Triangle-free subcubic graphs with minimum bipartite density

Baogang Xu*
School of Math. & Computer Science
Nanjing Normal University
122 Ninghai Road, Nanjing, 210097, China
Email: baogxu@njnu.edu.cn

Xingxing Yu†
School of Mathematics
Georgia Institute of Technology
Atlanta, GA 30332-0160, USA
Email:yu@math.gatech.edu
and
Center for Combinatorics, LPMC
Nankai University
Tianjin, 300071, China

Abstract

A graph is subcubic if its maximum degree is at most 3. The bipartite density of a graph G is $\max\{\varepsilon(H)/\varepsilon(G) : H$ is a bipartite subgraph of $G\}$, where $\varepsilon(H)$ and $\varepsilon(G)$ denote the numbers of edges in H and G, respectively. It is an NP-hard problem to determine the bipartite density of any given triangle-free cubic graph. Bondy and Locke gave a polynomial time algorithm which, given a triangle-free subcubic graph G, finds a bipartite subgraph of G with at least $\frac{4}{5}\varepsilon(G)$ edges; and showed that the Petersen graph and the dodecahedron are the only triangle-free cubic graphs with bipartite density $\frac{4}{5}$. Bondy and Locke further conjectured that there are precisely seven triangle-free subcubic graphs with bipartite density $\frac{4}{5}$. We prove this conjecture of Bondy and Locke. Our result will be used in a forthcoming paper to solve a problem of Bollobás and Scott related to judicious partitions.

Keywords: triangle-free; subcubic; bipartite subgraph; bipartite density
AMS 2000 Subject Classifications: 05C35; 05C75

---

*Supported by Nanjing Normal University, NSFC, and Georgia Institute of Technology.
†Partially supported by NSF, NSA, and NSFC Project 10628102

1 Introduction

The Maximum Bipartite Subgraph Problem on a graph G is that of finding a bipartite subgraph of G with the maximum number of edges (called *maximum* bipartite subgraph). This is the unweighted version of the Max-Cut problem, since the edges in a maximum bipartite subgraph form an edge cut. The Max-Cut problem is one of the Karp's original NP-complete problems [11], and it remains NP-complete for the unweighted version (see also [5, 7]). It is shown in [1] that it is NP-hard to approximate the max-cut problem on cubic graphs beyond the ratio of 0.997. On the other hand, the Max-Cut problem is polynomial time solvable for planar graphs, see [9, 13]. Goemans and Williamson [8] used semidefinite programming and hyperplane rounding to give a randomized algorithm with expected performance guarantee of 0.87856. Feige, Karpinski and Langberg [6] gave a similar randomized algorithm that improves this bound to .921 for subcubic graphs. A graph is subcubic if it has maximum degree at most three.

Yannakakis [15] showed that the Maximum Bipartite Subgraph Problem is NP-hard even when restricted to triangle-free cubic graphs. In this paper, we study the maximum bipartite subgraph problem for triangle-free subcubic graphs. For convenience, we let

$$\mathcal{G} = \{\text{connected, triangle-free, subcubic multigraphs}\}.$$

For a graph G, we follow [3] to denote by $\varepsilon(G)$ the number of edges of G, and let

$$\mathcal{B}(G) = \{\text{maximum bipartite subgraphs of } G\}.$$

We define the *bipartite density* of G as

$$b(G) = \max\{\frac{\varepsilon(B)}{\varepsilon(G)} \ : \ B \text{ is a bipartite subgraph of } G\}.$$

Erdös [4] proved that if G is $2m$-colorable then $b(G) \geq \frac{m}{2m-1}$. As a consequence, if G is a cubic graph then $b(G) \geq \frac{2}{3}$. Stanton [14] and Locke [12] further showed that if G is a cubic graph and $G \neq K_4$ then $b(G) \geq \frac{7}{9}$. Hopkins and Stanton [10] proved $b(G) \geq \frac{4}{5}$ if G is a triangle-free cubic graph. Bondy and Locke [3] gave a polynomial time algorithm which, given a graph $G \in \mathcal{G}$, finds a bipartite subgraph of G with at least $\frac{4}{5}\varepsilon(G)$ edges; and they further proved that the Petersen graph and the dodecahedron (shown in Figure 1) are the only cubic graphs with bipartite density $\frac{4}{5}$.

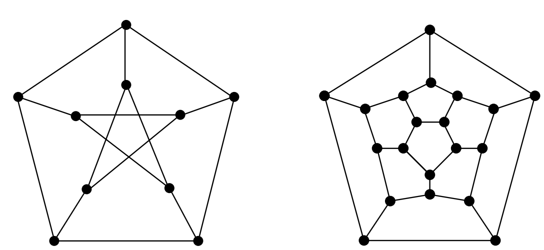

Figure 1: The Petersen graph and the dodecahedron.

Theorem 1.1 *(Bondy and Locke [3]) If $G \in \mathcal{G}$ then $b(G) \geq \frac{4}{5}$. Furthermore, if $G \in \mathcal{G}$ is cubic and $b(G) = \frac{4}{5}$, then G is either the Petersen graph or the dodecahedron.*

It is not hard to see that the graphs in Figure 2 are in $\mathcal{G}$ and have bipartite density $\frac{4}{5}$. Bondy and Locke [3] conjectured that the graphs in Figures 1 and 2 are precisely those in $\mathcal{G}$ with bipartite density $\frac{4}{5}$.

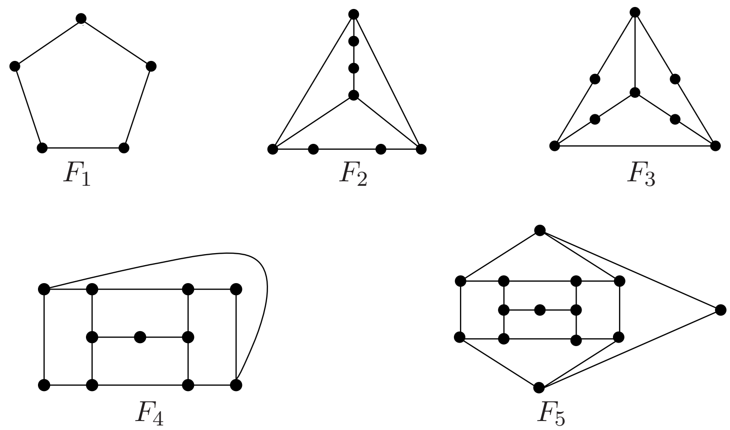

Figure 2: Triangle-free subcubic graphs with bipartite density $\frac{4}{5}$.

The main result of this paper is the following theorem, which establishes the conjecture of Bondy and Locke. For convenience, we use F_6 and F_7 to denote the Petersen graph and the dodecahedron, respectively.

Theorem 1.2 *If $G \in \mathcal{G}$ and $b(G) = \frac{4}{5}$, then $G \in \{F_i : 1 \le i \le 7\}$.*

Note the drawings of F_4 and F_5 in Figure 2; they are different from those in [3]. This is to illustrate a common structure of F_4 and F_5, which will be useful when proving Theorem 1.2.

It is pointed out in [3] that Theorem 1.2 is equivalent to the statement that the graphs in Figures 1 and 2 are precisely those in $\mathcal{G}$ which admit an m-covering by 5-cycles for some positive integer m. An *m-covering* of a graph is a collection of subgraphs of G such that every edge belongs to exactly m of these subgraphs.

For any bipartite graph B, we use $V_1(B)$ and $V_2(B)$ to denote a partition of $V(B)$ such that every edge of B has exactly one end in each $V_i(B)$. Bollobás and Scott [2] observed that the Petersen graph admits a maximum bipartite subgraph B such that $V_1(B)$ is an independent set; and they commented that the partition $V_1(B), V_2(B)$ of the Petersen graph is some way from judicious. (For a graph G, a partition V_1, V_2 of $V(G)$ is *judicious* if $\max\{\varepsilon(G[V_1]), \varepsilon(G[V_2])\}$ is close to be minimum among all bipartitions of $V(G)$, where for $i = 1, 2$, $G[V_i]$ denotes the subgraph of G induced by V_i). Bollobás and Scott [2] asked the following question.

Problem 1.3 *What are those cubic graphs G with $b(G) = \frac{4}{5}$ such that for some maximum bipartite subgraph B of G, $V_1(B)$ is independent.*

We observe that the dodecahedron admits a maximum bipartite subgraph B such that $V_1(B)$ is independent. See Figure 3. If we delete the edges joining vertices represented by solid circles, the result is a maximum bipartite subgraph of the dodecahedron, where $V_1(B)$ consists of those vertices represented by solid squares.

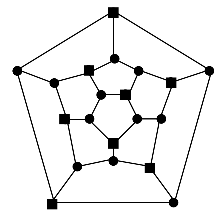

Figure 3: A maximum bipartite subgraph of the dodecahedron.

Interested readers may verify that each graph in Figure 2 also contains a maximum bipartite subgraph B with $V_1(B)$ independent. Hence, the following is a direct consequence of Theorem 1.2, which answers Problem 1.3 for triangle-free graphs. (In a forthcoming paper, we shall completely solve Problem 1.3.)

Corollary 1.4 *The graphs F_i, $1 \le i \le 7$, are precisely those in $\mathcal{G}$ that have bipartite density $\frac{4}{5}$ and contain a maximum bipartite subgraph B with $V_1(B)$ independent.*

To prove Theorem 1.2, it suffices to show that if $G \in \mathcal{G}$ and G is not cubic, then G is one of the graphs in Figure 2. We first prove, in section 2, several simple lemmas about graphs in $\mathcal{G}$ that have bipartite density $\frac{4}{5}$. These lemmas show that certain configurations are forbidden for graphs in $\mathcal{G}$ with bipartite density $\frac{4}{5}$. In section 3, we show that if $G \in \mathcal{G}$ contains two adjacent vertices of degree 2, then $b(G) = \frac{4}{5}$ implies $G \in \{F_1, F_2\}$. In section 4, we show that if $G \in \mathcal{G}$ has a vertex of degree 3 which is adjacent to two vertices of degree 2, then $b(G) = \frac{4}{5}$ implies $G = F_3$ or G is not a minimum counter example to Theorem 1.2. We show in section 5 that if no two vertices of degree 2 are adjacent or share a common neighbor, then $G \in \{F_4, F_5\}$ or G is not a minimum counter example to Theorem 1.2. The proof of Theorem 1.2 is completed in section 6.

For convenience, we use $A := B$ to rename B to A. Let G be a graph and $S \subseteq V(G) \cup E(G)$. Then $G - S$ denotes the graph obtained from G by deleting S and edges of G incident with vertices in S. For any subgraph H of G, we use $H + S$ to denote the subgraph of G with vertex set $V(H) \cup (S \cap V(G))$ and edge set $E(H) \cup \{uv \in S \cap E(G) : \{u, v\} \subseteq V(H) \cup (S \cap V(G))\}$. When $S = \{s\}$, we simply write $G - s := G - S$ and $H + s := H + S$. In the case of $H + S$, if G is not given then we implicitly assume that G is a multigraph containing both H and S.

Let G be a graph, and $v_1, \ldots, v_k$ vertices of G. We use $A(v_1, \ldots, v_k)$ to denote the set consisting of $v_i, 1 \le i \le k$, and all edges of G with at least one end in $\{v_1, \ldots, v_k\}$. A vertex of G is said to be a *k-vertex* if it has degree k in G. For any vertex v of G, we use $N_G(v)$ (or $N(v)$ if there is no ambiguity) to denote the set of neighbors of v in G.

