A Family of q-Dyson Style Constant Term Identities

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Abstract

By generalizing Gessel-Xin's Laurent series method for proving the Zeilberger-Bressoud q-Dyson Theorem, we establish a family of q-Dyson style constant term identities. These identities give explicit formulas for certain coefficients of the q-Dyson product, including three conjectures of Sills' as special cases and generalizing Stembridge's first layer formulas for characters of $SL(n,\mathbb{C})$.

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1 Introduction

1.1 Notation

Throughout this paper, we let n be a nonnegative integer, and use the following symbols:

$\mathbf{a} := (a_0, a_1, \dots, a_n)$,

$a := a_1 + a_2 + \cdots + a_n$,

$\mathbf{x} := (x_0, x_1, \dots, x_n)$,

$(z)_n := (1-z)(1-zq)\cdots(1-zq^{n-1})$,

$$D_n(\mathbf{x},\mathbf{a},q) := \prod_{0\le i<j\le n} \left(\frac{x_i}{x_j}\right)_{a_i} \left(\frac{x_j}{x_i}q\right)_{a_j}, \qquad (q\text{-Dyson product})$$

$\operatorname*{CT}_{\mathbf{x}} F(\mathbf{x})$ means to take the constant term in the x's of the series $F(\mathbf{x})$.

Since our main objective in this paper is to evaluate the constant term of the form

$$\frac{x_{j_1}^{p_1}\cdots x_{j_\nu}^{p_\nu}}{x_{i_1}x_{i_2}\cdots x_{i_m}} D_n(\mathbf{x},\mathbf{a},q),$$

it is convenient for us to define:

$I_0 := \{i_1, i_2, \dots, i_m\}$ is a set with $0 = i_1 < i_2 < \cdots < i_m < n$,
$I := I_0 \setminus \{i_1\} = \{i_2, \dots, i_m\}$,
$T := \{t_1, \dots, t_d\}$ is a d-element subset of I_0 or I with $t_1 < t_2 < \cdots < t_d$,
$\sigma(T) := a_{t_1} + a_{t_2} + \cdots + a_{t_d}$,

$$w_i := \begin{cases} a_i, & for \quad i \notin T; \\ 0, & for \quad i \in T, \end{cases}$$

$w := w_1 + w_2 + \cdots + w_n = a - \sigma(T)$.

1.2 Main results

In 1962, Freeman Dyson [5] conjectured the following identity:

Theorem 1.1 (Dyson's Conjecture). *For nonnegative integers $a_0, a_1, \dots, a_n$,*

$$\operatorname*{CT}_{\mathbf{x}} \prod_{0 \le i \ne j \le n} \left(1 - \frac{x_i}{x_j}\right)^{a_i} = \frac{(a_0 + a_1 + \cdots + a_n)!}{a_0!\, a_1! \cdots a_n!}.$$

Dyson's conjecture was first proved independently by Gunson [8] and by Wilson [18]. An elegant recursive proof was published by Good [7].

George Andrews [1] conjectured the q-analog of the Dyson conjecture in 1975:

Theorem 1.2. (Zeilberger-Bressoud). *For nonnegative integers $a_0, a_1, \dots, a_n$,*

$$\operatorname*{CT}_{\mathbf{x}} D_n(\mathbf{x}, \mathbf{a}, q) = \frac{(q)_{a+a_0}}{(q)_{a_0} (q)_{a_1} \cdots (q)_{a_n}}.$$

Andrews' q-Dyson conjecture attracted much interest [3, 9, 14, 15, 17], and was first proved, combinatorially, by Zeilberger and Bressoud [21] in 1985. Recently, Gessel and Xin [6] gave a very different proof by using properties of formal Laurent series and of polynomials. The coefficients of the Dyson and q-Dyson product are researched in [4, 10, 12, 13, 16]. In the equal parameter case, the identity reduces to Macdonald's constant term conjecture [11] for root systems of type A.

The main results of this paper are the following q-Dyson style constant term identities:

Theorem 1.3 (Main Theorem). *Let $i_1, \dots, i_m$ and $j_1, \dots, j_\nu$ be distinct integers satisfying $0 = i_1 < i_2 < \cdots < i_m < n$ and $0 < j_1 < \cdots < j_\nu \le n$. Then*

$$\operatorname*{CT}_{\mathbf{x}} \frac{x_{j_1}^{p_1} \cdots x_{j_\nu}^{p_\nu}}{x_{i_1} x_{i_2} \cdots x_{i_m}} D_n(\mathbf{x}, \mathbf{a}, q) = \frac{(q)_{a+a_0}}{(q)_{a_0} (q)_{a_1} \cdots (q)_{a_n}} \sum_{\varnothing \ne T \subseteq I_0} (-1)^d q^{L(T)} \frac{1 - q^{\sigma(T)}}{1 - q^{1+a_0+a-\sigma(T)}}, \tag{1.1}$$

where the p's are positive integers with $\sum_{i=1}^{\nu} p_i = m$ and

$$L(T) = \sum_{l \in I_0} \sum_{i=l}^{n} w_i - \sum_{l=1}^{\nu} p_l \sum_{i=j_l}^{n} w_i. \tag{1.2}$$

We remark that the cases $i_1 > 0$ or $i_m = n$ or both can be evaluated using the above theorem and Lemma 2.1. The equal parameter case of the above results are called by Stembridge [16] "the first layer formulas for characters of $SL(n,\mathbb{C})$". The following three Corollaries are the simplified, but equivalent, version of Sills' conjectures [12]. They are all special cases of Theorem 1.3. When $m=1$, we obtain

Corollary 1.4 (Conjecture 1.2, [12]). *Let r be a fixed integer with $0 < r \le n$ and $n \ge 1$. Then*

$$\operatorname*{CT}_{\mathbf{x}} \frac{x_r}{x_0} D_n(\mathbf{x},\mathbf{a},q) = -q^{\sum_{k=1}^{r-1} a_k} \left(\frac{1-q^{a_0}}{1-q^{a+1}}\right) \frac{(q)_{a+a_0}}{(q)_{a_0}(q)_{a_1}\cdots(q)_{a_n}}. \tag{1.3}$$

When $m=2$ and $p_1=2$, we obtain

Corollary 1.5 (Conjecture 1.5, [12]). *Let r,t be fixed integers with $1 \le t < r \le n$ and $n \ge 2$. Then*

$$\begin{aligned} &\operatorname*{CT}_{\mathbf{x}} \frac{x_r^2}{x_0 x_t} D_n(\mathbf{x},\mathbf{a},q) \\ &= q^{\widetilde{L}(r,t)} \left(\frac{(1-q^{a_0})(1-q^{a_t})\Big((1-q^{a_0+a+1}) + q^{a_t}(1-q^{a+1-a_t})\Big)}{(1-q^{a+1-a_t})(1-q^{a+1})(1-q^{a_0+a+1-a_t})}\right) \frac{(q)_{a+a_0}}{(q)_{a_0}(q)_{a_1}\cdots(q)_{a_n}}, \end{aligned} \tag{1.4}$$

where $\widetilde{L}(r,t) = 2\sum_{k=t+1}^{r-1} a_k + \sum_{k=1}^{t-1} a_k$.

When $m=2$ and $p_1=p_2=1$, we obtain

Corollary 1.6 (Conjecture 1.7, [12]). *Let r,s,t be fixed integers with $1 \le r < s \le n, t < s$ and $n \ge 3$. Then*

$$\begin{aligned} &\operatorname*{CT}_{\mathbf{x}} \frac{x_r x_s}{x_0 x_t} D_n(\mathbf{x},\mathbf{a},q) \\ &= q^{\widetilde{L}(r,s,t)} \left(\frac{(1-q^{a_0})(1-q^{a_t})\Big((1-q^{a_0+a+1}) + q^{M(r,s,t)}(1-q^{a+1-a_t})\Big)}{(1-q^{a+1-a_t})(1-q^{a+1})(1-q^{a_0+a+1-a_t})}\right) \frac{(q)_{a+a_0}}{(q)_{a_0}(q)_{a_1}\cdots(q)_{a_n}}, \end{aligned} \tag{1.5}$$

where

$$\widetilde{L}(r,s,t) = \begin{cases} \sum_{k=1}^{r-1} a_k + \sum_{k=t+1}^{s-1} a_k, & \text{if} \quad r<t<s; \\ \sum_{k=r}^{s-1} a_k + \sum_{k=1}^{t-1} a_k + 2\sum_{k=t+1}^{r-1} a_k, & \text{if} \quad t<r<s, \end{cases}$$

and

$$M(r,s,t) = \begin{cases} 1+a+a_0, & \text{if} \quad r<t<s; \\ a_t, & \text{if} \quad t<r<s. \end{cases}$$

When letting q approach 1 from the left, we get

Theorem 1.7. *Let $i_1,\dots,i_m$ and $j_1,\dots,j_\nu$ be distinct integers with $0 = i_1 < \cdots < i_m < n$ and $0 < j_1 < \cdots < j_\nu \le n$. Then*

$$\operatorname*{CT}_{\mathbf{x}} \frac{x_{j_1}^{p_1}\cdots x_{j_\nu}^{p_\nu}}{x_{i_1}x_{i_2}\cdots x_{i_m}} \prod_{0\le i\ne j\le n} \left(1-\frac{x_i}{x_j}\right)^{a_i} = \frac{(a_0+a_1+\cdots+a_n)!}{a_0!\,a_1!\cdots a_n!} \sum_{\varnothing\ne T\subseteq I_0} (-1)^d \frac{\sigma(T)}{1+a+a_0-\sigma(T)},$$

where the p's are positive integers with $\sum_{i=1}^{\nu} p_i = m$.

The proof of Theorem 1.3 is along the same line of Gessel and Xin's proof of Theorem 1.2 [6], but *we evaluate, rather than prove, the constant terms needed.* First of all, the underlying idea is the well-known fact that proving the equality of two polynomials of degree at most d, it suffices to prove that they are equal at $d+1$ points. As is often the case, points at which the polynomials vanish are most easily dealt with.

