. Hence $l_S(1) \leq l_S(v_1) \leq 28 < 33 = m_3(7,1) - 2$.

Subcase 1.2.2. $p_1 = 3$. Then $\exp_D(v_1) \leq p_1(n-2) + 1 = 3 \times 5 + 1 = 16$.

Suppose $p_2 = 3$. Then $q_1 \leq 2$. We claim that $q_2 \leq 5$. If $q_1 = 1$, then $q_2 \leq 4$ since D contains no cycle of length 6 or 7. If $q_1 = 2$, we can assume that $C_1 = (v_1, v_5, v_6)$, $C_2 = (v_2, v_3, v_4)$ and (v_1, v_7) and (v_7, v_2) are two arcs of D. Since $p_1 = p_2$, by symmetry, we can assume that the length of the shortest path from C_2 to C_1 is also 2. Note that (v_4, v_7) must not be an arc of D and $(v_2, v_7) \in E(D)$ implies that the digraph induced by vertex set $\{v_1, v_5, v_6, v_7\}$ is minimally strong. Thus $q_2 \leq 5$. Therefore $r(v_1) \leq p_1p_2 + q_1 + q_2 \leq 3 \times 3 + 2 + 5 = 16$ and so $l_S(v_1) \leq 16 + 16 = 32$ by Lemma 2.12. Thus $l_S(1) \leq l_S(v_1) \leq 32 < 33 = m_3(7,1) - 2$.

Suppose $p_2 = 4$. Then $q_1 = 1$ and so $q_2 \leq 4$ since D contains no cycle of length 6 or 7. If $q_2 \leq 3$, then $r(v_1) \leq p_1p_2 + q_1 + q_2 \leq 3 \times 4 + 1 + 3 = 16$ and so $l_S(v_1) \leq 16 + 16 = 32$ by Lemma 2.12. Hence $l_S(1) \leq l_S(v_1) \leq 32 < 33 = m_3(7,1) - 2$. If $q_2 = 4$, then without loss of generality, we can assume that D consists of two cycles $C_1 = (v_1, v_6, v_7)$, $C_2 = (v_2, v_3, v_4, v_5)$ and two additional arcs (v_1, v_2) and (v_4, v_7). Since $L(D) = \{3, 4, 5\}$, by (2.3), we have $\exp_D(v_1) \leq \phi(3,4,5) + \max_{v_i \in V(D)} d_{L(D)}(v_1, v_i) \leq 3 + 6 = 9$. Thus $r(v_1) \leq p_1p_2 + q_1 + q_2 = 3 \times 4 + 1 + 4 = 17$ and so $l_S(v_1) \leq 9 + 17 = 26$ by Lemma 2.12. Hence $l_S(1) \leq l_S(v_1) \leq 26 < 33 = m_3(7,1) - 2$.

Case 2. C_1 and C_2 have some common vertices.

Subcase 2.1. $p_1 = p_2$.

Then C_1 and C_2 are also a pair of $SSSD$ walks of length p_1. Let $x \in V(C_1) \bigcap V(C_2)$. Then $r(x) \le p_1 \le n-2$. By Lemma 2.9, we have $\exp(D) \le n+s(n-3) \le n+(n-2)(n-3) = n^2-4n+6$. Thus by Lemma 2.12, $l_S(1) \le l_S(x) \le \exp_S(x)+r(x) \le \exp(S)+r(x) \le (n^2-4n+6)+(n-2) = n^2-3n+4 < 2n^2-12n+19$.

In the following cases, we will consider the situation $p_1 \ne p_2$. It is clear that $|V(C_1) \bigcap V(C_2)| \ge p_1+p_2-n$; and for any $u \in V(C_1) \bigcap V(C_2)$, we have $r(u) \le p_1 p_2$.

Subcase 2.2. $p_2 = n-2, p_1 = n-3$.

Subcase 2.2.1. $s = n-4$, i.e., $L(D) = \{n-2, n-3, n-4\}$.

Subcase 2.2.1.1. $n > 8$. It follows from Lemma 2.5 that $d_{L(D)} = \max\limits_{x,y \in V(D)} d_{L(D)}(x,y) \le n$. By (2.2), we have $\phi(n-2,n-3,n-4) \le \lfloor \frac{(n-4)^2}{2} \rfloor$. Then by (2.4), we obtain

$$\exp(D) \le \phi(n-2,n-3,n-4) + d_{L(D)} \le \lfloor \frac{(n-4)^2}{2} \rfloor + n.$$

Let $x \in V(C_1) \bigcap V(C_2)$; then by Lemma 2.12, we have $l_S(x) \le \exp_D(x)+r(x) \le \exp(D)+r(x) \le \lfloor \frac{(n-4)^2}{2} \rfloor + n + (n-2)(n-3) = \lfloor \frac{n^2}{2} \rfloor + n^2 - 8n + 14$. Thus $l_S(1) \le l_S(x) \le \lfloor \frac{n^2}{2} \rfloor + n^2 - 8n + 14 < 2n^2 - 12n + 19$.