2 Several forbidden configurations

We show in this section that graphs in $\mathcal{G}$ with bipartite density $\frac{4}{5}$ do not contain certain configurations. First, it is easy to see that if $G \in \mathcal{G}$ then the minimum degree of G must be at least 2. Indeed, Lemma 3.1 of [3] says a bit more; and we state it and include its proof.

Lemma 2.1 *Let $G \in \mathcal{G}$ and assume $b(G) = \frac{4}{5}$. Then G is 2-connected.*

Proof. Suppose G is not 2-connected. Then since G is subcubic, G has a cut edge, say uv. Let G_u, G_v denote the components of $G - uv$ containing u, v, respectively. Clearly, $G_u, G_v \in \mathcal{G}$. By Theorem 1.1, $b(G_u) \geq \frac{4}{5}$ and $b(G_v) \geq \frac{4}{5}$. Let $B_u \in \mathcal{B}(G_u)$ and $B_v \in \mathcal{B}(G_v)$. Then $B := (B_u \cup B_v) + uv$ is a bipartite subgraph of G, and

$$\begin{aligned} \varepsilon(B) &= \varepsilon(B_u) + \varepsilon(B_v) + 1 \\ &\geq \frac{4}{5}\varepsilon(G_u) + \frac{4}{5}\varepsilon(G_v) + 1 \\ &> \frac{4}{5}\varepsilon(G). \end{aligned}$$

This implies $b(G) > \frac{4}{5}$, a contradiction. ∎

Lemma 2.1 will be used frequently in later proofs. Suppose $G \in \mathcal{G}$, $b(G) = \frac{4}{5}$, and G has maximum degree 2. Then it follows from Lemma 2.1 that G is a cycle of length 5. Hence, we have

Lemma 2.2 *Let $G \in \mathcal{G}$ and $b(G) = \frac{4}{5}$, and assume that G has maximum degree 2. Then $G = F_1$.*

The next lemma shows that, with the exception of F_1, for any graph in $\mathcal{G}$ with bipartite density $\frac{4}{5}$, no 2-vertex is adjacent to two 2-vertices.

Lemma 2.3 *Let $G \in \mathcal{G}$ and $b(G) = \frac{4}{5}$. Then $G = F_1$, or every 2-vertex of G is adjacent to at least one 3-vertex.*

Proof. Suppose the assertion of the lemma is false. Then $G \neq F_1$, and G has a 2-vertex x that is adjacent to two 2-vertices u and v. See Figure 4. Since G is 2-connected (by Lemma 2.1) and the maximum degree of G is 3 (by Lemma 2.2), we may assume without loss of generality that v is adjacent to a 3-vertex w in G. Let s and t be the neighbors of w other than v, and let $u' \neq x$ be the other neighbor of u.

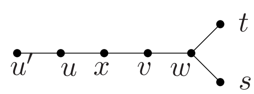

Figure 4: Vertices x, u, v and their neighbors.

Let $A := A(u, v, w, x)$. Clearly, $G - A$ is subcubic and triangle-free, and $\varepsilon(G - A) = \varepsilon(G) - 6$. Since G is 2-connected, $G - A$ must be connected. So $G - A \in \mathcal{G}$. Let $B' \in \mathcal{B}(G - A)$. Then by Theorem 1.1, $\varepsilon(B') \geq \frac{4}{5}\varepsilon(G - A) \geq \frac{4}{5}(\varepsilon(G) - 6)$. Without loss of generality, we may assume that $t \in V_1(B')$. Define

$$B := \begin{cases} B' + (A - \{uu'\}), & \text{if } s \in V_1(B'); \\ B' + (A - \{tw\}), & \text{if } s \in V_2(B') \text{ and } u' \in V_1(B'); \\ B' + (A - \{sw\}), & \text{if } s \in V_2(B') \text{ and } u' \in V_2(B'). \end{cases}$$

Then B is a bipartite subgraph of G, and $\varepsilon(B) = \varepsilon(B') + 5 \geq \frac{4}{5}(\varepsilon(G) - 6) + 5 > \frac{4}{5}\varepsilon(G)$. So $b(G) > \frac{4}{5}$, a contradiction. ∎

We now show that in a subcubic graph with bipartite density $\frac{4}{5}$, no 3-vertex can have three 2-vertices as neighbors.

Lemma 2.4 *Let $G \in \mathcal{G}$ and $b(G) = \frac{4}{5}$, and let x be a 3-vertex of G. Then, x is adjacent to at most two 2-vertices. Furthermore, if x is adjacent to two 2-vertices, say u and v, then neither u nor v is adjacent to a 2-vertex.*

Proof. By Lemma 2.1, G is 2-connected. First, assume that x is adjacent to three 2-vertices, say u, v and w. See Figure 5(a). Let u', v' and w' be the neighbors of u, v and w, respectively, which are all different from x. Let $A := A(u, v, w, x)$. Clearly, $G - A$ is subcubic and triangle-free, and $\varepsilon(G - A) = \varepsilon(G) - 6$. Since G is 2-connected, $G - A$ must be connected. So $G - A \in \mathcal{G}$. Let $B' \in \mathcal{B}(G - A)$. By Theorem 1.1, $\varepsilon(B') \geq \frac{4}{5}\varepsilon(G - A) = \frac{4}{5}(\varepsilon(G) - 6)$. Without loss of generality, we may assume $\{u', v'\} \subseteq V_1(B')$. Let $B := B' + (A - \{ww'\})$. Then B is a bipartite subgraph of G, and $\varepsilon(B) = \varepsilon(B') + 5 \geq \frac{4}{5}(\varepsilon(G) - 6) + 5 > \frac{4}{5}\varepsilon(G)$; contradicting the assumption that $b(G) = \frac{4}{5}$. This proves the first assertion of the lemma.

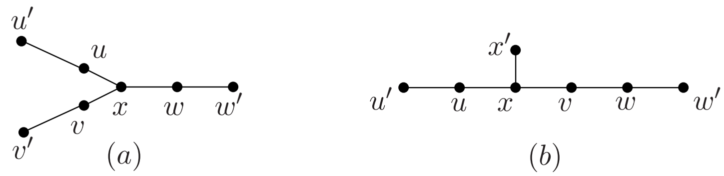

Figure 5: 3-Vertex x and its neighbors.

To prove the second assertion of the lemma, we assume for a contradiction that x is adjacent to two 2-vertices u and v, and v is adjacent to a 2-vertex w. See Figure 5(b). Let w' be the neighbor of w different from v, u' be the neighbor of u different from x, and x' be the neighbor of x not in $\{u, v\}$.

Again, let $A := A(u, v, w, x)$. Then, $G - A$ is subcubic and triangle-free, and $\varepsilon(G - A) = \varepsilon(G) - 6$. Since G is 2-connected, $G - A$ must be connected. Hence $G - A \in \mathcal{G}$. Let $B' \in \mathcal{B}(G - A)$. By Theorem 1.1, $\varepsilon(B') \geq \frac{4}{5}\varepsilon(G - A) = \frac{4}{5}(\varepsilon(G) - 6)$. Without loss of generality, assume that $u' \in V_1(B')$. Define

$$B := \begin{cases} B' + (A - \{xx'\}), & \text{if } w' \in V_2(B'); \\ B' + (A - \{ww'\}), & \text{if } w' \in V_1(B') \text{ and } x' \in V_2(B'); \\ B' + (A - \{uu'\}), & \text{if } w' \in V_1(B') \text{ and } x' \in V_1(B'). \end{cases}$$

Then B is a bipartite subgraph of G, and $\varepsilon(B) = \varepsilon(B') + 5 \geq \frac{4}{5}(\varepsilon(G) - 6) + 5 > \frac{4}{5}\varepsilon(G)$. So $b(G) > \frac{4}{5}$, a contradiction. ∎

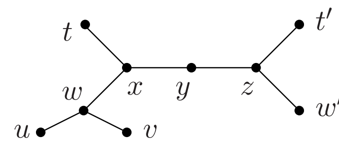

Figure 6: A forbidden configuration.

We now show that if $G \in \mathcal{G}$ and $b(G) = \frac{4}{5}$, then under some technical condition, G does not contain the configuration shown in Figure 6, where w, x, y, z are different from all other vertices, and their degrees in G are exactly those shown in Figure 6.

Lemma 2.5 *Let $G \in \mathcal{G}$ and $b(G) = \frac{4}{5}$, let y be a 2-vertex of G, and let $x, z \in N(y)$ be 3-vertices. Let $N(x) - \{y\} = \{t, w\}$, and assume that w is a 3-vertex and $zw \notin E(G)$. Then one of the following holds:*

(i) there exists $G' \in \mathcal{G}$ such that $b(G') = \frac{4}{5}$, $G' \notin \{F_i : 1 \le i \le 7\}$, and $|V(G')| < |V(G)|$; or

(ii) $N(t) \cap N(z) \neq \emptyset$.

Proof. Since $b(G) = \frac{4}{5}$, G is 2-connected (by Lemma 2.1). Let $w', t' \in N(z) - \{y\}$. See Figure 6. If $tt' \in E(G)$ or $tw' \in E(G)$, then (ii) holds. So we may assume that

(1) $tt', tw' \notin E(G)$.

Note that we allow $t \in \{t', w'\}$. Let $A := A(w, x, y)$, and let $G' := (G - A) + tz$. Clearly, G' is subcubic and $\varepsilon(G') = \varepsilon(G) - 5$. By (1), G' is triangle-free. Since G is 2-connected, G' is connected. Hence $G' \in \mathcal{G}$, and by Theorem 1.1, $b(G') \ge \frac{4}{5}$. Choose an arbitrary B' from $\mathcal{B}(G')$. Then $\varepsilon(B') \ge \frac{4}{5}\varepsilon(G') = \frac{4}{5}(\varepsilon(G) - 5)$. Note that $t \in V_i(B')$ for some $i \in \{1, 2\}$. Hence, we have

(2) $z \in V_{3-i}(B')$ if $tz \in E(B')$, and $z \in V_i(B')$ if $tz \notin E(B')$ (by maximality of B').

Let $u, v \in N(w) - \{x\}$. See Figure 6. Note that $\{t', w'\}$ and $\{u, v\}$ need not be disjoint. Define

$$B := \begin{cases} (B' - tz) + (A - \{wx\}), & \text{if } tz \in E(B') \text{ and } \{u, v\} \subseteq V_i(B'); \\ (B' - tz) + A, & \text{if } tz \in E(B') \text{ and } \{u, v\} \subseteq V_{3-i}(B'); \\ (B' - tz) + (A - \{wu\}), & \text{if } tz \in E(B'), u \in V_i(B') \text{ and } v \in V_{3-i}(B'); \\ (B' - tz) + (A - \{wv\}), & \text{if } tz \in E(B'), u \in V_{3-i}(B') \text{ and } v \in V_i(B'); \\ B' + (A - \{xt\}), & \text{if } tz \notin E(B') \text{ and } \{u, v\} \subseteq V_i(B'); \\ B' + (A - \{yz\}), & \text{if } tz \notin E(B') \text{ and } \{u, v\} \subseteq V_{3-i}(B'); \\ B' + (A - \{wu, yz\}), & \text{if } tz \notin E(B'), u \in V_i(B') \text{ and } v \in V_{3-i}(B'); \\ B' + (A - \{wv, yz\}), & \text{if } tz \notin E(B'), u \in V_{3-i}(B') \text{ and } v \in V_i(B'). \end{cases}$$

It is straightforward to verify that B is a bipartite subgraph of G. Moreover, $\varepsilon(B) = \varepsilon(B') + 4$, or $\varepsilon(B) = \varepsilon(B') + 5$. We claim that

(3) for any $B' \in \mathcal{B}(G')$, $\varepsilon(B) = \varepsilon(B') + 4$; and $b(G') = \frac{4}{5}$.