It is routine to show that after fixing parameters $a_1, \dots, a_n$, the constant term is a polynomial of degree at most d in the variable q^{a_0}. Then we can apply the Gessel-Xin's technique to show that the equality holds when the polynomial vanishes. The proof then differs in showing the equality at the extra points: The q-Dyson conjecture needs one extra point, which can be shown by induction; Corollaries 1.4, 1.5, and 1.6 need one, two and two extra points respectively; Theorem 1.3 needs many extra points. *For the constant terms at these extra points, we develop a new technique in evaluating, not proving, them based on several natural extensions of Gessel and Xin's work. Once obtained enough values of the polynomial, one can use Lagrange's interpolation formula to obtain an explicit formula. However, such a formula will not be as nice as that given in Theorem 1.3, which is obtained by guessing and proving.*

This paper is organized as follows. In section 2, our main result, Theorem 1.3, is established under the assumption of two main lemmas. The first lemma is for the vanishing points and the second one is for the extra points, and they take us the next three sections to prove. Then by specializing our main theorem, we prove Sills' three conjectures. In section 3, we introduce the field of iterated Laurent series and partial fraction decompositions as basic tools for evaluating constant terms. We also introduce basic notions and lemmas of [6] in a generalized form. These are essential for proving the two main lemmas. In section 4, we deal with some general q-Dyson style constant terms and prove our first main lemma. Section 5 includes new techniques and complicated computations for our second main lemma. It is a continuation of section 4.

2 The proofs and the consequences

Dyson's conjecture, Andrews' q-Dyson conjecture, and their relatives are all constant terms of certain Laurent polynomials. However, larger rings and fields will encounter when evaluating them. We closely follow the notation in [6]. In order to prove our Main Theorem, we make several generalizations that need to go into details to explain.

We first work in the ring of Laurent polynomials to see that some seemingly more complicated cases can be solved by Theorem 1.3.

Define an action π on Laurent polynomials by

$$\pi\big(F(x_0, x_1, \dots, x_n)\big) = F(x_1, x_2, \dots, x_n, x_0/q).$$

By iterating, if $F(x_0, x_1, \dots, x_n)$ is homogeneous of degree 0, then

$$\pi^{n+1}\big(F(x_0, x_1, \dots, x_n)\big) = F(x_0/q, x_1/q, \dots, x_n/q) = F(x_0, x_1, \dots, x_n),$$

so that in particular π is a cyclic action on $D_n(\mathbf{x}, \mathbf{a}, q)$.

Lemma 2.1. *Let $L(\mathbf{x})$ be a Laurent polynomial in the x's. Then*

$$\operatorname*{CT}_{\mathbf{x}} L(\mathbf{x})\, D_n(\mathbf{x}, \mathbf{a}, q) = \operatorname*{CT}_{\mathbf{x}}\ \pi\big(L(\mathbf{x})\big) D_n\big(\mathbf{x}, (a_n, a_0, \dots, a_{n-1}), q\big). \tag{2.1}$$

By iterating (2.1) and renaming the parameters, evaluating $\mathrm{CT}_{\mathbf{x}}\, L(\mathbf{x})\, D_n(\mathbf{x},\mathbf{a},q)$ *is equivalent to evaluating* $\mathrm{CT}_{\mathbf{x}}\, \pi^k(L(\mathbf{x}))\, D_n(\mathbf{x},\mathbf{a},q)$ *for any integer* k.

Proof. It is straightforward to check that

$$\pi\big(D_n(\mathbf{x},\mathbf{a},q)\big) = D_n\big(\mathbf{x},(a_n,a_0,\dots,a_{n-1}),q\big).$$

Note that an equivalent form was observed by Kadell [10, Equation 5.12]. Therefore, equation (2.1) follows by the above equality and the fact

$$\operatorname*{CT}_{\mathbf{x}} F(x_0,x_1,\dots,x_n) = \operatorname*{CT}_{\mathbf{x}} \pi\big(F(x_0,x_1,\dots,x_n)\big).$$

The second part of the lemma is obvious. □

Next we work in the ring of Laurent series in x_0 with coefficients Laurent polynomials in $x_1,x_2,\dots,x_n$. *The following lemma is a natural extension of Lemma 3.1 in [6].*

Lemma 2.2. *Let* $L(x_1,\dots,x_n)$ *be a Laurent polynomial independent of* a_0 *and* x_0. *Then for fixed nonnegative integers* $a_1,\dots,a_n$ *and* $k\le a$, $k\in\mathbb{Z}$ *the constant term*

$$\operatorname*{CT}_{\mathbf{x}} x_0^k L(x_1,\dots,x_n) D_n(\mathbf{x},\mathbf{a},q) \tag{2.2}$$

is a polynomial in q^{a_0} *of degree at most* $a-k$.

Proof. It is easy to prove that

$$\left(\frac{x_0}{x_j}\right)_{a_0}\left(\frac{x_j}{x_0}q\right)_{a_j} = q^{\binom{a_j+1}{2}}\left(-\frac{x_j}{x_0}\right)^{a_j}\left(\frac{x_0}{x_j}q^{-a_j}\right)_{a_0+a_j}$$

for all integers a_0, where both sides are regarded as Laurent series in x_0. Rewrite (2.2) as

$$\operatorname*{CT}_{\mathbf{x}}\ x_0^k L_1(x_1,\dots,x_n)\prod_{j=1}^{n} q^{\binom{a_j+1}{2}}\left(-\frac{x_j}{x_0}\right)^{a_j}\left(\frac{x_0}{x_j}q^{-a_j}\right)_{a_0+a_j}, \tag{2.3}$$

where $L_1(x_1,\dots,x_n)$ is a Laurent polynomial in $x_1,\dots,x_n$ independent of x_0 and a_0.

The well-known q-binomial theorem [2, Theorem 2.1] is the identity

$$\frac{(bz)_\infty}{(z)_\infty} = \sum_{k=0}^{\infty}\frac{(b)_k}{(q)_k}z^k. \tag{2.4}$$

Setting $z=uq^n$ and $b=q^{-n}$ in (2.4), we obtain

$$(u)_n = \frac{(u)_\infty}{(uq^n)_\infty} = \sum_{k=0}^{\infty} q^{k(k-1)/2}\begin{bmatrix} n\\ k\end{bmatrix}(-u)^k \tag{2.5}$$

for all integers n, where $\begin{bmatrix} n\\ k\end{bmatrix} = \frac{(q)_n}{(q)_k(q)_{n-k}}$ is the q-binomial coefficient.

Using (2.5), we see that for $1\le j\le n$,

$$q^{\binom{a_j+1}{2}}\left(-\frac{x_j}{x_0}\right)^{a_j}\left(\frac{x_0}{x_j}q^{-a_j}\right)_{a_0+a_j} = \sum_{k_j\ge 0} C(k_j)\begin{bmatrix} a_0+a_j\\ k_j\end{bmatrix} x_0^{k_j-a_j}x_j^{a_j-k_j},$$

where $C(k_j)=(-1)^{k_j+a_j}q^{\binom{a_j+1}{2}+\binom{k_j}{2}-k_ja_j}$.

Expanding the product in (2.3) and taking constant term in x_0, we see that (2.2) becomes

$$\sum_{\mathbf{k}} \begin{bmatrix} a_0+a_1 \\ k_1 \end{bmatrix} \begin{bmatrix} a_0+a_2 \\ k_2 \end{bmatrix} \cdots \begin{bmatrix} a_0+a_n \\ k_n \end{bmatrix} \operatorname*{CT}_{x_1,\dots,x_n} L_2(x_1,\dots,x_n;\mathbf{k}), \tag{2.6}$$

where $L_2(x_1,\dots,x_n;\mathbf{k})$ is a Laurent polynomial in $x_1,\dots,x_n$ independent of a_0 and the sum ranges over all sequences $\mathbf{k}=(k_1,\dots,k_n)$ of nonnegative integers satisfying $k_1+k_2+\cdots+k_n=a-k$. Since $\left[\begin{smallmatrix} a_0+a_i \\ k_i \end{smallmatrix}\right]$ is a polynomial in q^{a_0} of degree k_i, each summand in (2.6) is a polynomial in q^{a_0} of degree at most $k_1+k_2+\cdots+k_n=a-k$, and so is the sum. □

Lemma 2.2 reduces the proof of Theorem 1.3 to evaluating the constant term at enough values of the q^{a_0}'s. This is accomplished by the following Main Lemmas 1 and 2. Their proofs will be given in the next three sections, using the field of iterated Laurent series [20].

Lemma 2.3 (Main Lemma 1). *If a_0 belongs to the set $\{0,-1,\dots,-(a+1)\}\setminus\{-(a-\sigma(T)+1)\mid T\subseteq I\}$, then*

$$\operatorname*{CT}_{\mathbf{x}} \frac{x_{j_1}^{p_1}\cdots x_{j_\nu}^{p_\nu}}{x_{i_1}x_{i_2}\cdots x_{i_m}} D_n(\mathbf{x},\mathbf{a},q)=0. \tag{2.7}$$

Lemma 2.4 (Main Lemma 2). *If a_0 belongs to the set $\{-(a-\sigma(T)+1)\mid T\subseteq I\}$, then*

$$\operatorname*{CT}_{\mathbf{x}} \frac{x_{j_1}^{p_1}\cdots x_{j_\nu}^{p_\nu}}{x_{i_1}x_{i_2}\cdots x_{i_m}} D_n(\mathbf{x},\mathbf{a},q)=\sum_T (-1)^{w+d}q^{L^*(T)}\frac{(q)_w(q)_{a-w}}{(q)_{a_1}\cdots(q)_{a_n}}, \tag{2.8}$$

where the sum ranges over all $T\subseteq I$ such that $-(a-\sigma(T)+1)=a_0$ and

$$L^*(T)=\sum_{l\in I}\sum_{i=l}^{n} w_i-\sum_{l=1}^{\nu} p_l\sum_{i=j_l}^{n} w_i-\binom{w+1}{2}-1. \tag{2.9}$$

The following lemma shows that Main Lemmas 1 and 2 coincide with our Main Theorem, *the formula in which is obtained by guessing.*

Lemma 2.5. *If a_0 belongs to the set $\{-(a-\sigma(T)+1)\mid T\subseteq I\}$, then*

$$\frac{(q)_{a+a_0}}{(q)_{a_0}(q)_{a_1}\cdots(q)_{a_n}}\sum_{\varnothing\neq T\subseteq I_0}(-1)^d q^{L(T)}\frac{1-q^{\sigma(T)}}{1-q^{1+a_0+a-\sigma(T)}}=\sum_T(-1)^{w+d}q^{L^*(T)}\frac{(q)_w(q)_{a-w}}{(q)_{a_1}\cdots(q)_{a_n}}, \tag{2.10}$$

where the last sum ranges over all $T\subseteq I$ such that $-(a-\sigma(T)+1)=a_0$, $L^*(T)$ is defined as in (2.9), and $L(T)$ is defined as in (1.2).

