On the Determination Problem for P_4-Transformation of Graphs *

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Abstract

We prove that the P_4-transformation is one-to-one on the set of graphs with minimum degree at least 3, and if graphs G and G' have minimum degree at least 3 then any isomorphism from the P_4-graph $P_4(G)$ to the P_4-graph $P_4(G')$ is induced by a vertex-isomorphism from G to G' unless G and G' both belong to a special family of graphs.

1 Introduction

Broersma and Hoede [3] generalized the concept of line graphs and introduced the concept of path graphs. We follow their terminology and give the following definition. Let P_k and C_k denote a path and a cycle with k vertices, respectively. Denote by $\Pi_k(G)$ the set of all P_k's in G ($k \geq 1$). The *path graph* $P_k(G)$ of a graph G has vertex set $\Pi_k(G)$ and edges joining pairs of vertices that represent two paths P_k, the union of which forms either a path P_{k+1} or a cycle C_k in G. A graph is called a *P_k-graph* if it is isomorphic to $P_k(H)$ for some graph H. If $k = 2$, then the P_2-graph is exactly

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the line graph. The way of describing a line graph stresses the adjacency concept, whereas the way of describing a path graph stresses the concept of path generation by consecutive paths.

For a graph transformation, there are two general problems, which are formulated by Grünbaum [4]. We state them here for the P_4-transformation.

Characterization Problem: Characterize those graphs that are the P_4-graph of some graph.

Determination Problem: Determine which graphs have a given graph as their P_4-graphs.

For line graphs, there is a well known result concerning the determination problem: If G and G' are connected and have isomorphic line graphs, then G and G' are isomorphic unless one is $K_{1,3}$ and the other is K_3. This result is due to Whitney [19]. For the P_3-transformation, Broersma and Hoede [3] found two pairs and two classes of nonisomorphic connected graphs with isomorphic connected P_3-graphs. These examples show that Whitney's result on line graphs has no similar counterpart with respect to P_3-graphs. In [8], Li proved that the P_3-transformation is one-to-one on all graphs with minimum degree $\delta \geq 4$. Later in [9], Li obtained the same result for all graphs with minimum degree $\delta \geq 3$. Then Aldred, Ellingham, Hemminger and Jipsen [1] completely solved the determination problem for $k = 3$, and proposed a question to investigate the same problem for $k \geq 4$. Li and Zhao [11] proved that for $k \geq 4$ the P_k-transformation is one-to-one on all graphs with minimum degree $\delta \geq k$. In [10] Li and Zhao also showed that the P_4-transformation is one-to-one on all graphs with minimum degree $\delta = 3$ and satisfying one of two other conditions. In this paper, we obtain a stronger result that the P_4-transformation is one-to-one on all graphs with minimum degree $\delta \geq 3$. For more literature related to the path graphs we refer the reader(s) to [7, 12, 13, 14, 15, 16, 17, 18].

2 Preliminaries

In what follows, all graphs are connected and simple. For any graph G and any vertex u in G, let $N(u)$ denote the neighborhood of u in G and let $deg(u)$ denote the degree of u. We write $u \sim v$ if u and v are adjacent in G, and $u \not\sim v$ otherwise. For α in $\Pi_4(G)$, define $N(\alpha)$, $deg(\alpha)$, $\alpha \sim \beta$ and $\alpha \not\sim \beta$ in $P_4(G)$ similarly. For a nonnegative integer d, we denote by $\mathcal{G}_d$ the class of all connected graphs with minimum degree at least d.

We will follow the treatment of [8] for P_3-graphs, which in turn reflects Jung's ideas in [6] and Beineke-Hemminger's treatment in [5]. We introduce the following notation and obtain the corresponding results.

A *vertex-isomorphism* from G to $G^{'}$ is a bijection $f : V(G) \to V(G^{'})$ such that two vertices are adjacent in G if and only if their images are adjacent in $G^{'}$. We let $\Gamma(G, G^{'})$ denote the set of all vertex-isomorphisms of G to $G^{'}$.

An *edge-isomorphism* from G to $G^{'}$ is a bijection $f : E(G) \to E(G^{'})$ such that two edges are adjacent in G if and only if their images are adjacent in $G^{'}$. Obviously, an edge-isomorphism of two graphs is exactly a vertex-isomorphism of their line graphs. Let $\Gamma_e(G, G^{'})$ denote the set of all edge-isomorphisms of G to $G^{'}$. We shorten $\Gamma(P_4(G), P_4(G^{'}))$ to $\Gamma_4(G, G^{'})$ and call the members P_4-*isomorphisms* from G to $G^{'}$.