For otherwise, $\varepsilon(B) = \varepsilon(B') + 5$, or $b(G') > \frac{4}{5}$. If the former occurs, then $\varepsilon(B) = \varepsilon(B') + 5 \ge \frac{4}{5}(\varepsilon(G) - 5) + 5 > \frac{4}{5}\varepsilon(G)$, contradicting the assumption that $b(G) = \frac{4}{5}$. Now assume $b(G') > \frac{4}{5}$. Then $\varepsilon(B) \ge \varepsilon(B') + 4 > \frac{4}{5}(\varepsilon(G) - 5) + 4 = \frac{4}{5}\varepsilon(G)$, which implies $b(G) > \frac{4}{5}$, a contradiction.

By (3) and by the definition of B above,

(4) for any $B' \in \mathcal{B}(G')$ and for any $i \in \{1, 2\}$, $\{u, v, z\} \nsubseteq V_i(B')$, and $\{u, v\} \nsubseteq V_{3-i}(B')$ or $\{t, z\} \nsubseteq V_i(B')$.

Since $b(G') = \frac{4}{5}$ and G' is connected, it follows from Lemma 2.1 that G' is 2-connected. So u and v must be 2-vertices in G'. Since G is triangle-free, $uv \notin E(G')$. Because z is a 3-vertex in G and since $zw \notin E(G)$ and $tz \in E(G')$, z is also a 3-vertex in G'. To summarize, we have

(5) u and v are 2-vertices in G', $uv \notin E(G')$, $tz \in E(G')$, and z is a 3-vertex in G'.

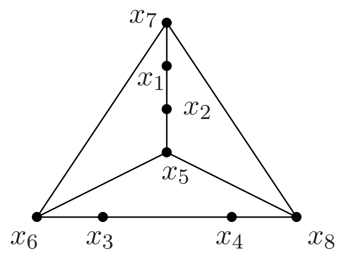

Figure 7: $G' = F_2$.

Since G' has a 2-vertex, $G' \notin \{F_6, F_7\}$. Note that $G' \neq F_1$ since z is a 3-vertex of G'. So if $G' \notin \{F_2, F_3, F_4, F_5\}$, then (i) holds. Therefore, we may assume $G' \in \{F_2, F_3, F_4, F_5\}$; and we have four cases to consider.

Case 1. $G' = F_2$.

See Figure 7, where the vertices of G' are labeled as $x_1, \ldots, x_8$. By (5) and by symmetry, we may assume that $u = x_1$ and $v = x_3$. Again by (5), $z \in \{x_5, x_6, x_7, x_8\}$. Define bipartite subgraph B' of G' as follows.

$$B' := \begin{cases} G' - \{x_6x_7, x_5x_8\}, & \text{if } z \in \{x_5, x_8\}; \\ G' - \{x_6x_3, x_5x_2\}, & \text{if } z = x_6; \\ G' - \{x_7x_1, x_4x_8\}, & \text{if } z = x_7. \end{cases}$$

Then $B' \in \mathcal{B}(G')$ and $\{u, v, z\} \subseteq V_i(B')$ for some $i \in \{1, 2\}$, contradicting (4).

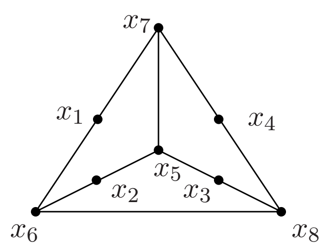

Figure 8: $G' = F_3$

Case 2. $G' = F_3$.

See Figure 8, where the vertices of G' are labeled as $x_1, \ldots, x_8$. Suppose t is a 3-vertex in G. Then t is a 3-vertex in G'. Let $B' := G' - \{x_5x_7, x_6x_8\}$. Now $B' \in \mathcal{B}(G')$ with $V_1(B') = \{x_1, x_2, x_3, x_4\}$ and $V_2(B') = \{x_5, x_6, x_7, x_8\}$. It follows from (5) that $\{u, v\} \subseteq V_1(B')$ and $\{t, z\} \subseteq V_2(B')$, contradicting (4). So we assume t is a 2-vertex in G. Then t is also a 2-vertex in G'.

By (5) and symmetry we may assume that $\{u, v\} = \{x_1, x_2\}$ or $\{u, v\} = \{x_1, x_3\}$.

Suppose $\{u, v\} = \{x_1, x_2\}$. Then $z \neq x_6$, since t is a 2-vertex in G' and $tz \in E(G')$. Define $B' := G' - \{x_1x_7, x_3x_5\}$. Then $B' \in \mathcal{B}(G')$, with $V_1(B') = \{x_1, x_2, x_7, x_8\}$ and $V_2(B') = \{x_3, x_4, x_5, x_6\}$. So $\{u, v, z\} \subseteq V_1(B')$ when $z \in \{x_7, x_8\}$, and $\{u, v\} \subseteq V_1(B')$ and $\{t, z\} \subseteq V_2(B')$ when $z = x_5$ (in which case, $t = x_3$). This contradicts (4).

So $\{u, v\} = \{x_1, x_3\}$. Define

$$B' := \begin{cases} G' - \{x_1x_7, x_3x_8\}, & \text{if } z \in \{x_7, x_8\}; \\ G' - \{x_1x_6, x_3x_5\}, & \text{if } z \in \{x_5, x_6\}. \end{cases}$$

Then $B' \in \mathcal{B}(G')$ and $\{u, v, z\} \subseteq V_i(B')$ for some $i \in \{1, 2\}$, contradicting (4).

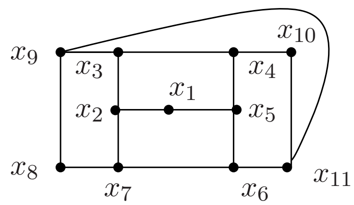

Figure 9: $G' = F_4$

Case 3. $G' = F_4$.

See Figure 9, where the vertices of G' are labeled as $x_1, \dots, x_{11}$. By (5) and by symmetry, we may assume $u = x_1$ and $v = x_{10}$. Also by (5), $z \notin \{x_1, x_8, x_{10}\}$. We define a bipartite subgraph B' of G' as follows.

$$B' := \begin{cases} G' - \{x_3x_9, x_4x_5, x_6x_7\}, & \text{if } z \in \{x_3, x_6, x_7, x_9\}; \\ G' - \{x_1x_5, x_7x_8, x_{10}x_{11}\}, & \text{if } z \in \{x_5, x_{11}\}; \\ G' - \{x_1x_2, x_4x_{10}, x_7x_8\}, & \text{if } z \in \{x_2, x_4\}. \end{cases}$$

Then, $B' \in \mathcal{B}(G')$, and $\{u, v, z\} \subseteq V_i(B')$ for some $i \in \{1, 2\}$, contradicting (4).

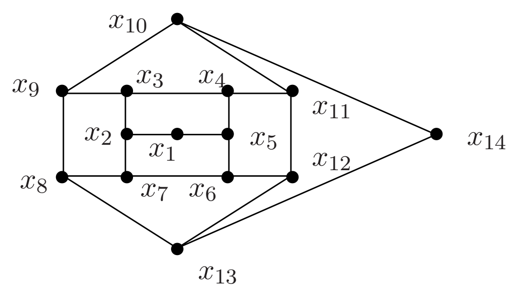

Figure 10: $G' = F_5$

Case 4. $G' = F_5$.

See Figure 10, where the vertices of G' are labeled as $x_1, \dots, x_{14}$. By (5), $\{u, v\} = \{x_1, x_{14}\}$, and $z \notin \{x_1, x_{14}\}$. Define a bipartite subgraph B' of G' as follows.

$$B' := \begin{cases} G' - \{x_2x_7, x_3x_4, x_6x_{12}, x_{10}x_{14}\}, & \text{if } z \in \{x_3, x_4, x_6, x_8, x_{10}, x_{12}\}; \\ G' - \{x_2x_3, x_4x_{11}, x_6x_7, x_{13}x_{14}\}, & \text{if } z \in \{x_7, x_9, x_{11}, x_{13}\}; \\ G' - \{x_1x_2, x_3x_9, x_6x_{12}, x_{10}x_{14}\}, & \text{if } z = x_2; \\ G' - \{x_1x_5, x_3x_9, x_6x_{12}, x_{13}x_{14}\}, & \text{if } z = x_5. \end{cases}$$

Then $B' \in \mathcal{B}(G')$, and $\{u, v, z\} \subseteq V_i(B')$ for some $i \in \{1, 2\}$. This contradicts (4). ∎

3 The graph F_2

We show in this section that F_1 and F_2 are the only graphs in $\mathcal{G}$ that have bipartite density $\frac{4}{5}$ and contain two adjacent 2-vertices.

Suppose that $G \in \mathcal{G}$ and $b(G) = \frac{4}{5}$, and assume that $G \neq F_1$. By Lemma 2.1, G is 2-connected. Let u, v be two adjacent 2-vertices in G, $x \in N(u) - \{v\}$, and $y \in N(v) - \{u\}$. By Lemma 2.3, both x and y are 3-vertices. Let $N(x) - \{u\} = \{x_1, x_2\}$ and $N(y) - \{v\} = \{y_1, y_2\}$. See Figure 11.

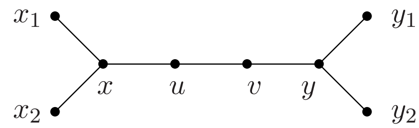

Figure 11: Adjacent 2-vertices and their neighbors.

Lemma 3.1 *$xy \notin E(G)$.*

Proof. Otherwise, we may assume by symmetry that $y = x_2$ and $x = y_2$. Let $A := A(u, v, x, y)$. Then $G - A$ is subcubic and triangle-free, and $\varepsilon(G - A) = \varepsilon(G) - 6$. Since G is 2-connected, $G - A$ must be connected. So $G - A \in \mathcal{G}$. Let $B' \in \mathcal{B}(G - A)$. Then by Theorem 1.1, $\varepsilon(B') \geq \frac{4}{5}\varepsilon(G - A) = \frac{4}{5}(\varepsilon(G) - 6)$. Clearly, $B := B' + (A - \{yy_1\})$ is a bipartite subgraph of G, and $\varepsilon(B) = \varepsilon(B') + 5 \geq \frac{4}{5}(\varepsilon(G) - 6) + 5 > \frac{4}{5}\varepsilon(G)$. This implies $b(G) > \frac{4}{5}$, a contradiction. ∎

Lemma 3.2 *$\{x_1, x_2\} \cap \{y_1, y_2\} \neq \emptyset$.*

Proof. Suppose $\{x_1, x_2\} \cap \{y_1, y_2\} = \emptyset$. Let $A := A(u, v)$. Then $G' := (G - A) + xy$ is subcubic and triangle-free. Since G is 2-connected, G' must be connected. So $G' \in \mathcal{G}'$. Note that $\varepsilon(G') = \varepsilon(G) - 2$. Let $B' \in \mathcal{B}(G')$. Then by Theorem 1.1, $\varepsilon(B') \geq \frac{4}{5}\varepsilon(G') = \frac{4}{5}(\varepsilon(G) - 2)$. Define

$$B = \begin{cases} (B' - xy) + A, & \text{if } xy \in E(B'); \\ B' + (A - \{uv\}), & \text{if } xy \notin E(B'). \end{cases}$$

Then B is a bipartite subgraph of G, and $\varepsilon(B) = \varepsilon(B') + 2 \geq \frac{4}{5}(\varepsilon(G) - 2) + 2 > \frac{4}{5}\varepsilon(G)$. So $b(G) > \frac{4}{5}$, a contradiction. ∎

By symmetry, we may assume that $x_1 = y_1$, which must be a 3-vertex in G (by Lemma 2.4). So let t be the neighbor of x_1 other than x and y. Since G is triangle-free, $t \neq x_2$ and $t \neq y_2$.