If a_0 belongs to the set $\{0,-1,\dots,-(a+1)\}\setminus\{-(a-\sigma(T)+1)\mid T\subseteq I\}$, then the left-hand side of (2.10) vanishes.

Proof. Let LHS and RHS denote the left-hand side and the right-hand side of (2.10) respectively. By definition, $L(T)=L(T\cup\{0\})+a_0$ for any $T\subseteq I$. This fact will be used.

If $a_0=0$, then simplifying gives

$$LHS=\frac{(q)_a}{(q)_{a_1}\cdots(q)_{a_n}}\sum_{T\subseteq I_0}(-1)^d q^{L(T)}\frac{1-q^{\sigma(T)}}{1-q^{1+a-\sigma(T)}},$$

where we have added the vanishing term corresponding to $T=\varnothing$. The sum equals 0 since for every $T\subseteq I$, when pairing the summand for T and the summand for $T\cup\{0\}$, we have

$$(-1)^d q^{L(T)}\frac{1-q^{\sigma(T)}}{1-q^{1+a-\sigma(T)}}+(-1)^{d+1}q^{L(T\cup\{0\})}\frac{1-q^{\sigma(T\cup\{0\})}}{1-q^{1+a-\sigma(T\cup\{0\})}}=0.$$

If $a_0=-a-1$, then the sum for RHS has only one term corresponding to $T=\varnothing$. For LHS, simplifying gives

$$LHS=\frac{(q)_{-1}}{(q)_{-a-1}(q)_{a_1}\cdots(q)_{a_n}}\sum_{\varnothing\neq T\subseteq I_0}(-1)^{d+1}q^{L(T)+\sigma(T)}.$$

Since for any $T\subseteq I$, we have

$$\begin{aligned}(-1)^{d+1}q^{L(T)+\sigma(T)}&+(-1)^{d+2}q^{L\big(T\cup\{0\}\big)+\sigma\big(T\cup\{0\}\big)}\\&=(-1)^{d+1}q^{\big(L(T)+\sigma(T)\big)}\big(1-q^{-a_0+a_0}\big)=0,\end{aligned}$$

LHS reduces to only one term corresponding to $T=\{0\}$, which is

$$\begin{aligned}LHS=&(-1)^2q^{L(\{0\})+a_0}\frac{(q)_{-1}}{(q)_{-a-1}(q)_{a_1}\cdots(q)_{a_n}}=q^{L(\{0\})+a_0}\frac{\big(1-\frac{1}{q}\big)\cdots\big(1-\frac{1}{q^a}\big)}{(q)_{a_1}\cdots(q)_{a_n}}\\=&(-1)^a q^{L(\{0\})-a-1-\binom{a+1}{2}}\frac{(q)_a}{(q)_{a_1}\cdots(q)_{a_n}}=(-1)^a q^{L^*(\varnothing)}\frac{(q)_a}{(q)_{a_1}\cdots(q)_{a_n}}=RHS.\end{aligned}$$

Now consider the cases $a_0=-1,\dots,-a$. Since the factor $(q)_{a_0+a}/(q)_{a_0}=(1-q^{a_0+1})\cdots(1-q^{a_0+a})$ of LHS vanishes for $a_0=-1,-2,\dots,-a$, the summand with respect to T has no contribution unless the denominator $1-q^{1+a_0+a-\sigma(T)}=0$, i.e., $a_0=-\big(a+1-\sigma(T)\big)$. Therefore, $LHS=0$ if a_0 does not belong to $\{-(a-\sigma(T)+1)\mid T\subseteq I\}$. If it is not the case, then only those terms with $-(a-\sigma(T)+1)=a_0$ have contributions. Such T can not contain 0, for otherwise we may deduce that $a+1-\sigma(T\setminus\{0\})=0$, which is impossible. Therefore it suffices to show that for every subset $T\subseteq I$ we have

$$\frac{(q)_{a+a_0}}{(q)_{a_0}\cdots(q)_{a_n}}(-1)^d q^{L(T)}\frac{1-q^{\sigma(T)}}{1-q^{1+a_0+a-\sigma(T)}}\Big|_{a_0=-w-1}=\frac{(q)_w(q)_{a-w}}{(q)_{a_1}\cdots(q)_{a_n}}(-1)^{w+d}q^{L^*(T)}.\qquad(2.11)$$

Since $L(T)|_{a_0=-w-1}=L^*(T)+\binom{w+1}{2}$, the left-hand side of (2.11) equals

$$(-1)^d q^{L^*(T)+\binom{w+1}{2}}\frac{\big[(1-q^{-w})\cdots(1-q^{-1})\big]\big[(1-q)\cdots(1-q^{a-w})\big]}{(q)_{a_1}\cdots(q)_{a_n}}=(-1)^{w+d}q^{L^*(T)}\frac{(q)_w(q)_{a-w}}{(q)_{a_1}\cdots(q)_{a_n}},$$

which is the right-hand side of (2.11). □

Proof of Theorem **1.3**. We prove the theorem by showing that both sides of (1.1) are polynomials in q^{a_0} of degree no more than $a+1$, and that they agree at the $a+2$ values corresponding to $a_0=0,-1,\dots,-a-1$. The latter statement follows by Main Lemma 1, Main Lemma 2, and Lemma 2.5. We now prove the former statement to complete the proof.

Applying Lemma 2.2 in the case $k=-1$ and $L(x_1,\dots,x_n)=x_{j_1}^{p_1}\cdots x_{j_\nu}^{p_\nu}/(x_{i_2}\cdots x_{i_m})$, we see that the constant term in (1.1) is a polynomial in q^{a_0} of degree at most $a+1$. The right-hand side of (1.1) can be written as

$$\sum_{\varnothing\neq T\subseteq I_0}(-1)^d q^{L(T)}\frac{1-q^{\sigma(T)}}{1-q^{a_0+1+a-\sigma(T)}}\frac{(1-q^{a_0+1})(1-q^{a_0+2})\cdots(1-q^{a_0+a})}{(q)_{a_1}(q)_{a_2}\cdots(q)_{a_n}}.$$

This is a polynomial in q^{a_0} of degree no more than $a+1$, as can be seen by checking the two cases: If $0 \notin T$ then the degree of $q^{L(T)}$ in q^{a_0} is 1 and $1-q^{a_0+1+a-\sigma(T)}$ cancels with the numerator so that the summand has degree a in q^{a_0}; Otherwise the summand has degree $a+1$ in q^{a_0}. □

The $m=0$ case of Theorem 1.3 reduces to the Zeilberger-Bressoud q-Dyson Theorem. Comparing with the proof of Theorem 1.2 in [6], the new part is Lemma 2.4, where we give explicit formula for the non-vanishing case $a_0=-a-1$. This gives a proof without using induction on n.

Proof of Corollary **1.4**. Applying the Main Theorem for $I_0=\{0\}$ gives

$$L(\{0\}) = \sum_{i=0}^{n} w_i - \sum_{i=r}^{n} w_i = \sum_{i=1}^{n} a_i - \sum_{i=r}^{n} a_i = \sum_{i=1}^{r-1} a_i.$$

Substituting the above into (1.1) and simplifying, we obtain Corollary 1.4. □

Proof of Corollary **1.5**. Applying the Main Theorem for $I_0=\{0,t\}$ and $p_1=2$ gives

$$\begin{aligned} L(\{0\}) &= \sum_{i=1}^{n} a_i + \sum_{i=t}^{n} a_i - 2\sum_{i=r}^{n} a_i, \\ L(\{t\}) &= \sum_{i=0}^{n} a_i + \sum_{i=t}^{n} a_i - 2\sum_{i=r}^{n} a_i - 2a_t, \\ L(\{0,t\}) &= \sum_{i=1}^{n} a_i + \sum_{i=t}^{n} a_i - 2\sum_{i=r}^{n} a_i - 2a_t. \end{aligned}$$

Substituting the above into (1.1) and simplifying, we obtain Corollary 1.5. □

Proof of Corollary **1.6**. Applying the Main Theorem for $I_0=\{0,t\}$ and $p_1=p_2=1$ gives

$$\begin{aligned} L(\{0\}) &= \sum_{i=1}^{n} a_i + \sum_{i=t}^{n} a_i - \sum_{i=r}^{n} a_i - \sum_{i=s}^{n} a_i, \\ L(\{t\}) &= \begin{cases} \sum_{i=0}^{n} a_i + \sum_{i=t}^{n} a_i - \sum_{i=r}^{n} a_i - \sum_{i=s}^{n} a_i - a_t, & \text{if } r<t<s, \\ \sum_{i=0}^{n} a_i + \sum_{i=t}^{n} a_i - \sum_{i=r}^{n} a_i - \sum_{i=s}^{n} a_i - 2a_t, & \text{if } t<r<s, \end{cases} \\ L(\{0,t\}) &= \begin{cases} \sum_{i=1}^{n} a_i + \sum_{i=t}^{n} a_i - \sum_{i=r}^{n} a_i - \sum_{i=s}^{n} a_i - a_t, & \text{if } r<t<s, \\ \sum_{i=1}^{n} a_i + \sum_{i=t}^{n} a_i - \sum_{i=r}^{n} a_i - \sum_{i=s}^{n} a_i - 2a_t, & \text{if } t<r<s. \end{cases} \end{aligned}$$

Substituting the above into (1.1) and simplifying, we obtain Corollary 1.6. □

3 Constant term evaluations and basic lemmas

From now on, we let $K=\mathbb{C}(q)$, and assume that all series are in the field of iterated Laurent series $K\langle\langle x_n, x_{n-1}, \dots, x_0\rangle\rangle = K((x_n))((x_{n-1}))\cdots((x_0))$. This means that all series are regarded first as Laurent series in x_0, then as Laurent series in x_1, and so on. The reason for choosing $K\langle\langle x_n, x_{n-1}, \dots, x_0\rangle\rangle$ as a working field has been explained in [6]. For more detailed account of the properties of this field, with other applications, see [19] and [20].