Subcase 2.2.1.2. $n = 7$. Let $V(D) = \{v_1, v_2, \cdots, v_7\}$. Since D is a primitive, minimally strong digraph, it is clear that $|V(C_1) \bigcap V(C_2)| = 3$. Without loss of generality, we assume that $C_2 = (v_1, v_2, v_3, v_4, v_5)$ and $C_1 = (v_1, v_2, v_3, v_6)$. Because D contains a cycle of length $n-4 = 3$, then we have $E(D) = E(C_1) \cup E(C_2) \cup \{(v_j, v_7), (v_7, v_i)\}$, where (v_i, v_j) is an arc of C_1 or C_2. Let $C_3 = (v_i, v_j, v_7)$. If there exists a vertex $u \in \{v_1, v_2, v_3\}$ such that u also meets C_3, then we have $\max\limits_{x \in V(D)} d_{\{3,4,5\}}(u,x) \le n-1 = 6$ and so $\exp_D(u) \le \phi(3,4,5) + 6 = 9$ by (2.3). Otherwise, we have $v_i = v_4$ and $v_j = v_5$. Thus $\max\limits_{x \in V(D)} d_{\{3,4,5\}}(v_3, x) = d_{\{3,4,5\}}(v_3, v_6) = 6$ and $\exp_D(v_3) \le \phi(3,4,5) + 6 = 9$ by (2.3). Hence, there exists a vertex $u \in \{v_1, v_2, v_3\}$ such that $\exp_D(u) \le 9$. Since $r(v_i) \le (n-2)(n-3) = 5 \times 4 = 20$ for $i = 1, 2, 3$, by Lemma 2.12, we have $l_S(1) \le l_S(u) \le \exp_D(u) + r(u) \le 20 + 9 < 33 = m_3(7,1) - 2$.

Subcase 2.2.1.3. $n = 8$. Similar to the proof of Subcase 2.2.1.2, we can show that $l_S(1) \le (n-2)(n-3) + \phi(4,5,6) + n - 1 = 6 \times 5 + 8 + 7 = 45 < 51 = m_3(8,1) - 2$.

Subcase 2.2.2. $s \le n-5$.

Now $|V(C_1) \bigcap V(C_2)| \ge p_1 + p_2 - n = n - 5$. If $|V(C_1) \bigcap V(C_2)| = n-5$, then D must be isomorphic to H_n, which is a contradiction. So $|V(C_1) \bigcap V(C_2)| \ge n-4$. Let $x \in V(C_1) \bigcap V(C_2)$ with $\exp_D(x) = \min\{\exp_D(u) : u \in V(C_1) \bigcap V(C_2)\}$. Then $\exp_D(x) \le \exp_D(5)$. Since D is not isomorphic to $D_{n,s}$, by Lemma 2.9, we have $\exp_D(1) \le (n-5)(n-3)+1$. Thus by Lemma 2.3, $\exp_D(x) \le \exp_D(5) \le \exp_D(1) + 4 \le n^2 - 8n + 20$. Since $r(x) \le (n-2)(n-3)$, by Lemma 2.12, we have $l_S(x) \le \exp_D(x) + r(x) \le 2n^2 - 13n + 26$. Hence $l_S(1) \le l_S(x) \le 2n^2 - 13n + 26 < 2n^2 - 12n + 19$ for $n > 7$.

Suppose $n = 7$. Since D is a primitive, minimally strong digraph, we have $s = n-5 = 2$ and $|V(C_1) \bigcap V(C_2)| = 3$. Without loss of generality, we assume that $C_2 = (v_1, v_2, v_3, v_4, v_5)$ and

$C_1 = (v_1, v_2, v_3, v_6)$. Because D contains a cycle of length 2, we get that $E(D) = E(C_1) \cup E(C_2) \cup \{(v_i, v_7), (v_7, v_i)\}$, where $i \in \{1, 2, 3, 4, 5, 6\}$. Now $L(D) = \{2, 4, 5\}$; by using the method similar to Subcase 2.2.1.2, we can show that $l_S(1) \leq (n-2)(n-3) + \phi(2,4,5) + n - 1 = 5 \times 4 + 4 + 6 = 30 < 33 = m_3(7,1) - 2$.

Subcase 2.3. $p_2 = n-2, p_1 = n-4$. Now, n is odd by Lemma 2.1.

Let $x \in V(C_1) \bigcap V(C_2)$ with $\exp_D(x) = \min\{\exp_D(u) : u \in V(C_1) \bigcap V(C_2)\}$. Then $r(x) \leq (n-2)(n-4)$. Since $|V(C_1) \bigcap V(C_2)| \geq p_1 + p_2 - n = n - 6$, we have $\exp_D(x) \leq \exp_D(7)$.

Subcase 2.3.1. $s = n - 4$.

