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Yet another generalization of Postnikov's hook length formula for binary trees

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Abstract. — We discover another one-parameter generalization of Postnikov's hook length formula for binary trees. The particularity of our formula is that the hook length h_v appears as an exponent. As an application, another simple hook length formula for binary trees is derived when the underlying parameter takes the value 1/2.

1. Introduction

Consider the set $\mathcal{B}(n)$ of all binary trees with n vertices. For each vertex v of $T \in \mathcal{B}(n)$, the *hook length* of v, denoted by h_v, or just h for short, is the number of descendants of v (including v). The *hook length multi-set* of T, denoted by $\mathcal{H}(T)$, is the multi-set of all hook lengths of T. The following hook length formula for binary trees

$$\sum_{T\in\mathcal{B}(n)} \prod_{h\in\mathcal{H}(T)} \bigl(1+\frac{1}{h}\bigr) = \frac{2^n}{n!}(n+1)^{n-1} \tag{1}$$

was discovered by Postnikov [Po04]. Further combinatorial proofs and extensions have been proposed by several authors [CY08, GS06, MY07, Se08]. In particular, Lascoux conjectured the following one-parameter generalization

$$\sum_{T\in\mathcal{B}(n)} \prod_{h\in\mathcal{H}(T)} \bigl(x+\frac{1}{h}\bigr) = \frac{1}{(n+1)!}\prod_{k=0}^{n-1}((n+1+k)x+n+1-k), \tag{2}$$

which was subsequently proved by Du-Liu [DL08]. The latter generalization appears to be very natural, because the *left-hand side* of (2) can be obtained from the left-hand side of (1) by replacing 1 by x.

It is also natural to look for an extension of (1) by introducing a new variable z in the *right-hand side*, namely by replacing $2^n(n+1)^{n-1}/n!$ by $2^n z(n+z)^{n-1}/n!$. It so happens that the corresponding left-hand side is also a sum on binary trees, but this time the hook length h_v appears as an exponent. The purpose of this note is to prove the following theorem.

Theorem 1. *For each positive integer n we have*

$$\sum_{T\in\mathcal{B}(n)}\prod_{h\in\mathcal{H}(T)}\frac{(z+h)^{h-1}}{h(2z+h-1)^{h-2}}=\frac{2^n z}{n!}(n+z)^{n-1}. \tag{3}$$

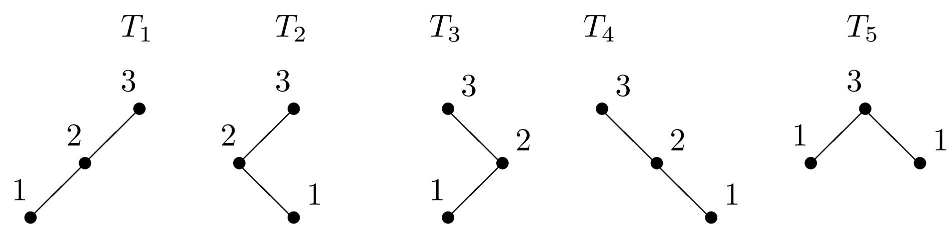
With $z=1$ in (3) we recover Postnikov's identity (1). The following corollary is derived from our identity (3) by taking $z=1/2$.

Corollary 2. *For each positive integer n we have*

$$\sum_{T\in\mathcal{B}(n)}\prod_{h\in\mathcal{H}(T)}\bigl(1+\frac{1}{2h}\bigr)^{h-1}=\frac{(2n+1)^{n-1}}{n!}. \tag{4}$$

2. Proof of Theorem 1

Let us take an example before proving Theorem 1. There are five binary trees with $n=3$ vertices, labeled by their hook lengths:



The hook lengths of T_1, T_2, T_3, T_4 are all the same $1,2,3$; but the hook lengths of T_5 are $1,1,3$. The left-hand side of (3) is then equal to

$$4\times\frac{1}{(2z)^{-1}}\cdot\frac{(z+2)^1}{2}\cdot\frac{(z+3)^2}{3(2z+1)}+\frac{1}{(2z)^{-1}}\cdot\frac{1}{(2z)^{-1}}\cdot\frac{(z+3)^2}{3(2z+1)}=\frac{2^3z(z+3)^2}{3!}.$$