We emphasize that the field of rational functions is a subfield of $K\langle\langle x_n, x_{n-1}, \ldots, x_0\rangle\rangle$, so that every rational function is identified with its unique iterated Laurent series expansion. The series expansions of $1/(1-q^k x_i/x_j)$ will be especially important. If $i<j$ then

$$\frac{1}{1-q^k x_i/x_j} = \sum_{l=0}^{\infty} q^{kl} x_i^l x_j^{-l}.$$

However, if $i>j$ then this expansion is not valid and instead we have the expansion

$$\frac{1}{1-q^k x_i/x_j} = \frac{1}{-q^k x_i/x_j(1-q^{-k}x_j/x_i)} = \sum_{l=0}^{\infty} -q^{-k(l+1)} x_i^{-l-1} x_j^{l+1}.$$

The constant term of the series $F(\mathbf{x})$ in x_i, denoted by $\mathrm{CT}_{x_i} F(\mathbf{x})$, is defined to be the sum of those terms in $F(\mathbf{x})$ that are free of x_i. It follows that

$$\mathop{\mathrm{CT}}_{x_i} \frac{1}{1-q^k x_i/x_j} = \begin{cases} 1, & \text{if } i<j, \\ 0, & \text{if } i>j. \end{cases} \tag{3.1}$$

We shall call the monomial $M = q^k x_i/x_j$ *small* if $i<j$ and *large* if $i>j$. Thus the constant term in x_i of $1/(1-M)$ is 1 if M is small and 0 if M is large.

An important property of the constant term operators defined in this way is their commutativity:

$$\mathop{\mathrm{CT}}_{x_i} \mathop{\mathrm{CT}}_{x_j} F(\mathbf{x}) = \mathop{\mathrm{CT}}_{x_j} \mathop{\mathrm{CT}}_{x_i} F(\mathbf{x}).$$

Commutativity implies that the constant term in a set of variables is well-defined, and this property will be used in our proof of the two Main Lemmas. (Note that, by contrast, the constant term operators in [22] do not commute.)

The *degree* of a rational function of x is the degree in x of the numerator minus the degree in x of the denominator. For example, if $i \neq j$ then the degree of $1 - x_j/x_i = (x_i - x_j)/x_i$ is 0 in x_i and 1 in x_j. A rational function is called *proper* (resp. *almost proper*) in x if its degree in x is negative (resp. zero).

Let

$$F = \frac{p(x_k)}{x_k^d \prod_{i=1}^m (1 - x_k/\alpha_i)} \tag{3.2}$$

be a rational function of x_k, where $p(x_k)$ is a polynomial in x_k, and the α_i are distinct monomials, each of the form $x_t q^s$. Then the partial fraction decomposition of F with respect to x_k has the following form:

$$F = p_0(x_k) + \frac{p_1(x_k)}{x_k^d} + \sum_{j=1}^m \frac{1}{1-x_k/\alpha_j} \left(\frac{p(x_k)}{x_k^d \prod_{i=1, i\neq j}^m (1 - x_k/\alpha_i)} \right)\Bigg|_{x_k=\alpha_j}, \tag{3.3}$$

where $p_0(x_k)$ is a polynomial in x_k, and $p_1(x_k)$ is a polynomial in x_k of degree less than d.

The following lemma is the basic tool in extracting constant terms.

Lemma 3.1. *Let F be as in* (3.2) *and* (3.3). *Then*

$$\mathop{\mathrm{CT}}_{x_k} F = p_0(0) + \sum_j \big(F\,(1 - x_k/\alpha_j)\big)\Big|_{x_k=\alpha_j}, \tag{3.4}$$

where the sum ranges over all j such that x_k/α_j is small. In particular, if F is proper in x_k, then $p_0(x_k) = 0$; if F is almost proper in x_k, then $p_0(x_k) = (-1)^m \prod_{i=1}^m \alpha_i \,\mathrm{LC}_{x_k}\, p(x_k)$, where LC_{x_k} means to take the leading coefficient with respect to x_k.

Lemma 3.1 is the general form of [6, Lemma 4.1] and the proof is also straightforward. The new observation is that we have explicit formulas not only for proper F but also for almost proper F. Such explicit formulas are useful in predicting the final result when iterating Lemma 3.1.

The following slight *extension* of [6, Lemma 4.2] plays an important role in our argument.

Lemma 3.2. *Let $a_1, \dots, a_s$ be nonnegative integers. Then for any positive integers $k_1, \dots, k_s$ with $1 \le k_i \le a_1 + \cdots + a_s + 1$ for all i, either $1 \le k_i \le a_i$ for some i or $-a_j \le k_i - k_j \le a_i - 1$ for some $i < j$, except only when $k_i = a_i + \cdots + a_s + 1$ for $i = 1, \dots, s$.*

Proof. The basic idea is the same as of [6, Lemma 4.2]. Assume $k_1, \dots, k_s$ to satisfy that for all i, $a_i < k_i \le a_1 + \cdots + a_s + 1$, and for all $i < j$, either $k_i - k_j \ge a_i$ or $k_i - k_j \le -a_j - 1$. Then we need to show that $k_i = a_i + \cdots + a_s + 1$ for $i = 1, \dots, s$.

We construct a tournament on $1, 2, \dots, s$ with numbers on the arcs as follows: For $i < j$, if $k_i - k_j \ge a_i$ then we draw an arc $i \xleftarrow{a_i} j$ from j to i and if $k_i - k_j \le -1 - a_j$ then we draw an arc $i \xrightarrow{a_j+1} j$ from i to j.

We call an arc from u to v an *ascending arc* if $u < v$ and a *descending arc* if $u > v$. We note two facts: (i) the number on an arc from u to v is less than or equal to $k_v - k_u$, and (ii) the number on an ascending arc is always positive.

A consequence of (i) is that for any directed path from e to f, the sum along the arcs is less than or equal to $k_f - k_e$. It follows that the sum along a cycle is non-positive. But any cycle must have at least one ascending arc, and by (ii) the number on this arc is positive, and so the sum along the cycle is positive. Thus there can be no cycles.

Therefore the tournament we have constructed is transitive, and hence defines a total ordering $\to$ on $1, 2, \dots, s$. Assume the total ordering is given by $i_1 \to i_2 \to \cdots \to i_{s-1} \to i_s$. Then $k_{i_s} - k_{i_1} \ge a_{i_2} + a_{i_3} + \cdots + a_{i_s}$. This implies that

$$\begin{aligned} k_{i_s} &\ge k_{i_1} + a_{i_2} + a_{i_3} + \cdots + a_{i_s} \\ &\ge a_{i_1} + 1 + a_{i_2} + a_{i_3} + \cdots + a_{i_s} \\ &= a_1 + a_2 + \cdots + a_s + 1, \end{aligned} \tag{3.5}$$

By assumption, $1 \le k_i \le a_1 + \cdots + a_s + 1$ for all i, so $k_{i_s} = a_1 + a_2 + \cdots + a_s + 1$. But for the equality in (3.5) to hold, we must have $k_{i_1} = a_{i_1} + 1$, and there are no arcs of the form $i_{l-1} \xrightarrow{a_{i_l}+1} i_l$ (i.e., $i_{l-1} < i_l$) for $l = 2, 3, \dots, s$. It follows that the total ordering $i_1 \to i_2 \to \cdots \to i_{s-1} \to i_s$ is actually $s \to (s-1) \to \cdots \to 2 \to 1$. One can then deduce that

$$k_{i_l} = a_{i_1} + \cdots + a_{i_l} + 1, \quad \text{for} \quad l = 1, \dots, s.$$

This completes our proof. □

4 The general setup and the proof of Main Lemma 1

Fix a monomial $M(\mathbf{x}) = \prod_{i=0}^{n} x_i^{b_i}$ with $\sum_{i=0}^{n} b_i = 0$. We derive general properties for q-Dyson style constant terms, and specialize $M(\mathbf{x})$ for the proofs of our main lemmas.

Define $Q(h)$ to be

$$Q(h) := M(\mathbf{x}) \prod_{j=1}^{n} \left(\frac{x_0}{x_j}\right)_{-h} \left(\frac{x_j}{x_0}q\right)_{a_j} \prod_{1\le i<j\le n} \left(\frac{x_i}{x_j}\right)_{a_i} \left(\frac{x_j}{x_i}q\right)_{a_j}. \tag{4.1}$$

If $h \ge 0$, then

$$Q(h) = \prod_{i=0}^{n} x_i^{b_i} \prod_{j=1}^{n} \frac{(x_j q/x_0)_{a_j}}{\left(1-\frac{x_0}{x_j q}\right)\left(1-\frac{x_0}{x_j q^2}\right)\cdots\left(1-\frac{x_0}{x_j q^h}\right)} \prod_{1\le i<j\le n} \left(\frac{x_i}{x_j}\right)_{a_i} \left(\frac{x_j}{x_i}q\right)_{a_j}. \tag{4.2}$$

We are interested in the constant term of $Q(h)$ for $h = 0, 1, 2, \dots, a+1$.