Since D is not isomorphic to $D_{n,s}$, by Lemma 2.9, we have $\exp_D(1) \leq (n-4)(n-3) + 1 = n^2 - 7n + 13$. Thus by Lemma 2.3, $\exp_D(x) \leq \exp_D(7) \leq \exp_D(1) + 6 \leq n^2 - 7n + 19$. Consequently, $l_S(x) \leq \exp_D(x) + r(x) \leq n^2 - 7n + 19 + (n-2)(n-4)$ by Lemma 2.12. Hence $l_S(1) \leq l_S(x) \leq 2n^2 - 13n + 27 < 2n^2 - 12n + 19$ for $n \geq 9$.

Suppose $n = 7$. We only need to consider two cases $L(D) = \{n-2, n-4\}$ and $L(D) = \{n-2, n-3, n-4\}$. If $L(D) = \{n-2, n-4\}$, then $p_1 = n-4 \geq 3$ and $p_1 + p_2 = 2n - 6 > n$. So by Lemma 2.6, $\exp_D(x) \leq \exp(D) \leq n + p_1(p_2 - 2) = 7 + 3(5-2) = 16$. Since $r(x) \leq (n-2)(n-4) = 5 \times 3 = 15$, then $l_S(x) \leq 16 + 15 = 31$ by Lemma 2.12. Hence $l_S(1) \leq l_S(x) \leq 31 < 33 = m_3(7,1) - 2$. If $L(D) = \{n-2, n-3, n-4\}$, then $|V(C_1) \bigcap V(C_2)| = 2$. Let $V(D) = \{v_1, v_2, \cdots, v_7\}$. Without loss of generality, we assume that $C_2 = (v_1, v_2, v_3, v_4, v_5)$ and $C_1 = (v_1, v_2, v_6)$. Let C_3 be a cycle of D with length 4, then $|V(C_3) \bigcap V(C_2)| \geq 2$. Thus $\max\limits_{x \in V(D)} d_{\{3,4,5\}}(v_1, x) \leq |V(C_2)| + n - 1 = 5 + 6 = 11$. By (2.3), $\exp_D(v_1) \leq \phi(3,4,5) + \max\limits_{x \in V(D)} d_{\{3,4,5\}}(v_1, x) \leq 3 + 11 = 14$. Since $r(v_1) \leq (n-2)(n-4) = 5 \times 3 = 15$, by Lemma 2.12, we have $l_S(v_1) \leq 14 + 15 = 29$. Hence $l_S(1) \leq l_S(v_1) \leq 29 < 33 = m_3(7,1) - 2$.

Subcase 2.3.2. $s \leq n - 5$.

Since D is not isomorphic to $D_{n,s}$, by Lemma 2.9, we have $\exp_D(1) \leq (n-5)(n-3) + 1 = n^2 - 8n + 16$. Thus by Lemma 2.3, $\exp_D(x) \leq \exp_D(7) \leq \exp_D(1) + 6 \leq n^2 - 8n + 22$. Consequently, $l_S(x) \leq \exp_D(x) + r(x) \leq n^2 - 8n + 22 + (n-2)(n-4)$ by Lemma 2.12. Hence $l_S(1) \leq l_S(x) \leq 2n^2 - 14n + 30 < 2n^2 - 12n + 19$.

Subcase 2.4. $p_2 = n-3, p_1 = n-4$.

Let $x \in V(C_1) \bigcap V(C_2)$ with $\exp_D(x) = \min\{\exp_D(u) : u \in V(C_1) \bigcap V(C_2)\}$. Then $r(x) \leq (n-3)(n-4)$. Since $|V(C_1) \bigcap V(C_2)| \geq p_1 + p_2 - n = n - 7$, we have $\exp_D(x) \leq \exp_D(8)$. Because D is not isomorphic to $D_{n,s}$, by Lemma 2.9, we have $\exp_D(1) \leq (n-4)(n-3) + 1 = n^2 - 7n + 13$. Thus by Lemma 2.3, $\exp_D(x) \leq \exp_D(8) \leq \exp_D(1) + 7 \leq n^2 - 7n + 20$. Consequently, $l_S(x) \leq \exp_D(x) + r(x) \leq n^2 - 7n + 20 + (n-3)(n-4)$ by Lemma 2.12. Hence $l_S(1) \leq l_S(x) \leq 2n^2 - 14n + 32 < 2n^2 - 12n + 19$.

Subcase 2.5. $p_2 \leq n-2, p_1 \leq n-5$.

By Lemma 2.9, we have $\exp(D) \leq n + (n-5)(n-3) = n^2 - 7n + 15$. Let $x \in V(C_1) \bigcap V(C_2)$. Then $r(x) \leq (n-2)(n-5)$ and so $l_S(x) \leq \exp_D(x) + r(x) \leq \exp(D) + r(x) \leq 2n^2 - 14n + 25$ by Lemma 2.12. Hence $l_S(1) \leq l_S(x) \leq 2n^2 - 14n + 25 < 2n^2 - 12n + 19$.

Combining the above Cases, the proof of this theorem is completed. □

Acknowledgements

I would like to thank the referee for many helpful comments and suggestions and for bringing Ref. [3] to my attention.

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