Lemma 3.3 *If $x_2 = y_2$ then $G = F_2$.*

Proof. Suppose $x_2 = y_2$. See Figure 12. Recall that we assume $x_1 = y_1$. Since G is 2-connected and x_1 is a 3-vertex, x_2 is a 3-vertex. We proceed to prove that $G = F_2$. Since G is triangle-free, $x_1x_2 \notin E(G)$. Let s be the neighbor of x_2 other than x and y. If $s = t$ then, since G is 2-connected, s must be a 2-vertex in G; and in this case $G - uv$ is bipartite, which implies $b(G) > \frac{4}{5}$, a contradiction. Therefore, $s \neq t$.

First, we assume $st \notin E(G)$. Let $A := A(u, v, x, y, x_1, x_2)$, and let $G' := (G - A) + \{q, sq, qt\}$, where q is a new vertex (not in G). Then $G' \in \mathcal{G}$ and $\varepsilon(G') = \varepsilon(G) - 7$. Let $B' \in \mathcal{B}(G')$. Then by Theorem 1.1, $\varepsilon(B') \geq \frac{4}{5}\varepsilon(G') = \frac{4}{5}(\varepsilon(G) - 7)$. By the maximality of B', at least one of qs and qt is in $E(B')$. So we may assume that $qs \in E(B')$ and $s \in V_1(B')$. Note that $t \in V_2(B')$ if $qt \notin E(B')$ (by maximality of B'), and $t \in V_1(B')$ if $qt \in E(B')$. Define

$$B := \begin{cases} (B' - qs) + (A - \{uv, tx_1\}), & \text{if } qt \notin E(B'); \\ (B' - q) + (A - \{uv\}), & \text{otherwise.} \end{cases}$$

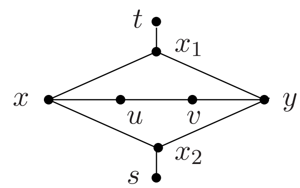

Figure 12: $x_1 = y_1$ and $x_2 = y_2$.

Then B is a bipartite subgraph of G, and $\varepsilon(B) = \varepsilon(B') + 6 \geq \frac{4}{5}(\varepsilon(G) - 7) + 6 > \frac{4}{5}\varepsilon(G)$. So $b(G) > \frac{4}{5}$, a contradiction.

Therefore, $st \in E(G)$. If both s and t are 2-vertices in G, then $G = F_2$. So we may assume one of $\{s, t\}$ is a 3-vertex. Then, since G is 2-connected, both s and t are 3-vertices in G. Let s', t' be the neighbors of s, t, respectively, not contained in $\{x_1, x_2, s, t\}$.

Let $A' := A(u, v, x, y, x_1, x_2, s, t)$. Then $G - A'$ is subcubic and triangle-free, and $\varepsilon(G - A') = \varepsilon(G) - 12$. Since G is 2-connected, $G - A'$ must be connected. So $G - A' \in \mathcal{G}$. Let $B' \in \mathcal{B}(G - A')$. By Theorem 1.1, $\varepsilon(B') \geq \frac{4}{5}\varepsilon(G - A') = \frac{4}{5}(\varepsilon(G) - 12)$. Define

$$B := \begin{cases} B' + (A - \{uv, st\}), & \text{if } \{s', t'\} \subseteq V_i(B') \text{ for some } i \in \{1, 2\}; \\ B' + (A - \{uv, tx_1\}), & \text{otherwise.} \end{cases}$$

Then B is a bipartite subgraph of G, and $\varepsilon(B) = \varepsilon(B') + 10 \geq \frac{4}{5}(\varepsilon(G) - 12) + 10 > \frac{4}{5}\varepsilon(G)$. Hence $b(G) > \frac{4}{5}$, a contradiction. ∎

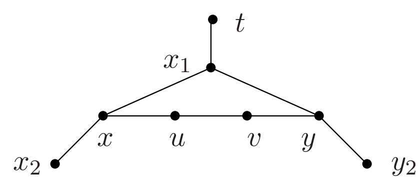

Figure 13: $x_1 = y_1$ and $x_2 \neq y_2$.

Therefore, we may assume $x_2 \neq y_2$. See Figure 13.

Lemma 3.4 $N(x_2) \cap N(y_2) \neq \emptyset$.

Proof. Suppose $N(x_2) \cap N(y_2) = \emptyset$. Let $A := A(u, v, x, y, x_1)$ and $G' := (G - A) + x_2y_2$. Then G' is subcubic and triangle-free, and $\varepsilon(G') = \varepsilon(G) - 7$. Since G is 2-connected, G' is connected. So $G' \in \mathcal{G}$. Let $B' \in \mathcal{B}(G')$. Then by Theorem 1.1, $\varepsilon(B') \geq \frac{4}{5}\varepsilon(G') = \frac{4}{5}(\varepsilon(G) - 7)$. Without loss of generality, we may assume $x_2 \in V_1(B')$. Then $y_2 \in V_2(B')$ if $x_2y_2 \in E(B')$, and $y_2 \in V_1(B')$ if $x_2y_2 \notin E(B')$ (by maximality of B'). Define

$$B := \begin{cases} (B' - x_2y_2) + (A - \{xx_1\}), & \text{if } x_2y_2 \in E(B') \text{ and } t \in V_1(B'); \\ (B' - x_2y_2) + (A - \{yx_1\}), & \text{if } x_2y_2 \in E(B') \text{ and } t \in V_2(B'); \\ B' + (A - \{uv, tx_1\}), & \text{otherwise.} \end{cases}$$

Then B is a bipartite subgraph of G, and $\varepsilon(B) = \varepsilon(B') + 6 \geq \frac{4}{5}(\varepsilon(G) - 7) + 6 > \frac{4}{5}\varepsilon(G)$. This implies $b(G) > \frac{4}{5}$, a contradiction. ∎

Lemma 3.5 $N(x_2) \cap N(t) \neq \emptyset \neq N(y_2) \cap N(t)$.

Proof. Suppose otherwise. By symmetry, we may assume $N(x_2) \cap N(t) = \emptyset$. Let $A := A(u, v, x, y, x_1)$ and $G' := (G - A) + tx_2$. Then G' is subcubic and triangle-free, and $\varepsilon(G') = \varepsilon(G) - 7$. Since G is 2-connected, G' is connected. So $G' \in \mathcal{G}$. Let $B' \in \mathcal{B}(G')$. By Theorem 1.1, $\varepsilon(B') \geq \frac{4}{5}\varepsilon(G') = \frac{4}{5}(\varepsilon(G) - 7)$. Without loss of generality, we may assume $x_2 \in V_1(B')$. Then $t \in V_2(B')$ if $tx_2 \in E(B')$, and $t \in V_1(B')$ if $tx_2 \notin E(B')$ (by maximality of B'). Define

$$B := \begin{cases} (B' - tx_2) + (A - \{uv\}), & \text{if } tx_2 \in E(B') \text{ and } y_2 \in V_1(B'); \\ (B' - tx_2) + (A - \{yx_1\}), & \text{if } tx_2 \in E(B') \text{ and } y_2 \in V_2(B'); \\ B' + (A - \{yy_2, xx_1\}), & \text{if } tx_2 \notin E(B'). \end{cases}$$

Then B is a bipartite subgraph of G, and $\varepsilon(B) = \varepsilon(B') + 6 > \frac{4}{5}\varepsilon(G)$. This implies $b(G) > \frac{4}{5}$, a contradiction. ∎

Lemma 3.6 *No vertex of G is adjacent to all of $\{x_2, y_2, t\}$.*

Proof. Otherwise, let w be a vertex of G such that $N(w) = \{x_2, y_2, t\}$. By Lemma 2.4, both x_2 and y_2 are 3-vertices of G. Let $s_1 \in N(x_2) - \{w, x\}$ and $s_2 \in N(y_2) - \{w, y\}$. See Figure 14.

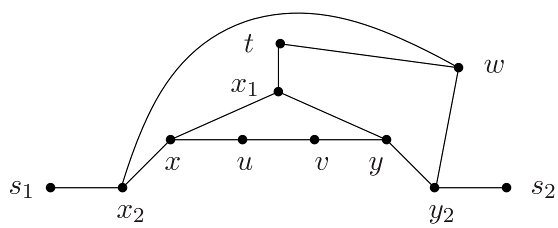

Figure 14: $N(w) = \{x_2, y_2, t\}$.

Let $A := A(u, v, x, y, x_1, x_2, y_2, t, w)$. Then $G - A$ is subcubic and triangle-free, $\varepsilon(G - A) = \varepsilon(G) - 13$ when t is a 2-vertex of G, and $\varepsilon(G - A) = \varepsilon(G) - 14$ when t is a 3-vertex of G. Since G is 2-connected, $G - A$ must be connected. So $G - A \in \mathcal{G}$. Let $B' \in \mathcal{B}(G - A)$. Then by Theorem 1.1, $\varepsilon(B') \geq \frac{4}{5}\varepsilon(G - A)$. Without loss of generality, we may assume $s_1 \in V_1(B')$.

Suppose that t is a 2-vertex of G. Define

$$B := \begin{cases} B' + (A - \{xx_2, x_1y\}), & \text{if } s_2 \in V_1(B'); \\ B' + (A - \{wx_2, x_1y\}), & \text{if } s_2 \in V_2(B'). \end{cases}$$

Then B is a bipartite subgraph of G, and $\varepsilon(B) = \varepsilon(B') + 11 \geq \frac{4}{5}(\varepsilon(G) - 13) + 11 > \frac{4}{5}\varepsilon(G)$. So $b(G) > \frac{4}{5}$, a contradiction.

Hence t is a 3-vertex of G, and let $s_3 \in N(t) - \{w, x_1\}$. Define

$$B := \begin{cases} B' + (A - \{xx_1, yy_2\}), & \text{if } \{s_2, s_3\} \subseteq V_1(B'); \\ B' + (A - \{x_1y, x_2w\}), & \text{if } \{s_2, s_3\} \subseteq V_2(B'); \\ B' + (A - \{wt, uv\}), & \text{if } s_2 \in V_1(B') \text{ and } s_3 \in V_2(B'); \\ B' + (A - \{xx_1, wy_2\}), & \text{if } s_2 \in V_2(B') \text{ and } s_3 \in V_1(B'). \end{cases}$$

Then B is a bipartite subgraph of G, and $\varepsilon(B) = \varepsilon(B') + 12 \geq \frac{4}{5}(\varepsilon(G) - 14) + 12 > \frac{4}{5}\varepsilon(G)$. Again $b(G) > \frac{4}{5}$, a contradiction. ∎

By Lemmas 3.4 and 3.5, let $w_1 \in N(t) \cap N(x_2)$, $w_2 \in N(x_2) \cap N(y_2)$, and $w_3 \in N(y_2) \cap N(t)$. By Lemma 3.6, w_1, w_2, w_3 are pairwise distinct. This, in particular, implies that x_2, y_2, t are 3-vertices of G. If none of $\{w_1, w_2, w_3\}$ is a 3-vertex of G, then $\varepsilon(G) = 14$ and $G - \{xx_1, yy_2\}$ is a bipartite subgraph of G, which implies $b(G) > \frac{4}{5}$, a contradiction. Hence, since G is 2-connected, at least two of $\{w_1, w_2, w_3\}$ are 3-vertices of G.