Since the degree in x_0 of $1 - x_j q^i/x_0$ is zero, the degree in x_0 of $Q(h)$ is $b_0 - nh$. Thus when $h > \frac{b_0}{n}$, $Q(h)$ is proper. Applying Lemma 3.1, we have

$$\operatorname*{CT}_{x_0} Q(h) = \sum_{\substack{0<r_1\le n,\\ 1\le k_1\le h}} Q(h \mid r_1; k_1), \tag{4.3}$$

where

$$Q(h \mid r_1; k_1) = Q(h)\left(1 - \frac{x_0}{x_{r_1} q^{k_1}}\right)\bigg|_{x_0 = x_{r_1} q^{k_1}}.$$

For each term in (4.3) we will extract the constant term in x_{r_1}, and then perform further constant term extractions, eliminating one variable at each step. In order to keep track of the terms we obtain, we introduce some notations from [6].

For any rational function F of $x_0, x_1, \dots, x_n$, and for sequences of integers $\mathbf{k} = (k_1, \dots, k_s)$ and $\mathbf{r} = (r_1, r_2, \dots, r_s)$ let $E_{\mathbf{r},\mathbf{k}}F$ be the result of replacing x_{r_i} in F with $x_{r_s} q^{k_s - k_i}$ for $i = 0, 1, \dots, s-1$, where we set $r_0 = k_0 = 0$. Then for $0 < r_1 < r_2 < \cdots < r_s \le n$ and $0 < k_i \le h$, we define

$$Q(h \mid \mathbf{r}; \mathbf{k}) = Q(h \mid r_1, \dots, r_s; k_1, \dots, k_s) = E_{\mathbf{r},\mathbf{k}}\left[Q(h) \prod_{i=1}^{s} \left(1 - \frac{x_0}{x_{r_i} q^{k_i}}\right)\right]. \tag{4.4}$$

Note that the product on the right-hand side of (4.4) cancels all the factors in the denominator of Q that would be taken to zero by $E_{\mathbf{r},\mathbf{k}}$.

Lemma 4.1. *Let $R = \{r_0, r_1, \dots, r_s\}$. Then the rational functions $Q(h \mid \mathbf{r}; \mathbf{k})$ have the following two properties:*

- **i** *If $1 \le k_i \le a_{r_1} + \cdots + a_{r_s}$ for all i with $1 \le i \le s$ and $h > \frac{b_0}{n}$, then $Q(h \mid \mathbf{r}; \mathbf{k}) = 0$.*
- **ii** *If $k_i > a_{r_1} + \cdots + a_{r_s}$ for some i with $1 \le i \le s < n$, and if*

$$h > a_{r_1} + \cdots + a_{r_s} + \frac{\sum_{i\in R} b_i}{n-s}, \tag{4.5}$$

then

$$\operatorname*{CT}_{x_s} Q(h \mid \mathbf{r};\mathbf{k}) = \sum_{\substack{r_s<r_{s+1}\le n,\\ 1\le k_{s+1}\le h}} Q(h \mid r_1,\dots,r_s,r_{s+1};k_1,\dots,k_s,k_{s+1}). \tag{4.6}$$

Proof of property **(i)**. By Lemma 3.2, either $1 \le k_i \le a_{r_i}$ for some i with $1 \le i \le s$, or $-a_{r_j} \le k_i - k_j \le a_{r_i} - 1$ for some $i < j$, since the exceptional case can not happen. If $1 \le k_i \le a_{r_i}$ then $Q(h \mid \mathbf{r};\mathbf{k})$ has the factor

$$E_{\mathbf{r},\mathbf{k}}\left[\left(\frac{x_{r_i}}{x_0}q\right)_{a_{r_i}}\right] = \left(\frac{x_{r_s}q^{k_s-k_i}}{x_{r_s}q^{k_s}}q\right)_{a_{r_i}} = (q^{1-k_i})_{a_{r_i}} = 0.$$

If $-a_{r_j} \le k_i - k_j \le a_{r_i} - 1$ where $i < j$ then $Q(h \mid \mathbf{r};\mathbf{k})$ has the factor

$$E_{\mathbf{r},\mathbf{k}}\left[\left(\frac{x_{r_i}}{x_{r_j}}\right)_{a_{r_i}}\left(\frac{x_{r_j}}{x_{r_i}}q\right)_{a_{r_j}}\right],$$

which is equal to

$$q^{\binom{a_{r_j}+1}{2}}\left(-\frac{x_{r_j}}{x_{r_i}}\right)^{a_{r_j}}\left(\frac{x_{r_i}}{x_{r_j}}q^{-a_{r_j}}\right)_{a_{r_i}+a_{r_j}} = q^{\binom{a_{r_j}+1}{2}}(-q^{k_i-k_j})^{a_{r_j}}(q^{k_j-k_i-a_{r_j}})_{a_{r_i}+a_{r_j}} = 0.$$

Proof of property **(ii).** Note that since $h \ge k_i$ for all i, the hypothesis implies that $h > a_{r_1} + \cdots + a_{r_s}$.

We first show that $Q(h \mid \mathbf{r};\mathbf{k})$ is proper in x_{r_s}. To do this we write $Q(h \mid \mathbf{r};\mathbf{k})$ as N/D, in which N (the "numerator") is

$$E_{\mathbf{r},\mathbf{k}}\left[\prod_{i=0}^{n} x_i^{b_i}\prod_{j=1}^{n}\left(\frac{x_j}{x_0}q\right)_{a_j}\cdot\prod_{\substack{1\le i,j\le n\\ j\ne i}}\left(\frac{x_i}{x_j}q^{\chi(i>j)}\right)_{a_i}\right],$$

and D (the "denominator") is

$$E_{\mathbf{r},\mathbf{k}}\left[\prod_{j=1}^{n}\left(\frac{x_0}{x_jq^h}\right)_h \Big/ \prod_{i=1}^{s}\left(1-\frac{x_0}{x_{r_i}q^{k_i}}\right)\right],$$

where $\chi(S)$ is 1 if the statement S is true, and 0 otherwise. Notice that $R = \{r_0, r_1, \dots, r_s\}$. Then the degree in x_{r_s} of

$$E_{\mathbf{r},\mathbf{k}}\left[\left(1-\frac{x_i}{x_j}q^m\right)\right]$$

is 1 if $i \in R$ and $j \notin R$, and is 0 otherwise, as is easily seen by checking the four cases. Clearly the degree in x_{r_s} of $E_{\mathbf{r},\mathbf{k}}\, x_i^{b_i}$ is b_i if $i \in R$ and is 0 otherwise. Thus the parts of N contributing to the degree in x_{r_s} are

$$E_{\mathbf{r},\mathbf{k}}\left[\prod_{i\in R} x_i^{b_i}\prod_{i=1}^{s}\prod_{j\ne r_0,\dots,r_s}\left(\frac{x_{r_i}}{x_j}q^{\chi(r_i>j)}\right)_{a_{r_i}}\right],$$

which has degree $(n-s)(a_{r_1}+\cdots+a_{r_s})+\sum_{i\in R} b_i$. The parts of D contributing to the degree in x_{r_s} are

$$E_{\mathbf{r},\mathbf{k}}\left[\prod_{j\neq r_0,\ldots,r_s}\left(\frac{x_0}{x_jq^h}\right)_h\right],$$

which has degree $(n-s)h$.

Thus the total degree of $Q(h\mid\mathbf{r};\mathbf{k})$ in x_{r_s} is

$$d_t=(n-s)(a_{r_1}+\cdots+a_{r_s}-h)+\sum_{i\in R} b_i. \tag{4.7}$$

The hypothesis (4.5) implies that $d_t<0$, so $Q(h\mid\mathbf{r};\mathbf{k})$ is proper in x_{r_s}. Next we apply Lemma 3.1. For any rational function F of x_{r_s} and integers j and k, let $T_{j,k}F$ be the result of replacing x_{r_s} with $x_jq^{k-k_s}$ in F. Since $x_{r_s}q^{k_s}/(x_jq^k)$ is small when $j>r_s$ and is large when $j<r_s$, Lemma 3.1 gives

$$\operatorname*{CT}_{x_s} Q(h\mid\mathbf{r};\mathbf{k})=\sum_{\substack{r_s<r_{s+1}\leq n\\ 1\leq k_{s+1}\leq h}} T_{r_{s+1},k_{s+1}}\left[Q(h\mid\mathbf{r};\mathbf{k})\left(1-\frac{x_{r_s}q^{k_s}}{x_{r_{s+1}}q^{k_{s+1}}}\right)\right]. \tag{4.8}$$

We must show that the right-hand side of (4.8) is equal to the right-hand side of (4.6). Set $\mathbf{r}'=(r_1,\ldots,r_s,r_{s+1})$ and $\mathbf{k}'=(k_1,\ldots,k_s,k_{s+1})$. Then the equality follows easily from the identity

$$T_{r_{s+1},k_{s+1}}\circ E_{\mathbf{r},\mathbf{k}}=E_{\mathbf{r}',\mathbf{k}'}. \tag{4.9}$$

To see that (4.9) holds, we have

$$(T_{r_{s+1},k_{s+1}}\circ E_{\mathbf{r},\mathbf{k}})\,x_{r_i}=T_{r_{s+1},k_{s+1}}\left[x_{r_s}q^{k_s-k_i}\right]=x_{r_{s+1}}q^{k_{s+1}-k_i}=E_{\mathbf{r}',\mathbf{k}'}\,x_{r_i},$$

and if $j\notin\{r_0,\ldots,r_s\}$ then $(T_{r_{s+1},k_{s+1}}\circ E_{\mathbf{r},\mathbf{k}})\,x_j=x_j=E_{\mathbf{r}',\mathbf{k}'}\,x_j$. □

Now we concentrate on proving our main lemmas. In what follows, unless specified otherwise, we assume that $M(\mathbf{x})=x_{j_1}^{p_1}\cdots x_{j_\nu}^{p_\nu}/(x_{i_1}x_{i_2}\cdots x_{i_m})$, where the j's are different from the i's, the p's are positive integers with $\sum_{i=1}^{\nu}p_i=m$, $n\geq j_\nu>\cdots>j_1>0$ and $n>i_m>\cdots>i_1=0$. Note that the assumptions $i_1=0$ and $i_m<n$ are supported by Lemma 2.1.