Let $y(x)$ be a formal power series in x such that

$$y(x)=e^{xy(x)}. \tag{5}$$

By the Lagrange inversion formula $y(x)^z$ has the following explicit expansion:

$$y(x)^z=\sum_{n\ge 0}z(n+z)^{n-1}\frac{x^n}{n!}. \tag{6}$$

Since $y^{2z}=(y^z)^2$ we have

$$\sum_{n\ge 0}2z(n+2z)^{n-1}\frac{x^n}{n!}=\bigl(\sum_{n\ge 0}z(n+z)^{n-1}\frac{x^n}{n!}\bigr)^2. \tag{7}$$

Comparing the coefficients of x^n on both sides of (7) yields the following Lemma.

Lemma 3. *We have*

$$\frac{2z(n+2z)^{n-1}}{n!} = \sum_{k=0}^{n} \frac{z(k+z)^{k-1}}{k!} \times \frac{z(n-k+z)^{n-k-1}}{(n-k)!}. \tag{8}$$

In fact, Lemma 3 can be obtained from Abel's celebrated generalization of the binomial formula by a simple change of variables (See [Mo70, p.12] or [Ri68, p.18]).

Proof of Theorem 1. Let

$$P(n) = \sum_{T\in\mathcal{B}(n)} \prod_{h\in\mathcal{H}(T)} \frac{(z+h)^{h-1}}{h(2z+h-1)^{h-2}}.$$

We show that $P(n)$ satisfies a weighted Catalan recurrence (see (9)). In fact, each binary tree T with n vertices is obtained by attaching a left tree and a right tree (with k and $n-k-1$ vertices) at the root v, which has hook length $h_v = n$. Hence $P(0) = 1$ and

$$P(n) = \sum_{k=0}^{n-1} P(k)P(n-1-k) \times \frac{(z+n)^{n-1}}{n(2z+n-1)^{n-2}} \quad (n \ge 1). \tag{9}$$

It is routine to verify that $P(n) = 2^n z(z+n)^{n-1}/n!$ for $n = 1, 2, 3$. Suppose that $P(k) = 2^k z(z+k)^{k-1}/k!$ for $k \le n-1$. From identity (9) and Lemma 3 we have

$$\begin{aligned} P(n) &= \sum_{k=0}^{n-1} \frac{2^k z(z+k)^{k-1}}{k!} \times \frac{2^{n-k-1} z(z+n-k-1)^{n-k-2}}{(n-k-1)!} \\ &\qquad\qquad \times \frac{(z+n)^{n-1}}{n(2z+n-1)^{n-2}} \\ &= \frac{2^n z}{n!}(z+n)^{n-1}. \end{aligned}$$

By induction, formula (3) is true for any positive integer n. ∎

3. Conclusion and Remarks

The present hook length formula was originally discovered by using the *expansion technique*, developed in [Ha08]. A unified formula that includes both the Lascoux-Du-Liu generalization (2) and the present generalization (3) has also been proved in [Ha08, Theorem 6.8]. In [Ya08] Yang has extended formula (3) to binomial families of trees.

The right-hand sides of (3) and (4) have been studied by other authors [GS06, DL08, MY07], but our formula has the following two major differences: (i) the hook length h_v appears as an exponent; (ii) the underlying set remains the set of binary trees, whereas in the above mentioned papers the summation has been changed to the set of m-ary trees or plane forests. It is interesting to compare Corollary 2 with the following results obtained by Du and Liu [DL08]. Note that the right-hand sides of formulas (4), (10) and (11) are all identical!

Proposition 4. *For each positive integer n we have*

$$\sum_{T\in\mathcal{T}(n)} \prod_{v\in I(T)} \Big(\frac{2}{3}+\frac{1}{3h_v}\Big) = \frac{(2n+1)^{n-1}}{n!}, \tag{10}$$

where $\mathcal{T}(n)$ is the set of all 3-ary trees with n internal vertices and $I(T)$ is the set of all internal vertices of T.

Proposition 5. *For each positive integer n we have*

$$\sum_{T\in\mathcal{F}(n)} \prod_{v\in T} \Big(2-\frac{1}{h_v}\Big) = \frac{(2n+1)^{n-1}}{n!}, \tag{11}$$

where $\mathcal{F}(n)$ is the set of all plane forests with n vertices.

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