Let $A := A(u, v, x, y, x_1, x_2, y_2, t, w_1, w_2, w_3)$. Then $G - A$ is subcubic and triangle-free, $\varepsilon(G - A) = \varepsilon(G) - 16$ when one of $\{w_1, w_2, w_3\}$ is a 2-vertex, and $\varepsilon(G - A) = \varepsilon(G) - 17$ when all of $\{w_1, w_2, w_3\}$ are 3-vertices. Since G is 2-connected, $G - A$ is connected. So $G - A \in \mathcal{G}$. Let $B' \in \mathcal{B}(G - A)$. Then by Theorem 1.1, $\varepsilon(B') \geq \frac{4}{5}\varepsilon(G - A)$. For each $i \in \{1, 2, 3\}$, if w_i is a 3-vertex then let s_i be the neighbor of w_i not contained in A.

Suppose exactly one of $\{w_1, w_2, w_3\}$ is a 2-vertex. Then $\varepsilon(G') = \varepsilon(G) - 16$. Define

$$B := \begin{cases} B' + (A - \{xx_1, yy_2, w_2s_2\}), & \text{if } w_1 \text{ or } w_3 \text{ is a 2-vertex;} \\ B' + (A - \{xx_1, yy_2, w_3s_3\}), & \text{if } w_2 \text{ is a 2-vertex.} \end{cases}$$

Then B is a bipartite subgraph of G, and $\varepsilon(B) = \varepsilon(B') + 13 \geq \frac{4}{5}(\varepsilon(G) - 16) + 13 > \frac{4}{5}\varepsilon(G)$. However, this implies $b(G) > \frac{4}{5}$, a contradiction.

Therefore, w_1, w_2, w_3 are all 3-vertices in G. Then, $\varepsilon(G') = \varepsilon(G) - 17$. Without loss of generality, we may assume that $s_1 \in V_1(B')$. Define

$$B := \begin{cases} B' + (A - \{xx_1, yy_2, w_3s_3\}), & \text{if } s_2 \in V_1(B'); \\ B' + (A - \{xx_1, yy_2, w_2s_2\}), & \text{if } s_3 \in V_1(B'); \\ B' + (A - \{xx_1, yy_2, w_1s_1\}), & \text{if } \{s_2, s_3\} \subseteq V_2(B'). \end{cases}$$

Then B is a bipartite subgraph of G, and $\varepsilon(B) = \varepsilon(B') + 14 \geq \frac{4}{5}(\varepsilon(G) - 17) + 14 > \frac{4}{5}\varepsilon(G)$. Again, $b(G) > \frac{4}{5}$, a contradiction. ∎

Summarizing the above lemmas, we have

Lemma 3.7 *If G contains two adjacent 2-vertices, then $G \in \{F_1, F_2\}$.*

4 The graph F_3

In this section, we show that if $G \in \mathcal{G}$, $b(G) = \frac{4}{5}$, and some 3-vertex of G is adjacent to two 2-vertices, then $G = F_3$, or there exists $G' \in \mathcal{G}$ such that $b(G') = \frac{4}{5}$, $G' \notin \{F_i : 1 \leq i \leq 7\}$, and $|V(G')| < |V(G)|$.

Let $G \in \mathcal{G}$ and $b(G) = \frac{4}{5}$. Then G is 2-connected (by Lemma 2.1). Let x be a 3-vertex of G with $N(x) = \{u, v, y\}$, and assume that both u and v are 2-vertices in G. Let u_1, v_1 be the neighbors of u, v, respectively, other than x. Since G is triangle-free, $y \notin \{u_1, v_1\}$. See Figure 15. By Lemma 2.4, u_1, v_1 and y are all 3-vertices in G.

Lemma 4.1 *$u_1 \neq v_1$.*

Proof. Otherwise, $u_1 = v_1$. Let $w \in N(u_1) - \{u, v\}$, and let $A := A(u, v, x, u_1)$. Then $G - A$ is subcubic and triangle-free, and $\varepsilon(G - A) = \varepsilon(G) - 6$. Since G is 2-connected, $G - A$ is connected. So $G - A \in \mathcal{G}$. Let $B' \in \mathcal{B}(G - A)$. Then by Theorem 1.1, $\varepsilon(B') \geq \frac{4}{5}\varepsilon(G - A) = \frac{4}{5}(\varepsilon(G) - 6)$. Define

$$B := \begin{cases} B' + A, & \text{if } \{w, y\} \subseteq V_i(B') \text{ for some } i \in \{1, 2\}; \\ B' + (A - \{xy\}), & \text{otherwise.} \end{cases}$$

Then B is a bipartite subgraph of G, and $\varepsilon(B) \geq \varepsilon(B') + 5 \geq \frac{4}{5}(\varepsilon(G) - 6) + 5 > \frac{4}{5}\varepsilon(G)$. Hence $b(G) > \frac{4}{5}$, a contradiction. ∎

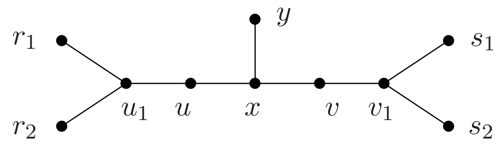

Figure 15: $u_1 \neq v_1$.

Let $N(u_1) - \{u\} = \{r_1, r_2\}$, and $N(v_1) - \{v\} = \{s_1, s_2\}$. See Figure 15.

Lemma 4.2 *$y \notin \{r_1, r_2\}$ and $y \in N(r_1) \cup N(r_2)$, and $y \notin \{s_1, s_2\}$ and $y \in N(s_1) \cup N(s_2)$.*

Proof. Suppose the assertion of the lemma is false. By symmetry, we may assume that $y \in \{r_1, r_2\}$ or $y \notin N(r_1) \cup N(r_2)$.

Let $A := A(u, v, x, v_1)$ and $G' = (G - A) + u_1y$. Then, G' is subcubic and $\varepsilon(G') = \varepsilon(G) - 6$. Since $y \in \{r_1, r_2\}$ or $y \notin N(r_1) \cup N(r_2)$, G' is triangle-free. Since G is 2-connected, G' must be connected. So $G' \in \mathcal{G}$. Let $B' \in \mathcal{B}(G')$. By Theorem 1.1, $\varepsilon(B') \geq \frac{4}{5}\varepsilon(G') = \frac{4}{5}(\varepsilon(G) - 6)$. Without loss of generality, we may assume $u_1 \in V_1(B')$. Then $y \in V_2(B')$ if $u_1y \in E(B')$, and $y \in V_1(B')$ if $u_1y \notin E(B')$ (by maximality of B'). Define

$$B := \begin{cases} (B' - u_1y) + (A - \{vx\}), & \text{if } u_1y \in E(B') \text{ and } \{s_1, s_2\} \subseteq V_i(B') \text{ for some } i \in \{1,2\}; \\ (B' - u_1y) + (A - \{v_1s_1\}), & \text{if } u_1y \in E(B'), s_1 \in V_1(B') \text{ and } s_2 \in V_2(B'); \\ (B' - u_1y) + (A - \{v_1s_2\}), & \text{if } u_1y \in E(B'), s_1 \in V_2(B') \text{ and } s_2 \in V_1(B'); \\ B' + (A - \{ux, vx\}), & \text{if } u_1y \notin E(B') \text{ and } \{s_1, s_2\} \subseteq V_i(B') \text{ for some } i \in \{1,2\}; \\ B' + (A - \{ux, v_1s_2\}), & \text{if } u_1y \notin E(B'), s_1 \in V_1(B') \text{ and } s_2 \in V_2(B'); \\ B' + (A - \{ux, v_1s_1\}), & \text{if } u_1y \notin E(B'), s_1 \in V_2(B') \text{ and } s_2 \in V_1(B'). \end{cases}$$

Now B is a bipartite subgraph of G, and $\varepsilon(B) = \varepsilon(B') + 5 \geq \frac{4}{5}(\varepsilon(G) - 6) + 5 > \frac{4}{5}\varepsilon(G)$. Hence, $b(G) > \frac{4}{5}$, a contradiction. ∎

Therefore, $y \notin \{r_1, r_2, s_1, s_2\}$, and we may assume by symmetry that $y \in N(r_1) \cap N(s_1)$.

Lemma 4.3 $r_1 \neq s_1$.

Proof. Suppose $r_1 = s_1$. Then $N(r_1) = \{u_1, v_1, y\}$. Since G is 2-connected and because y is a 3-vertex in G (by Lemma 2.4), $u_1v_1 \notin E(G)$. See Figure 16. Let $y' \in N(y) - \{r_1, x\}$.

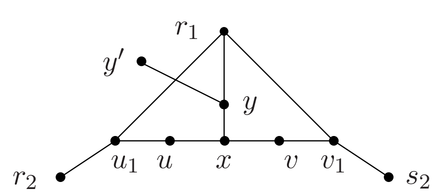

Figure 16: $r_1 = s_1$.

Let $A := A(u, v, x, y, r_1, u_1, v_1)$. Then, $G - A$ is subcubic and triangle-free, and $\varepsilon(G - A) = \varepsilon(G) - 11$. Since G is 2-connected, $G - A$ is connected. So $G - A \in \mathcal{G}$. Let $B' \in \mathcal{B}(G - A)$. By

Theorem 1.1, $\varepsilon(B') \geq \frac{4}{5}\varepsilon(G-A) = \frac{4}{5}(\varepsilon(G)-11)$. Without loss of generality, we may assume that $r_2 \in V_1(B')$. Define

$$B := \begin{cases} B' + (A - \{xy, v_1s_2\}), & \text{if } y' \in V_1(B'); \\ B' + (A - \{r_1y, v_1s_2\}), & \text{if } y' \in V_2(B'). \end{cases}$$

Clearly, B is a bipartite subgraph of G, and $\varepsilon(B) = \varepsilon(B')+9 \geq \frac{4}{5}(\varepsilon(G)-11)+9 > \frac{4}{5}\varepsilon(G)$. Hence, $b(G) > \frac{4}{5}$, a contradiction. ∎

Lemma 4.4 *If $u_1v_1 \in E(G)$ then $G = F_3$.*

Proof. Suppose $u_1v_1 \in E(G)$. See Figure 17. If both r_1 and s_1 are 2-vertices in G, then $G = F_3$. So we may assume that at least one of $\{r_1, s_1\}$ is a 3-vertex in G. Then since G is 2-connected, both r_1 and s_1 are 3-vertices in G. Let $r_1' \in N(r_1) - \{u_1, y\}$ and $s_1' \in N(s_1) - \{v_1, y\}$.

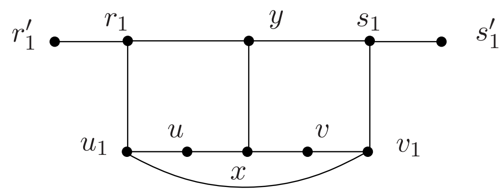

Figure 17: $u_1v_1 \in E(G)$.

Let $A := A(u, v, x, y, u_1, v_1, r_1, s_1)$. Then $G - A$ is subcubic and triangle-free, and $\varepsilon(G-A) = \varepsilon(G) - 12$. Since G is 2-connected, $G - A$ is connected. So $G - A \in \mathcal{G}$. Let $B' \in \mathcal{B}(G')$. By Theorem 1.1, $\varepsilon(B') \geq \frac{4}{5}\varepsilon(G - A) = \frac{4}{5}(\varepsilon(G) - 12)$. Without loss of generality, we may assume $r_1' \in V_1(B')$. Define

$$B := \begin{cases} B' + (A - \{xy, u_1v_1\}), & \text{if } s_1' \in V_1(B'); \\ B' + (A - \{r_1y, xv\}), & \text{if } s_1' \in V_2(B'). \end{cases}$$

Then B is a bipartite subgraph of G, and $\varepsilon(B) = \varepsilon(B') + 10 \geq \frac{4}{5}(\varepsilon(G) - 12) + 10 > \frac{4}{5}\varepsilon(G)$. So $b(G) > \frac{4}{5}$, a contradiction. ∎

Therefore, we may assume $u_1v_1 \notin E(G)$.