Lemma 4.2. *Let $M(\mathbf{x})$ be as above. If Lemma 4.1 does not apply, then there is a subset $T=\{t_1,t_2,\ldots,t_d\}$ of I such that: $h=a-\sigma(T)+1$, $Q(h\mid\mathbf{r};\mathbf{k})$ is almost proper in x_n, and $R=\{0,1,\ldots,\widehat{t_1},\ldots,\widehat{t_d},\ldots,n\}$, where $\widehat{t}$ denotes the omission of t.*

Proof. Since Lemma 4.1 does not apply, we must have $k_i>a_{r_1}+\cdots+a_{r_s}$ for some i with $1\leq i\leq s<n$. It follows that $h>a_{r_1}+\cdots+a_{r_s}$.

Let $T=I\setminus R$ denoted by $\{t_1,\ldots,t_d\}$. Then by (4.7), the degree in x_{r_s} of $Q(h\mid\mathbf{r};\mathbf{k})$ is given by

$$d_t=(n-s)(a_{r_1}+\cdots+a_{r_s}-h)+\sum_{i=1}^{\nu}p_i\chi(j_i\in R)-(m-d).$$

The hypothesis implies that $d_t \geq 0$. This is equivalent to

$$h - (a_{r_1} + \cdots + a_{r_s}) \leq \frac{\sum_{i=1}^{\nu} p_i \chi(j_i \in R) - (m-d)}{n-s}.$$

Notice that $s \leq n-d$ and $\sum_{i=1}^{\nu} p_i \chi(j_i \in R) \leq m$. It follows that

$$h - (a_{r_1} + \cdots + a_{r_s}) \leq \frac{\sum_{i=1}^{\nu} p_i \chi(j_i \in R) - (m-d)}{n-s} \leq \frac{m-(m-d)}{n-(n-d)} = 1,$$

and the equality holds only when $s = n-d$ and $\sum_{i=1}^{\nu} p_i \chi(j_i \in R) = m$. The former condition is sufficient, since if $s = n-d$ then every j_i belongs to R. Thus we can conclude that $h = a_{r_1} + \cdots + a_{r_s} + 1$ and $d_t = 0$. This is equivalent to say that $h = a - (a_{t_1} + \cdots + a_{t_d}) + 1$ and $Q(h \mid \mathbf{r}; \mathbf{k})$ is almost proper in x_{r_s}. Since $i_m < n$, we have $R = \{0, 1, \dots, \widehat{t_1}, \dots, \widehat{t_d}, \dots, n\}$. □

Proof of Main Lemma **1**. By definition (4.1) of $Q(h)$ we see that $\mathrm{CT}_{\mathbf{x}}\, Q(-a_0)$ equals the left-hand side of (2.7) if we take $M(\mathbf{x}) = x_{j_1}^{p_1} \cdots x_{j_\nu}^{p_\nu} / (x_{i_1} x_{i_2} \cdots x_{i_m})$.

Fix nonnegative integers $a_1, \dots, a_n$. Clearly if $a_0 = 0$, then the left-hand side of (2.7) is

$$\mathop{\mathrm{CT}}_{\mathbf{x}} \frac{x_{j_1}^{p_1} \cdots x_{j_\nu}^{p_\nu}}{x_0 x_{i_2} \cdots x_{i_m}} \prod_{j=1}^{n} \left(\frac{x_j}{x_0} q\right)_{a_j} \prod_{1 \leq i < j \leq n} \left(\frac{x_i}{x_j}\right)_{a_i} \left(\frac{x_j}{x_i} q\right)_{a_j}.$$

Since the above Laurent polynomial contains only negative powers in x_0, its constant term in x_0 equals zero.

Now we prove by induction on $n-s$ that

$$\mathop{\mathrm{CT}}_{\mathbf{x}} Q(h \mid \mathbf{r}; \mathbf{k}) = 0, \quad \text{if } h \in \{1, \dots, a+1\} \setminus \{a - \sigma(T) + 1 \mid T \subseteq I\}.$$

Note that taking constant term with respect to a variable that does not appear has no effect. Also note that $h \neq 1 + a - \sigma(\varnothing) = 1 + a_1 + \cdots + a_n$.

We may assume that $s \leq n$ and $0 < r_1 < \cdots < r_s \leq n$, since otherwise $Q(h \mid \mathbf{r}; \mathbf{k})$ is not defined. If $s = n$ then r_i must equal i for $i = 1, \dots, n$. Thus $Q(h \mid \mathbf{r}; \mathbf{k}) = Q(h \mid 1, 2, \dots, n; k_1, k_2, \dots, k_n)$, which is 0 by part (i) of Lemma 4.1 and the fact that $k_i \leq h \leq a_1 + \cdots + a_n$ for each i.

Now suppose $0 \leq s < n$. Since $b_0 = -1$, the condition $h > \frac{b_0}{n} = -\frac{1}{n}$ always holds. If part (i) of Lemma 4.1 applies, then $Q(h \mid \mathbf{r}; \mathbf{k}) = 0$. Otherwise, by Lemma 4.2, part (ii) of Lemma 4.1 applies and (4.6) holds. Therefore, applying $\mathrm{CT}_{\mathbf{x}}$ to both sides of (4.6) gives

$$\mathop{\mathrm{CT}}_{\mathbf{x}} Q(h \mid \mathbf{r}; \mathbf{k}) = \sum_{\substack{r_s < r_{s+1} \leq n \\ 1 \leq k_{s+1} \leq h}} \mathop{\mathrm{CT}}_{\mathbf{x}} Q(h \mid r_1, \dots, r_s, r_{s+1}; k_1, \dots, k_s, k_{s+1}).$$

By induction, every term on the right is zero. □

5 Proof of Main Lemma 2

The proof of Main Lemma 2 relies on Lemma 3.1 for almost proper rational functions. It involves complicated computations. By the proof of Main Lemma 1, Lemma 4.2 describes all cases for $\mathrm{CT}_{\mathbf{x}}\, Q(h \mid \mathbf{r}, \mathbf{k}) \neq 0$. To evaluate such cases, we need the following two lemmas.

Lemma 5.1.

$$\prod_{l=1}^{n} \frac{(q^{-\sum_{i=l}^{n} w_i})_{w_l}}{(q)_{\sum_{i=l}^{n} w_i}(q^{-\sum_{i=1}^{l-1} w_i})_{\sum_{i=1}^{l-1} w_i}} \prod_{1\le i<j\le n} \left(q^{-\sum_{l=i}^{j-1} w_l}\right)_{w_i} \left(q^{\sum_{l=i}^{j-1} w_l+1}\right)_{w_j}$$
$$=(-1)^{w} q^{-\binom{w+1}{2}} \frac{(q)_w}{(q)_{w_1}\cdots(q)_{w_n}}, \tag{5.1}$$

where $w = w_1 + \cdots + w_n$.

Proof. Denote the left-hand side of (5.1) by H_n and the right-hand side by G_n. Clearly we have $H_1 = G_1$. To show that $H_n = G_n$, it suffices to show that $H_n/H_{n-1} = G_n/G_{n-1}$ for $n \ge 2$. We have

$$\frac{H_n}{H_{n-1}} = \frac{(q^{-w_n})_{w_n}}{(q)_{w_n}(q^{-w_1-\cdots-w_{n-1}})_{w_1+\cdots+w_{n-1}}} \prod_{l=1}^{n-1} \frac{(q^{-w_l-\cdots-w_n})_{w_l}}{(q^{w_l+\cdots+w_{n-1}+1})_{w_n}(q^{-w_l-\cdots-w_{n-1}})_{w_l}}$$
$$\cdot \prod_{l=1}^{n-1} (q^{-w_l-\cdots-w_{n-1}})_{w_l} (q^{w_l+\cdots+w_{n-1}+1})_{w_n}$$
$$= \frac{(-1)^{w_n} q^{-\binom{w_n+1}{2}}}{(-1)^{w-w_n} q^{-\binom{w-w_n+1}{2}} (q)_{w-w_n}} \prod_{l=1}^{n-1} (-1)^{w_l} q^{-\binom{w_l+1}{2}-w_l(w_{l+1}+\cdots+w_n)} (q^{w_{l+1}+\cdots+w_n+1})_{w_l}.$$

Since it is straightforward to show that

$$\prod_{l=1}^{n-1} q^{-\binom{w_l+1}{2}-w_l(w_{l+1}+\cdots+w_n)} = q^{-\binom{w-w_n+1}{2}-w_n(w-w_n)}$$

and that

$$\prod_{l=1}^{n-1} (q^{w_{l+1}+\cdots+w_n+1})_{w_l} = (q^{w_n+1})_{w-w_n},$$

we have

$$\frac{H_n}{H_{n-1}} = (-1)^{w_n} q^{-\binom{w_n+1}{2}-w_n(w-w_n)} \frac{(q^{w_n+1})_{w-w_n}}{(q)_{w-w_n}},$$

which is equal to G_n/G_{n-1}. □

For fixed subset $T = \{t_1, t_2, \ldots, t_d\}$ of I, we let $h^* = a - \sigma(T) + 1 = w + 1$, $\mathbf{r}^* = (1, \ldots, \widehat{t_1}, \ldots, \widehat{t_d}, \ldots, n)$, and $\mathbf{k}^* = (k_1, \ldots, k_{n-d})$ with $k_l = \sum_{i=r_l}^{n} w_i + 1$. Let

$$N_l = \#\{t_j < l \mid t_j \in T\}, \tag{5.2}$$

where $\#S$ is the cardinality of the set S. Then $E_{\mathbf{r}^*,\mathbf{k}^*} x_i$ is $x_n q^{k_{n-d}-k_{i-N_i}}$ for $i \notin T$, and is x_i for $i \in T$. For $i \notin T$, we have $k_{n-d} - k_{i-N_i} = w_n - \sum_{l=i}^{n} w_l$.

Lemma 5.2. *Let T be a subset of I. Then*

$$\operatorname*{CT}_{\mathbf{x}} Q(h^* \mid \mathbf{r}^*;\mathbf{k}^*) = (-1)^{w+d} q^{L^*(T)} \frac{(q)_w (q)_{a-w}}{(q)_{a_1}\cdots(q)_{a_n}}, \tag{5.3}$$

where

$$L^*(T) = \sum_{l\in I}\sum_{i=l}^{n} w_i - \sum_{l=1}^{\nu} p_l \sum_{i=j_l}^{n} w_i - \binom{w+1}{2} - 1.$$

Proof. By Lemma 4.2, $Q(h^* \mid \mathbf{r}^*;\mathbf{k}^*)$ is almost proper in x_n. Let $R^* = \{r_1,\dots,r_s\} = \{1,\dots,n\}\setminus T$, $s = n-d$.

It is straightforward to check that for any $1 \le i < j \le n$

$$\operatorname*{LC}_{x_n} E_{\mathbf{r}^*,\mathbf{k}^*} \Big(\frac{x_i}{x_j}\Big)_{a_i} \Big(\frac{x_j}{x_i}q\Big)_{a_j} = \begin{cases} \Big(-\dfrac{1}{x_i}\Big)^{w_j} q^{\binom{w_j+1}{2}+(w_n-\sum_{l=j}^{n} w_l)w_j}, & \text{if } i\notin R^*, j\in R^*,\\ \Big(-\dfrac{1}{x_j}\Big)^{w_i} q^{\binom{w_i}{2}+(w_n-\sum_{l=i}^{n} w_l)w_i}, & \text{if } i\in R^*, j\notin R^*,\\ \big(q^{-\sum_{l=i}^{j-1} w_l}\big)_{w_i} \big(q^{\sum_{l=i}^{j-1} w_l+1}\big)_{w_j}, & \text{if } i,j\in R^*,\\ \Big(\dfrac{x_i}{x_j}\Big)_{a_i}\Big(\dfrac{x_j}{x_i}q\Big)_{a_j}, & \text{if } i,j\notin R^*. \end{cases} \tag{5.4}$$

For convenience, we always assume $i<j$ within this proof if i and j appears simultaneously.

Recall that $M(\mathbf{x}) = x_{j_1}^{p_1}\cdots x_{j_\nu}^{p_\nu}/(x_{i_1}x_{i_2}\cdots x_{i_m})$, we have

$$E_{\mathbf{r}^*,\mathbf{k}^*} M(\mathbf{x}) = \frac{x_n^m q^{\sum_{i=1}^{\nu} p_i(k_{n-d}-k_{j_i}-N_{j_i})}}{x_n^{m-d} q^{(m-d)k_{n-d}-\sum_{l\in I\setminus T} k_l - N_l} x_{t_1}\cdots x_{t_d}} = \frac{x_n^d q^{L_1(d)}}{x_{t_1}\cdots x_{t_d}}, \tag{5.5}$$

where

$$L_1(d) = dw_n + \sum_{l\in I\setminus T}\sum_{i=l}^{n} w_i - \sum_{i=1}^{\nu} p_i \sum_{l=j_i}^{n} w_l - 1. \tag{5.6}$$

It is easy to see that

$$\operatorname*{LC}_{x_n} E_{\mathbf{r}^*,\mathbf{k}^*} \prod_{l=1}^{n} \Big(\frac{x_l}{x_0}q\Big)_{a_l} = \prod_{l\in R^*} (q^{-\sum_{i=l}^{n} w_i})_{w_l}, \tag{5.7}$$

and that

$$E_{\mathbf{r}^*,\mathbf{k}^*} \frac{\prod_{i=1}^{n-d}\big(1-x_0/(x_{r_i}q^{k_i})\big)}{\prod_{l=1}^{n}\big(x_0/(x_l q^{h^*})\big)_{h^*}} = \frac{1}{\prod_{l\in R^*}(q)_{\sum_{i=l}^{n} w_i}(q^{-\sum_{i=1}^{l-1} w_i})_{\sum_{i=1}^{l-1} w_i}\prod_{l\notin R^*}(x_n q^{w_n-w}/x_l)_{w+1}}. \tag{5.8}$$

By the definition of $Q(h)$ in (4.2), we have

$$Q(h^* \mid \mathbf{r}^*;\mathbf{k}^*) = E_{\mathbf{r}^*,\mathbf{k}^*} M(x) \prod_{j=1}^{n} \frac{(x_j q/x_0)_{a_j}}{\big(x_0/(x_j q^{h^*})\big)_{h^*}} \prod_{1\le i<j\le n} \Big(\frac{x_i}{x_j}\Big)_{a_i}\Big(\frac{x_j}{x_i}q\Big)_{a_j} \prod_{i=1}^{n-d}\Big(1-x_0/(x_{r_i}q^{k_i})\Big). \tag{5.9}$$

Apply Lemma 3.1 with respect to x_n. Since $Q(h^* \mid \mathbf{r}^*; \mathbf{k}^*)$ has no small factors in the denominator, the summation part in (3.4) equals 0. Thus the result can be written as

$$\operatorname*{LC}_{x_n} E_{\mathbf{r}^*,\mathbf{k}^*} M(x) \prod_{j=1}^{n} \frac{(x_j q/x_0)_{a_j}}{\big(x_0/(x_j q^{h^*})\big)_{h^*}} \prod_{1\le i<j\le n} \Big(\frac{x_i}{x_j}\Big)_{a_i} \Big(\frac{x_j}{x_i} q\Big)_{a_j} \prod_{i=1}^{n-d} \Big(1 - x_0/(x_{r_i} q^{k_i})\Big).$$

Substituting (5.4), (5.5), (5.7), and (5.8) into the result, and then collecting similar terms, we can write

$$\operatorname*{CT}_{\mathbf{x}} Q(h^* \mid \mathbf{r}^*; \mathbf{k}^*) = q^{L_1(d)} A_1 A_2 \operatorname*{CT}_{\mathbf{x}} B_1 B_2. \tag{5.10}$$

Here $q^{L_1(d)} A_1$ is the collection of all powers in q (only from (5.8, 5.4)) given by

$$A_1 = \prod_{l\notin R^*} q^{\binom{w+1}{2} - (w+1)w_n} \prod_{i\notin R^*, j\in R^*} q^{\binom{w_j+1}{2} + (w_n - \sum_{l=j}^{n} w_l) w_j} \prod_{i\in R^*, j\notin R^*} q^{\binom{w_i}{2} + (w_n - \sum_{l=i}^{n} w_l) w_i};$$

A_2 is the collection of all q-factorials (only from (5.7, 5.8, 5.4)) given by

$$A_2 = \prod_{l\in R^*} \frac{(q^{-\sum_{i=l}^{n} w_i})_{w_l}}{(q)_{\sum_{i=l}^{n} w_i} (q^{-\sum_{i=1}^{l-1} w_i})_{\sum_{i=1}^{l-1} w_i}} \prod_{i,j\in R^*} (q^{-\sum_{l=i}^{j-1} w_l})_{w_i} (q^{\sum_{l=i}^{j-1} w_l + 1})_{w_j};$$

B_1 is the collection of all monomial factors (only from (5.5, 5.8, 5.4)) given by

$$B_1 = \frac{1}{x_{t_1}\cdots x_{t_d}} \prod_{l\notin R^*} (-1)^{w+1} x_l^{w+1} \prod_{i\notin R^*, j\in R^*} \big(-1/x_i\big)^{w_j} \prod_{i\in R^*, j\notin R^*} \big(-1/x_j\big)^{w_i} = (-1)^d; \tag{5.11}$$

and B_2 is the collection of all q-factorials containing variables (only from (5.4)) given by

$$B_2 = \prod_{i,j\notin R^*} (x_i/x_j)_{a_i} (x_j q/x_i)_{a_j} = D_d(x_{t_1}, \dots, x_{t_d}; a_{t_1}, \dots, a_{t_d}; q).$$

(Note that for the q-Dyson Theorem, $M(x) = 1$, $T = I = \varnothing$, and hence $B_2 = 1$, so we do not need the next paragraph for our alternative proof of Theorem 1.2.)

It follows by Theorem 1.2 and (5.11) that

$$\operatorname*{CT}_{\mathbf{x}} B_1 B_2 = \operatorname*{CT}_{\mathbf{x}} (-1)^d \prod_{i,j\notin R^*} (x_i/x_j)_{a_i} (x_j/x_i q)_{a_j} = (-1)^d \frac{(q)_{a-w}}{\prod_{l\in T} (q)_{a_l}}. \tag{5.12}$$

Recall that $w_i = 0$ if $i \notin R^*$. By Lemma 5.1 we have

$$A_2 = (-1)^w q^{-\binom{w+1}{2}} \frac{(q)_w}{\prod_{l\in R^*} (q)_{w_l}}. \tag{5.13}$$

Let $A_1 = q^{L_2(d)}$, where

$$L_2(d) = \sum_{l\notin R^*} \Big[\binom{w+1}{2} - (w+1)w_n\Big] + \sum_{i\notin R^*, j\in R^*} \Big[\binom{w_j+1}{2} + \Big(w_n - \sum_{l=j}^{n} w_l\Big) w_j\Big] + \sum_{i\in R^*, j\notin R^*} \Big[\binom{w_i+1}{2} + \Big(w_n - \sum_{l=i}^{n} w_l\Big) w_i - w_i\Big].$$

We claim that

$$L_2(d) = \widetilde{L}_2(d) = -dw_n + dw - \sum_{l\in T}\sum_{k=1}^{l-1} w_k. \tag{5.14}$$

It is clear that $L_2(0) = \widetilde{L}_2(0) = 0$. Therefore to show that $L_2(d) = \widetilde{L}_2(d)$ it suffices to show that $L_2(d) - L_2(d-1) = \widetilde{L}_2(d) - \widetilde{L}_2(d-1)$ for $d \geq 1$.