Lemma 4.5 *$r_1 \neq s_2$ and $r_2 \neq s_1$.*

Proof. Otherwise, we may assume by symmetry that $r_2 = s_1$, which must be a 3-vertex in G. See Figure 18. Then, since G is 2-connected, r_1 is a 3-vertex in G. If $r_1 = s_2$ then $G-xy$ is a bipartite subgraph of G, which implies $b(G) > \frac{4}{5}$, a contradiction. So $r_1 \neq s_2$. Let $r_1' \in N(r_1) - \{u_1, y\}$.

Let $A := A(u, v, x, y, u_1, v_1, r_1, s_1)$. Then $G - A$ is subcubic and triangle-free, and $\varepsilon(G - A) = \varepsilon(G) - 12$. Since G is 2-connected, $G - A$ is connected. So $G - A \in \mathcal{G}$.

Let $B' \in \mathcal{B}(G - A)$. By Theorem 1.1, $\varepsilon(B') \geq \frac{4}{5}\varepsilon(G - A) = \frac{4}{5}(\varepsilon(G) - 12)$. Define

$$B := \begin{cases} B' + (A - \{xy, v_1s_2\}), & \text{if } \{r_1', s_2\} \subseteq V_i(B') \text{ for some } i \in \{1, 2\}; \\ B' + (A - \{xy\}), & \text{otherwise.} \end{cases}$$

Clearly, B is a bipartite subgraph of G, and $\varepsilon(B) \geq \varepsilon(B') + 10 \geq \frac{4}{5}(\varepsilon(G) - 12) + 10 > \frac{4}{5}\varepsilon(G)$. Hence $b(G) > \frac{4}{5}$, a contradiction. ∎

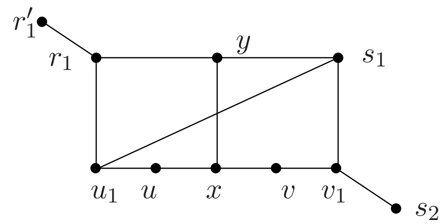

Figure 18: $r_2 = s_1$

Lemma 4.6 *At least one of $\{r_1, s_1\}$ is a 3-vertex in G.*

Proof. Suppose both r_1 and s_1 are 2-vertices of G. Let $A := A(u, v, x, y, u_1, v_1, r_1, s_1)$. Note that $G - A$ is subcubic and triangle-free, and $\varepsilon(G') = \varepsilon(G) - 11$. Since G is 2-connected, $G - A$ is connected. Hence $G - A \in \mathcal{G}$. Let $B' \in \mathcal{B}(G - A)$. By Theorem 1.1, $\varepsilon(B') \geq \frac{4}{5}\varepsilon(G - A) = \frac{4}{5}(\varepsilon(G) - 11)$. Define $B := B' + (A - \{xy, v_1s_2\})$. Then B is a bipartite subgraph of G, and $\varepsilon(B) \geq \varepsilon(B') + 9 \geq \frac{4}{5}(\varepsilon(G) - 11) + 9 > \frac{4}{5}\varepsilon(G)$. So $b(G) > \frac{4}{5}$, a contradiction. ∎

By symmetry, we may assume that r_1 is a 3-vertex of G. Since r_2 is adjacent to neither y nor v, $N(r_2) \cap N(x) = \emptyset$. So we derive from Lemma 2.5 (with u, u_1, x, r_1, r_2 playing the roles of y, x, z, w, t, respectively) that there exists $G' \in \mathcal{G}$ such that $b(G') = \frac{4}{5}$, $G' \notin \{F_i : 1 \leq i \leq 7\}$, and $|V(G')| < |V(G)|$. Summarizing the lemmas above, we have the following.

Lemma 4.7 *Let $G \in \mathcal{G}$ and $b(G) = \frac{4}{5}$, and assume that there is a 3-vertex in G that is adjacent to two 2-vertices of G. Then one of the following holds:*

(i) there exists $G' \in \mathcal{G}$ such that $b(G') = \frac{4}{5}$, $G' \notin \{F_i : 1 \leq i \leq 7\}$, and $|V(G')| < |V(G)|$; or

(ii) $G = F_3$.

5 The graphs F_4 and F_5

In this section we show that if $G \in \mathcal{G}$, $b(G) = \frac{4}{5}$, G contains a 2-vertex, and no two 2-vertices of G are adjacent or share a common neighbor, then $G \in \{F_4, F_5\}$, or there exists $G' \in \mathcal{G}$ such that $b(G') = \frac{4}{5}$, $G' \notin \{F_i : 1 \leq i \leq 7\}$, and $|V(G')| < |V(G)|$.

Let $G \in \mathcal{G}$ and $b(G) = \frac{4}{5}$. By Lemma 2.1, G is 2-connected. Let $x \in V(G)$ be a 2-vertex and let $N(x) = \{u, v\}$. Assume that both u and v are 3-vertices in G. Let $N(u) = \{x, u_1, u_2\}$ and $N(v) = \{x, v_1, v_2\}$. Moreover, assume u_1, u_2, v_1, v_2 are all 3-vetices in G. Then $G \notin \{F_1, F_2, F_3\}$.

We further assume that

($*$) there is no $G' \in \mathcal{G}$ such that $b(G') = \frac{4}{5}$, $G' \notin \{F_i : 1 \leq i \leq 7\}$, and $|V(G')| < |V(G)|$.

Lemma 5.1 *$\{u_1, u_2\} \cap \{v_1, v_2\} = \emptyset$, and $\{u_1v_1, u_2v_2\} \subseteq E(G)$ or $\{u_1v_2, u_2v_1\} \subseteq E(G)$.*

Proof. Suppose $\{u_1, u_2\} \cap \{v_1, v_2\} \neq \emptyset$. By symmetry we may assume that $u_1 = v_1$. See Figure 19(a). Since no two 2-vertices of G share a common neighbor, u_1 is a 3-vertex. Let $s \in N(u_1) - \{u, v\}$, and let $A := A(u, v, x, u_1)$. Then $G - A$ is subcubic and triangle-free, and $\varepsilon(G - A) = \varepsilon(G) - 7$. Since G is 2-connected, $G - A$ is connected. So $G - A \in \mathcal{G}$. Let $B' \in \mathcal{B}(G - A)$.

By Theorem 1.1, $\varepsilon(B') \geq \frac{4}{5}\varepsilon(G - A) = \frac{4}{5}(\varepsilon(G) - 7)$. Without loss of generality, we may assume $s \in V_1(B')$. Define

$$B := \begin{cases} B' + (A - \{su_1\}), & \text{if } \{u_2, v_2\} \subseteq V_i(B') \text{ for some } i \in \{1, 2\}; \\ B' + (A - \{uu_2\}), & \text{if } u_2 \in V_1(B') \text{ and } v_2 \in V_2(B'); \\ B' + (A - \{vv_2\}), & \text{if } u_2 \in V_2(B') \text{ and } v_2 \in V_1(B'). \end{cases}$$

Then B is a bipartite subgraph of G, and $\varepsilon(B) \geq \varepsilon(B') + 6 \geq \frac{4}{5}(\varepsilon(G) - 7) + 6 > \frac{4}{5}\varepsilon(G)$. This shows $b(G) > \frac{4}{5}$, a contradiction.

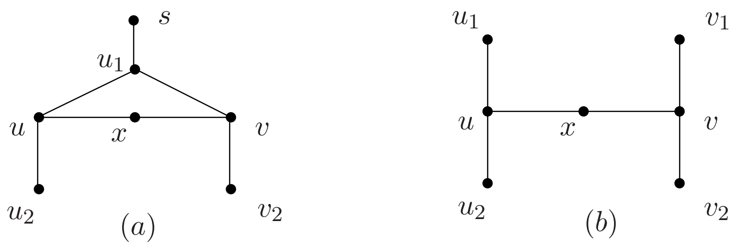

Figure 19: x and its neighbors.

So $\{u_1, u_2\} \cap \{v_1, v_2\} = \emptyset$. See Figure 19(b). Then $u_2v \notin E(G)$. Suppose $u_1v_1, u_1v_2 \notin E(G)$. Then $N(u_1) \cap N(v) = \emptyset$. Hence by Lemma 2.5 (with u_1, u_2, u, x, v as t, w, x, y, z, respectively), we derive a contradiction to $(*)$. So $u_1v_1 \in E(G)$ or $u_1v_2 \in E(G)$. Similarly, we can show $u_2v_1 \in E(G)$ or $u_2v_2 \in E(G)$; $v_1u_1 \in E(G)$ or $v_1u_2 \in E(G)$; and $v_2u_1 \in E(G)$ or $v_2u_2 \in E(G)$. Therefore, $u_1v_1, u_2v_2 \in E(G)$, or $u_1v_2, u_2v_1 \in E(G)$. ∎

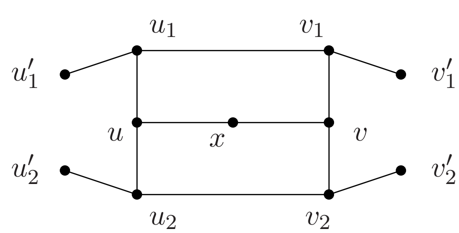

Figure 20: A 2-vertex in two 5-cycles.

We now assume that $\{u_1v_1, u_2v_2\} \subseteq E(G)$; for when $\{u_1v_2, u_2v_1\} \subseteq E(G)$, we simply exchange the notation of v_1 and v_2. Let $u'_1 \in N(u_1) - \{u, v_1\}$, $v'_1 \in N(v_1) - \{v, u_1\}$, $u'_2 \in N(u_2) - \{u, v_2\}$, and $v'_2 \in N(v_2) - \{v, u_2\}$. See Figure 20.

Lemma 5.2 *$u_1v_2, u_2v_1 \notin E(G)$.*

Proof. If $\{u_1v_2, u_2v_1\} \subseteq E(G)$, then $\varepsilon(G) = 10$ and $G - ux$ is a bipartite subgraph of G with 9 edges, which implies $b(G) > \frac{4}{5}$, a contradiction. So $u_1v_2 \notin E(G)$ or $v_1u_2 \notin E(G)$. By symmetry, we may assume $u_2v_1 \notin E(G)$. If $u_1v_2 \notin E(G)$, then the assertion of the lemma holds. So we may assume $u_1v_2 \in E(G)$.

Let $A := A(u, u_1, u_2, v, v_1, v_2, x)$. Then $G - A$ is subcubic and triangle-free, and $\varepsilon(G - A) = \varepsilon(G) - 11$. Since G is 2-connected, $G - A$ is connected. So $G - A \in \mathcal{G}$. Let $B' \in \mathcal{B}(G - A)$. Then by Theorem 1.1, $\varepsilon(B') \geq \frac{4}{5}\varepsilon(G-A) = \frac{4}{5}(\varepsilon(G)-11)$. Let $B := B' + (A - \{xv, v_1v_1'\})$. Then B is a bipartite subgraph of G and $\varepsilon(B) = \varepsilon(B') + 9 \geq \frac{4}{5}(\varepsilon(G)-11) + 9 > \frac{4}{5}\varepsilon(G)$. This, however, implies $b(G) > \frac{4}{5}$, a contradiction. ∎

Lemma 5.3 *$u_1' \neq u_2'$ and $u_1'u_2' \in E(G)$, and $v_1' \neq v_2'$ and $v_1'v_2' \in E(G)$.*

Proof. Otherwise, we may assume by symmetry that $u_1' = u_2'$ or $u_1'u_2' \notin E(G)$. Let $A := A(u, v, u_2, v_2, v_1, x)$, and let $G' := (G - A) + u_1u_2'$. Then G' is subcubic and $\varepsilon(G') = \varepsilon(G) - 10$. Since $u_1' = u_2'$ or $u_1'u_2' \notin E(G)$, G' is triangle-free. Note that G' need not be connected; but each component of G' is in $\mathcal{G}$.

Choose an arbitrary B' from $\mathcal{B}(G')$. By applying Theorem 1.1 to each component of G', $\varepsilon(B') \geq \frac{4}{5}\varepsilon(G') = \frac{4}{5}(\varepsilon(G)-10)$. Note that $u_1 \in V_i(B')$ for some $i \in \{1,2\}$. Then $u_2' \in V_{3-i}(B')$ if $u_1u_2' \in E(B')$, and $u_2' \in V_i(B')$ if $u_1u_2' \notin E(B')$ (by maximality of B'). Define

$$B := \begin{cases} (B' - u_1u_2') + (A - \{ux\}), & \text{if } u_1u_2' \in E(B') \text{ and } \{v_1', v_2'\} \subseteq V_i(B'); \\ (B' - u_1u_2') + (A - \{u_1v_1, u_2v_2\}), & \text{if } u_1u_2' \in E(B') \text{ and } \{v_1', v_2'\} \subseteq V_{3-i}(B'); \\ (B' - u_1u_2') + (A - \{u_1v_1, vv_2\}), & \text{if } u_1u_2' \in E(B'), v_1' \in V_{3-i}(B') \text{ and } v_2' \in V_i(B'); \\ (B' - u_1u_2') + (A - \{u_2v_2, vv_1\}), & \text{if } u_1u_2' \in E(B'), v_1' \in V_i(B') \text{ and } v_2' \in V_{3-i}(B'); \\ B' + (A - \{u_2u_2', ux\}), & \text{if } u_1u_2' \notin E(B') \text{ and } \{v_1', v_2'\} \subseteq V_i(B'); \\ B' + (A - \{u_2u_2', u_1v_1, u_2v_2\}), & \text{if } u_1u_2' \notin E(B') \text{ and } \{v_1', v_2'\} \subseteq V_{3-i}(B'); \\ B' + (A - \{u_2u_2', u_1v_1, vv_2\}), & \text{if } u_1u_2' \notin E(B'), v_1' \in V_{3-i}(B') \text{ and } v_2' \in V_i(B'); \\ B' + (A - \{u_2u_2', u_2v_2, vv_1\}), & \text{if } u_1u_2' \notin E(B'), v_1' \in V_i(B') \text{ and } v_2' \in V_{3-i}(B'). \end{cases}$$

Then, B is a bipartite subgraph of G. Moreover, $\varepsilon(B) = \varepsilon(B') + 9$ when $\{u_1, v_1', v_2'\} \subseteq V_i(B')$, and $\varepsilon(B) = \varepsilon(B') + 8$ otherwise.

We claim that $b(G') = \frac{4}{5}$ and, for each $B' \in \mathcal{B}(G')$ and for any $i \in \{1,2\}$, $\{u_1, v_1', v_2'\} \not\subseteq V_i(B')$. Suppose $b(G') > \frac{4}{5}$. Then $\varepsilon(B') > \frac{4}{5}\varepsilon(G') = \frac{4}{5}(\varepsilon(G)-10)$. Hence $\varepsilon(B) \geq \varepsilon(B') + 8 > \frac{4}{5}(\varepsilon(G) - 10) + 8 = \frac{4}{5}\varepsilon(G)$, which implies $b(G) > \frac{4}{5}$, a contradiction. Now suppose $\{u_1, v_1', v_2'\} \subseteq V_i(B')$ for some $i \in \{1,2\}$. Then $\varepsilon(B) = \varepsilon(B') + 9$. So $\varepsilon(B) = \varepsilon(B') + 9 \geq \frac{4}{5}(\varepsilon(G)-10) + 9 > \frac{4}{5}\varepsilon(G)$. Again, $b(G) > \frac{4}{5}$, a contradiction.

We further claim that G' is connected. For otherwise, since G is 2-connected, $\{u_1, u_2'\}$ is in a component of G', say G_1; and $\{v_1', v_2'\}$ is contained in the other component of G', say G_2. Note that $G_1, G_2 \in \mathcal{G}$. So $b(G_i) = \frac{4}{5}$ for $i = 1, 2$ (by Theorem 1.1 and since $b(G') = \frac{4}{5}$). Let $B_1 \in \mathcal{B}(G_1)$, and assume $u_1 \in V_1(B_1)$. Since v_1' and v_2' are not 3-vertices in G_2, G_2 is not cubic, and hence $G_2 \notin \{F_6, F_7\}$. So by $(*)$, $G_2 \in \{F_1, F_2, F_3, F_4, F_5\}$. Then, since v_1', v_2' are not 3-vertices in G_2, it is easy to check that there exists $B_2 \in \mathcal{B}(G_2)$ such that $\{v_1', v_2'\} \subseteq V_1(B_2)$. Therefore, $B' := B_1 \cup B_2 \in \mathcal{B}(G')$ such that $\{u_1, v_1', v_2'\} \subseteq V_1(B')$. But this contradicts the previous claim.

Therefore, $G' \in \mathcal{G}$. Since $b(G') = \frac{4}{5}$, G' must be 2-connected (by Lemma 2.1). Hence $v_1' \neq v_2'$. Since $u_1 \neq v_2'$ (by Lemma 5.2), u_1, v_1' and v_2' are pairwise distinct, and so, are all 2-vertices in G'. Therefore, $G' \neq F_5$ (which has only two 2-vertices) and $G' \notin \{F_6, F_7\}$ (which are cubic). Again by $(*)$, $G' \in \{F_1, F_2, F_3, F_4\}$. Note that since G is triangle-free, $u_1v_1' \notin E(G)$. Hence, $u_1v_1' \notin E(G')$.

Case 1. $G' = F_1$.

Then we may label the vertices of G' so that $G' = x_1x_2x_3x_4x_5x_1$. Without loss of generality, we may assume $u_1 = x_1$ and $u_2' \in x_2$. Note that $u_1u_2' \notin E(G)$; otherwise, G' would have multiple edges.

Suppose $\{v_1', v_2'\} = \{x_4, x_5\}$. Then x_3 is a 2-vertex in G (by definition of G'). Since $u_1u_2' \notin E(G)$, $x_2 = u_2'$ is a 2-vertex in G. Hence, x_2, x_3 are two adjacent 2-vertices in G. By Lemma 3.7, $G \in \{F_1, F_2\}$, a contradiction (since $G \notin \{F_1, F_2, F_3\}$).

So $\{v_1', v_2'\} \neq \{x_4, x_5\}$. Then $\{v_1', v_2'\} = \{x_3, x_5\}$ or $\{v_1', v_2'\} = \{x_3, x_4\}$. Define

$$B' := \begin{cases} G' - x_3x_4, & \text{if } \{v_1', v_2'\} = \{x_3, x_4\}; \\ G' - x_1x_5, & \text{if } \{v_1', v_2'\} = \{x_3, x_5\}. \end{cases}$$

Then $B' \in \mathcal{B}(G')$, and $\{u_1, v_1', v_2'\} \subseteq V_i(B')$ for some $i \in \{1, 2\}$, a contradiction.

Case 2. $G' = F_2$.

See Figure 7, where the vertices of G' are labeled as $x_1, \ldots, x_8$. By symmetry, let $v_1' = x_1$.

First, suppose $v_1'v_2' \in E(G')$. Then $v_2' = x_2$, and $u_1 \in \{x_3, x_4\}$. By symmetry, we may assume $u_1 = x_3$. Define $B' := G' - \{x_1x_2, x_3x_6\}$. Then $B' \in \mathcal{B}(G')$, and $\{u_1, v_1', v_2'\} \subseteq V_i(B')$ for some $i \in \{1, 2\}$, a contradiction.

Now assume $v_1'v_2' \notin E(G')$. Then we may assume by symmetry that $v_2' = x_3$. Since $u_1v_1' \notin E(G')$, $u_1 = x_4$. In this case, $B' := G' - \{x_1x_7, x_3x_4\} \in \mathcal{B}(G')$, and $\{u_1, v_1', v_2'\} \subseteq V_i(B')$ for some $i \in \{1, 2\}$, a contradiction.

Case 3. $G' = F_3$.

See Figure 8, where the vertices of G' are labeled as $x_1, \ldots, x_8$. By symmetry, we may assume $\{u_1, v_1', v_2'\} = \{x_1, x_2, x_3\}$. Then $B' := G' - \{x_5x_7, x_6x_8\} \in \mathcal{B}(G')$, and $\{u_1, v_1', v_2'\} \subseteq V_i(B')$ for some $i \in \{1, 2\}$, a contradiction.

Case 4. $G' = F_4$.

See Figure 9, where the vertices of G' are labeled as $x_1, \ldots, x_{11}$. Clearly, $\{u_1, v_1', v_2'\} = \{x_1, x_8, x_{10}\}$. Then $B' := G' - \{x_1x_5, x_7x_8, x_{10}x_{11}\} \in \mathcal{B}(G')$, and $\{u_1, v_1', v_2'\} \subseteq V_i(B')$ for some $i \in \{1, 2\}$, a contradiction. ∎

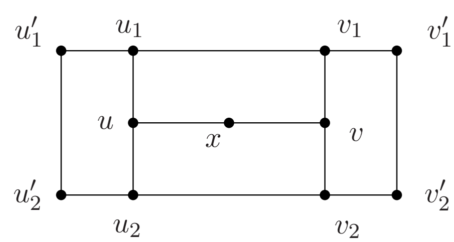

Figure 21: A common subgraph of F_4 and F_5.

Therefore, G must contain the configuration shown in Figure 21, where all vertices are distinct.