Since $w_i = 0$ for $i \in T$, we have

$$\begin{aligned} L_2(d) - L_2(d-1) =& \binom{w+1}{2} - (w+1)w_n + \sum_{j=t_d+1}^{n}\left[\binom{w_j+1}{2} + \Big(w_n - \sum_{l=j}^{n} w_l\Big)w_j\right] \\ &+ \sum_{i=1}^{t_d-1}\left[\binom{w_i+1}{2} + \Big(w_n - \sum_{l=i}^{n} w_l\Big)w_i - w_i\right] \\ =& \binom{w+1}{2} - (w+1)w_n + \sum_{j=1}^{n}\left[\binom{w_j+1}{2} + \Big(w_n - \sum_{l=j}^{n} w_l\Big)w_j\right] - \sum_{i=1}^{t_d-1} w_i. \end{aligned}$$

Simplifying the above equation, we obtain

$$\begin{aligned} L_2(d) - L_2(d-1) =& \binom{w+1}{2} - (w+1)w_n + \sum_{j=1}^{n}\binom{w_j+1}{2} + w_n w - \sum_{j=1}^{n}\sum_{l=j}^{n} w_l w_j - \sum_{i=1}^{t_d-1} w_i \\ =& \binom{w+1}{2} - w_n + \sum_{j=1}^{n}\binom{w_j+1}{2} - \sum_{i<j} w_i w_j - \sum_{i=1}^{n} w_i^2 - \sum_{i=1}^{t_d-1} w_i. \end{aligned}$$

Using the fact $\binom{w+1}{2} = \sum_{i=1}^{n}\binom{w_i+1}{2} + \sum_{i<j} w_i w_j$, we get

$$\begin{aligned} L_2(d) - L_2(d-1) =& - w_n + 2\sum_{j=1}^{n}\binom{w_j+1}{2} - \sum_{i=1}^{n} w_i^2 - \sum_{i=1}^{t_d-1} w_i \\ =& - w_n + w - \sum_{i=1}^{t_d-1} w_i, \end{aligned}$$

which equals $\widetilde{L}_2(d) - \widetilde{L}_2(d-1)$. Thus the claim follows.

Substituting (5.12), (5.13), and $A_1 = q^{L_2(d)}$ (with (5.14)) into (5.10) and simplifying yields

$$\mathrm{CT}_{\mathbf{x}} Q(h^* \mid \mathbf{r}^*; \mathbf{k}^*) = (-1)^{d+w} q^{L_1(d)+L_2(d)-\binom{w+1}{2}} \frac{(q)_w (q)_{a-w}}{(q)_{a_1}\cdots(q)_{a_n}}.$$

Therefore

$$\begin{aligned} L^*(T) =& L_1(d) + L_2(d) - \binom{w+1}{2} \\ =& dw_n + \sum_{l\in I\setminus T}\sum_{i=l}^{n} w_i - \sum_{i=1}^{\nu} p_i \sum_{l=j_i}^{n} w_l - 1 - dw_n + dw - \sum_{l\in T}\sum_{k=1}^{l-1} w_k - \binom{w+1}{2} \\ =& \sum_{l\in I\setminus T}\sum_{i=l}^{n} w_i - \sum_{i=1}^{\nu} p_i \sum_{l=j_i}^{n} w_l - 1 + dw - \sum_{l\in T}\sum_{k=1}^{l-1} w_k - \binom{w+1}{2}. \end{aligned}$$

Since dw can be written as $\sum_{l\in T}\sum_{k=1}^{n} w_k$, we have

$$L^*(T) = \sum_{l\in I\setminus T}\sum_{i=l}^{n} w_i - \sum_{i=1}^{\nu} p_i \sum_{l=j_i}^{n} w_l - 1 + \sum_{l\in T}\sum_{k=l}^{n} w_k - \binom{w+1}{2}$$
$$= \sum_{l\in I}\sum_{i=l}^{n} w_i - \sum_{i=1}^{\nu} p_i \sum_{l=j_i}^{n} w_l - 1 - \binom{w+1}{2}.$$

□

Proof of Main Lemma **2**. Applying Lemma 3.1 gives (4.3) as follows.

$$\operatorname*{CT}_{x_0} Q(h) = \sum_{\substack{0<r_1\le n,\\ 1\le k_1\le h}} Q(h \mid r_1; k_1).$$

Iteratively apply Lemma 4.1 to each summand when applicable. In each step, we need to deal with a sum of terms like $Q(h \mid r_1,\dots,r_s;k_1,\dots,k_s)$. For such summand, we apply Lemma 4.1 with respect to x_{r_s}. The summand is taken to 0 if part (i) applies, and is taken to a sum if part (ii) applies. In the latter case, the number of variables decreases by one. Since there are only $n+1$ variables, the iteration terminates. Note that if $r_s = n$ and part (ii) applies, the summand will be taken to 0. So finally we can write

$$\operatorname*{CT}_{\mathbf{x}} Q(h) = \operatorname*{CT}_{\mathbf{x}} \sum_{r_1,\dots,r_s,k_1,\dots,k_s} Q(h \mid r_1,\dots,r_s;k_1,\dots,k_s),$$

where the sum ranges over all r's and k's with $0<r_1<\cdots<r_s\le n, 1\le k_1,k_2,\dots,k_s\le h$ such that Lemma 4.1 does not apply. Note that we may have different s.

By Lemma 4.2, Lemma 4.1 does not apply only if there is a subset $T=\{t_1,\dots,t_d\}$ of I such that $(r_1,\dots,r_s)=(1,\dots,\widehat{t_1},\dots,\widehat{t_d},\dots,n)$, and $h=a-\sigma(T)+1$. So the sum becomes

$$\operatorname*{CT}_{\mathbf{x}} Q(h) = \operatorname*{CT}_{\mathbf{x}} \sum_{T} \sum_{1\le k_1,\dots,k_{n-d}\le h} Q(h \mid \mathbf{r}^*;\mathbf{k}),$$

where T ranges over all $T\subseteq I$ such that $a-\sigma(T)+1=h$.

For each fixed subset T of I as above, we show that almost every $Q(h \mid \mathbf{r}^*;\mathbf{k})$ vanishes. Notice that $E_{\mathbf{r}^*,\mathbf{k}} x_i = x_n^{k_{n-d}-k_{i-N_i}}$ for $i\notin T$ with N_i defined as in (5.2). Rename the parameters a_i by w_i for $i\notin T$, and set $w_i=0$ for $i\in T$. The expression becomes easy to describe.

If $1\le k_{i-N_i}\le w_i$ for some $i\notin T$, then $Q(h \mid \mathbf{r}^*;\mathbf{k})$ has the factor

$$E_{\mathbf{r}^*,\mathbf{k}}\left[\left(\frac{x_i}{x_0}q\right)_{a_i}\right] = \left(\frac{x_n q^{k_{n-d}-k_{i-N_i}}}{x_n q^{k_{n-d}}}q\right)_{w_i} = (q^{1-k_{i-N_i}})_{w_i} = 0.$$

If $-w_j \le k_{i-N_i}-k_{j-N_j}\le w_i-1$, where $i<j$ and $i,j\notin T$, then $Q(h \mid \mathbf{r}^*;\mathbf{k})$ has the factor

$$E_{\mathbf{r}^*,\mathbf{k}}\left[\left(\frac{x_i}{x_j}\right)_{a_i}\left(\frac{x_j}{x_i}q\right)_{a_j}\right] = E_{\mathbf{r}^*,\mathbf{k}}\left[\left(\frac{x_i}{x_j}\right)_{w_i}\left(\frac{x_j}{x_i}q\right)_{w_j}\right],$$

which is equal to

$$E_{\mathbf{r}^*,\mathbf{k}}\left[q^{\binom{w_j+1}{2}}\left(-\frac{x_j}{x_i}\right)^{w_j}\left(\frac{x_i}{x_j}q^{-w_j}\right)_{w_i+w_j}\right]=q^{\binom{w_j+1}{2}}(-q^{k_{i-N_i}-k_{j-N_j}})^{w_j}(q^{k_{j-N_j}-k_{i-N_i}-w_j})_{w_i+w_j}=0.$$

If neither of the above two cases happen, then by Lemma 3.2 for the case $s=n-d$, we see that $\mathbf{k}$ must equal $\mathbf{k}^*$ given by

$$\mathbf{k}^*=\left(\sum_{i=r_1}^{n}w_i+1,\sum_{i=r_2}^{n}w_i+1,\ldots,\sum_{i=r_{n-d}}^{n}w_i+1\right).$$

Therefore, for every T, all $Q(h\mid\mathbf{r}^*;\mathbf{k})$ vanish except for $Q(h\mid\mathbf{r}^*;\mathbf{k}^*)$. It follows that

$$\operatorname*{CT}_{\mathbf{x}}Q(h^*)=\operatorname*{CT}_{\mathbf{x}}\sum_T Q(h^*\mid\mathbf{r}^*;\mathbf{k}^*)=\sum_T\operatorname*{CT}_{\mathbf{x}}Q(h^*\mid\mathbf{r}^*;\mathbf{k}^*).$$

Thus the proof is completed by Lemma 5.2. □

6 Concluding Remark

For the equal parameter case, Stembridge [16] studied the constant terms for general monomials $M(\mathbf{x})$ and obtained recurrence formulas. However, explicit formulas are obtained only for $M(\mathbf{x})=x_{j_1}^{p_1}\cdots x_{j_\nu}^{p_\nu}/(x_{i_1}x_{i_2}\cdots x_{i_m})$, just as we discussed. These formulas are called first layer formulas. For the unequal parameter case, our method may be used to evaluate the constant terms for monomials like $M(\mathbf{x})=x_sx_t/x_0^2$, but the explicit formula will be too complicated. We can expect that other types of q-Dyson style constant terms can be solved in a similar way.

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