Lemma 5.4 *$N(u_1') \cap N(v_1') \neq \emptyset$ and $N(u_2') \cap N(v_2') \neq \emptyset$.*

Proof. Suppose the assertion of the lemma is false. Let us assume by symmetry that $N(u_1') \cap N(v_1') = \emptyset$. Let $A := A(u, v, u_1, v_1, u_2, v_2, x)$ and $G' := (G - A) + u_1'v_1'$. Then G' is subcubic and $\varepsilon(G') = \varepsilon(G) - 11$. Since $N(u_1') \cap N(v_1') = \emptyset$, G' is triangle-free. Since G is 2-connected, G' is connected. So $G' \in \mathcal{G}$. Let $B' \in \mathcal{B}(G')$. By Theorem 1.1, $\varepsilon(B') \geq \frac{4}{5}\varepsilon(G') = \frac{4}{5}(\varepsilon(G) - 11)$. Without loss of generality, we may assume that $u_1' \in V_1(B')$. Then $v_1' \in V_2(B')$ if $u_1'v_1' \in E(B')$, and $v_1' \in V_1(B')$ if $u_1'v_1' \notin E(B')$ (by maximality of B'). Define

$$B := \begin{cases} (B' - u_1'v_1') + (A - \{u_2v_2, vv_1\}), & \text{if } u_1'v_1' \in E(B') \text{ and } \{u_2', v_2'\} \subseteq V_1(B'); \\ (B' - u_1'v_1') + (A - \{u_2v_2, uu_1\}), & \text{if } u_1'v_1' \in E(B') \text{ and } \{u_2', v_2'\} \subseteq V_2(B'); \\ (B' - u_1'v_1') + (A - \{ux\}), & \text{if } u_1'v_1' \in E(B'), u_2' \in V_1(B') \text{ and } v_2' \in V_2(B'); \\ (B' - u_1'v_1') + (A - \{uu_1, vv_2\}), & \text{if } u_1'v_1' \in E(B'), u_2' \in V_2(B') \text{ and } v_2' \in V_1(B'); \\ B' + (A - \{u_2v_2, u_1v_1\}), & \text{if } u_1'v_1' \notin E(B') \text{ and } \{u_2', v_2'\} \subseteq V_1(B'); \\ B' + (A - \{uu_1, u_2v_2, v_1v_1'\}), & \text{if } u_1'v_1' \notin E(B') \text{ and } \{u_2', v_2'\} \subseteq V_2(B'); \\ B' + (A - \{ux, v_1v_1'\}), & \text{if } u_1'v_1' \notin E(B'), u_2' \in V_1(B') \text{ and } v_2' \in V_2(B'); \\ B' + (A - \{ux, u_1u_1'\}), & \text{if } u_1'v_1' \notin E(B'), u_2' \in V_2(B') \text{ and } v_2' \in V_1(B'). \end{cases}$$

Then, B is a bipartite subgraph of G, and $\varepsilon(B) \geq \varepsilon(B') + 9 \geq \frac{4}{5}(\varepsilon(G) - 11) + 9 > \frac{4}{5}\varepsilon(G)$. So $b(G) > \frac{4}{5}$, a contradiction. ∎

Therefore, let $w_1 \in N(u_1') \cap N(v_1')$ and $w_2 \in N(u_2') \cap N(v_2')$.

Lemma 5.5 *If $w_1 \in \{u_2', v_2'\}$, then $G = F_4$.*

Proof. Suppose $w_1 \in \{u_2', v_2'\}$. By symmetry, we assume that $w_1 = u_2'$. In this case, $u_2'v_1' \in E(G)$, and so, $u_1'v_2' \notin E(G)$; for otherwise, $\varepsilon(G) = 16$ and $G - \{u_1u_1', xv, v_2v_2'\}$ is bipartite, which implies $b(G) > \frac{4}{5}$, a contradiction. Hence $w_2 = v_1'$.

If u_1', v_2' are 2-vertices in G, then $G = F_4$. So we may assume that at least one of u_1', v_2' is a 3-vertex in G. Since G is 2-connected, both u_1' and v_2' are 3-vertices in G. Let $u_1'' \in N(u_1') - \{u_1, u_2'\}$, and $v_2'' \in N(v_2') - \{v_2, v_1'\}$.

Let $A := A(u, u_1, u_2, u_1', u_2', v, v_1, v_2, v_1', v_2', x)$. Then $G - A$ is subcubic and triangle-free, and $\varepsilon(G - A) = \varepsilon(G) - 17$. Since G is 2-connected, $G - A$ is connected. So $G - A \in \mathcal{G}$. Let $B' \in \mathcal{B}(G - A)$. By Theorem 1.1, $\varepsilon(B') \geq \frac{4}{5}\varepsilon(G - A) = \frac{4}{5}(\varepsilon(G) - 17)$. Without loss of generality, we assume that $u_1'' \in V_1(B')$. Define

$$B := \begin{cases} B' + (A - \{uu_2, vv_1, u_2'v_1'\}), & \text{if } v_2'' \in V_1(B'); \\ B' + (A - \{u_1u_1', u_2v_2, vv_1\}), & \text{if } v_2'' \in V_2(B'). \end{cases}$$

Then, B is a bipartite subgraph of G, and $\varepsilon(B) = \varepsilon(B') + 14 \geq \frac{4}{5}(\varepsilon(G) - 17) + 14 > \frac{4}{5}\varepsilon(G)$. However, this implies $b(G) > \frac{4}{5}$, a contradiction. ∎

Lemma 5.6 *If $w_1 \notin \{u_2', v_2'\}$, $G = F_5$.*

Proof. Suppose $w_1 \notin \{u_2', v_2'\}$. Then $w_2 \notin \{u_1', v_1'\}$. If both w_1 and w_2 are 2-vertices in G, then $\varepsilon(G) = 18$ and $G - \{u_1u_1', xv, v_2v_2'\}$ is bipartite, which shows $b(G) > \frac{4}{5}$, a contradiction. So at least one of $\{w_1, w_2\}$ is a 3-vertex in G. Then, since G is 2-connected, both w_1 and w_2 are 3-vertices in G. Let $w_1' \in N(w_1) - \{u_1', v_1'\}$ and $w_2' \in N(w_2) - \{u_2', v_2'\}$. If $w_1' = w_2'$, then $G = F_5$ (since G is 2-connected). So we may assume $w_1' \neq w_2'$.

Let $A := A(u, v, u_1, u_2, v_1, v_2, u_1', v_1', u_2', v_2', w_1, w_2, x)$. Then, $G - A$ is subcubic and triangle-free, and $\varepsilon(G - A) = \varepsilon(G) - 20$. Since G is 2-connected, $G - A$ is connected. So $G - A \in \mathcal{G}$. Let $B' \in \mathcal{B}(G - A)$. By Theorem 1.1, $\varepsilon(B') \geq \frac{4}{5}\varepsilon(G - A) = \frac{4}{5}(\varepsilon(G) - 20)$. Without loss of generality, we assume that that $w_1' \in V_1(B')$.

Suppose $\varepsilon(B') > \frac{4}{5}\varepsilon(G - A)$. Then $B := B' + (A - \{u_1u_1', xv, v_2v_2', w_2w_2'\})$ is a bipartite subgraph of G, and $\varepsilon(B) = \varepsilon(B') + 16 > \frac{4}{5}(\varepsilon(G) - 20) + 16 = \frac{4}{5}\varepsilon(G)$. This implies $b(G) > \frac{4}{5}$, a contradiction.

So $\varepsilon(B') = \frac{4}{5}\varepsilon(G - A)$. Since w'_1, w'_2 cannot be 3-vertices in $G - A$, it follows from $(*)$ that $G - A \in \{F_i : 1 \leq i \leq 5\}$. This implies that w'_1, w'_2 are 2-vertices in $G - A$. Therefore, it is easy to check that there exists $B'' \in \mathcal{B}(G - A)$ such that $\{w'_1, w'_2\} \not\subseteq V_i(B'')$ for any $i \in \{1, 2\}$. Then, $B := B'' + (A - \{u_1u'_1, xv, v_2v'_2\})$ is a bipartite subgraph of G, and $\varepsilon(B) = \varepsilon(B'') + 17 \geq \frac{4}{5}(\varepsilon(G) - 20) + 17 > \frac{4}{5}\varepsilon(G)$. This shows $b(G) > \frac{4}{5}$, a contradiction. ∎

Summarizing the above lemmas, we have

Lemma 5.7 *Let $G \in \mathcal{G}$ with $b(G) = \frac{4}{5}$. Suppose G contains a 2-vertex, but no two 2-vertices of G are adjacent or share a common neighbor. Then one of the following holds:*

(i) there exists $G' \in \mathcal{G}$ such that $b(G') = \frac{4}{5}$, $G' \notin \{F_i : 1 \leq i \leq 7\}$, and $|V(G')| < |V(G)|$; or

(ii) $G \in \{F_4, F_5\}$.

6 Completing the proof of Theorem 1.2

We complete the proof of Theorem 1.2. Suppose the assertion of Theorem 1.2 is false. Let $G \in \mathcal{G}$ and $b(G) = \frac{4}{5}$ such that

(1) $G \notin \{F_i : 1 \leq i \leq 7\}$, and

(2) subject to (1), $|V(G)|$ is minimum.

If G contains no 2-vertex, then by Theorem 1.1, $G \in \{F_6, F_7\}$, contradicting (1). So G contains a 2-vertex.

Suppose the maximum degree of G is 2. Then by Lemma 2.2, $G = F_1$, contradicting (1). So G must also have a 3-vertex.

If G contains a 2-vertex whose neighbors are all 2-vertices, then by Lemma 2.3, $G = F_1$, contradicting (1). If G contains two adjacent 2-vertices, then by Lemma 3.7, $G \in \{F_1, F_2\}$, contradicting (1) again. If G contains two 2-vertices which share a common neighbor, then by Lemma 4.7, we derive a contradiction to (1) or (2). Therefore, no two 2-vertices of G are adjacent or share a common neighbor. Now by Lemma 5.7, we derive a contradiction to (1) or (2). ∎

We conclude this paper with the following problem suggested by an anonymous referee: For any fixed integer $k > 0$, is there an integer $f(k)$ such that there are at most $f(k)$ triangle-free subcubic (or cubic) graphs G containing a bipartite subgraph with exactly $\frac{4}{5}\varepsilon(G) + k$ edges? If the answer is affirmative, what is the smallest $f(k)$?

References

[1] P. Berman and M. Karpinski, On some tighter inapproximability results, (extended abstract) *Lecture Notes in Computer Science*, Springer, Berlin, Heidelberg, **1644** (1999) 200–209.

[2] B. Bollobás and A. D. Scott, Problems and results on judicious partitions, *Random Structure and Algorithm*, **21**(2002)414-430.

[3] J. A. Bondy and S. C. Locke, Largest bipartite subgraphs in triangle-free graphs with maximum degree three, *J. Graph Theory* **10** (1986) 477-504.

[4] P. Erdös, Problems and results in graph theory and combinatorial analysis, *Graph Theory and Related Topics* (Proceedings of Waterloo Conference, dedicated to W. T. Tutte on his sixties birthday, Waterloo, 1977), Academic, New York (1979) 153–163.

[5] S. Even and Y. Shiloach, NP-completeness of several arrangement problems, Technical Report 43, Department of Computer Science, Technion, Haifa, Israel (1975).

[6] U. Feige, M. Karpinski, and M. Langberg, Improved approximation of max-cut on graphs of bounded degree, *J. Algorithms* **43** (2002) 201–219.

[7] M. R. Garey, D. S. Johnson, and L. J. Stockmeyer, Some simplified NP-complete graph problems, *Theor. Comput. Sci* **1** (1976) 237–267.

[8] M. X. Goemans and D. P. Williamson, Improved approximation algorithms for maximum cut and satisfiability using semidefinite programming, *J. ACM* **42** (1995) 1115–1145.

[9] F. Hadlock, Finding a maximum cut of a planar graph in polynomial time, *SIAM J. Computing* **4** (1975) 221–225.

[10] G. Hopkins and W. Staton, Extremal bipartite subgraphs of cubic triangle-free graphs, *J. Graph Theory* **6** (1982) 115–121.

[11] R. M. Karp, Reducibility among combinatorial problems, in *Complexity of Computer Computations* (R. Miller and J. Thatcher, eds.) Plenum Press, New York (1972) 85–103.

[12] S. C. Locke, Maximum k-colourable subgraphs, *J. Graph Theory* **6** (1982) 123–132.

[13] G. I. Orlova and Y. G. Dorfman, Finding the maximal cut in a graph, *Eng. Cyber. 10* (1972) 502–506.

[14] W. Stanton, Edge deletions and the chromatic number, *Ars Combinatoria* **10** (1980) 103–106.

[15] M. Yannakakis, Node- and Edge-Deletion NP-Complete Problems, *STOC* (1978)253-264