Let $f \in \Gamma(G, G^{'})$ and $x_1x_2 \cdots x_k$ be a P_k-path in G, then $f(x_1)f(x_2) \cdots f(x_k)$ is a P_k-path in $G^{'}$. Define a mapping $f^{'} : \Pi_k(G) \to \Pi_k(G^{'})$ by $f^{'}(x_1x_2 \cdots x_k) = f(x_1)f(x_2) \cdots f(x_k)$ and call $f^{'}$ the *mapping induced by a vertex-isomorphism* f. Let $\Gamma^{'}_k(G, G^{'}) = \{f^{'} | f \in \Gamma(G, G^{'})\}$. Note that $f^{'}$ is not defined for a connected graph in general unless it has at least one P_k-path.

For $f \in \Gamma_e(G, G^{'})$, define a mapping $f^* : \Pi_4(G) \to \Pi_4(G^{'})$ by $f^*(tuvw) = f(tu)f(uv)f(vw)$ for a P_4-path $tuvw$ in G, and call f^* the *mapping induced by an edge-isomorphism* f. We let $\Gamma^*(G, G^{'}) = \{f^* \mid f \in \Gamma_e(G, G^{'})\}$. Note that f^* is not defined for a connected graph in general unless it has at least one P_4-path.

If $P_4 = tuvw$, then the edge uv is called *middle edge* of the P_4 and $tuvw = wvut$. We let $S(uv)$ denote the set of all P_4-paths with a common middle edge uv. Any subset of $S(uv)$ is called a *double star* at the edge uv. A mapping $f : \Pi_4(G) \to \Pi_4(G')$ is called *double star-preserving at the edge uv* if the set $f(S(uv))$ is a double star in G', and if the set $f(S(uv))$ is a double star in G' for every edge uv of G, then f is called *double star-preserving.*

Theorem 2.1 ([5]) *If G and H are connected graphs, then*
(1) $\Gamma_2'(G, H) \subseteq \Gamma_e(G, H)$;
(2) *the mapping $T : \Gamma(G, H) \to \Gamma_2'(G, H)$ given by $T(f) = f'$ is one-to-one.*

Theorem 2.2 ([5]) *If G and H are connected graphs, then, except for the four cases shown in Figure 1, each edge-isomorphism of G onto H is induced by an isomorphism of G onto H.*

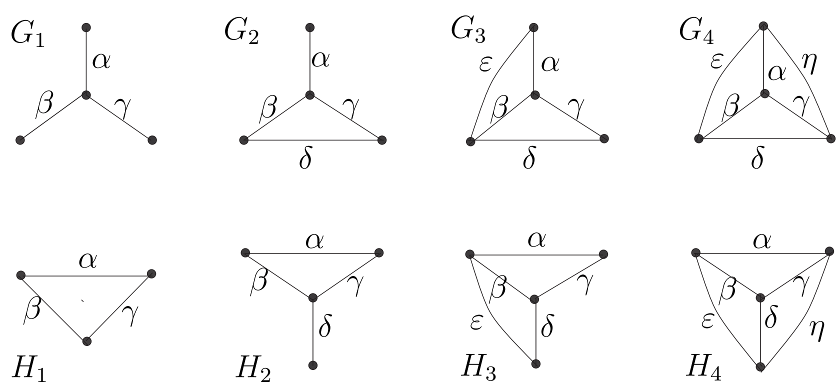

Figure 1

From [10], we have the following two results.

Theorem 2.3 ([10]) *If $G, G' \in \mathcal{G}_3$, then*
(1) $\Gamma^*(G, G') \subseteq \Gamma_4(G, G')$.
(2) *the mapping $T : \Gamma_e(G, G') \to \Gamma^*(G, G')$ given by $T(f) = f^*$ is one-to-one.*

Theorem 2.4 ([10]) *Let $G, G' \in \mathcal{G}_3$ and let $f : \Pi_4(G) \to \Pi_4(G')$ be a bijective mapping. Then f is induced by an edge-isomorphism from G to G' if and only if f and f^{-1} are double star-preserving P_4-isomorphisms.*

We let $E_1(G) = \{uv \in E(G) \mid uv$ is a common edge of two triangles with $deg(u) = deg(v) = 3$ in $G\}$, and denote by $E_2(G) = E(G)\backslash E_1(G)$. For $\alpha \in \Pi_4(G)$ and $f \in \Gamma_4(G, G')$, we let α' denote $f(\alpha)$.

Lemma 2.5 *Let $G, G' \in \mathcal{G}_3$ and let f be a P_4-isomorphism from G to G'. Then f is double star-preserving at every edge of $E_2(G)$ if and only if for every P_3-path tuv of G, $f(x_1tuv), \cdots, f(x_rtuv)$ have a common middle edge and $f(tuvy_1), \cdots, f(tuvy_s)$ have a common middle edge, where $N(t)\backslash\{u, v\} = \{x_1, \cdots, x_r\}$, $N(v)\backslash\{t, u\} = \{y_1, \cdots, y_s\}$, and $r \geq 1$, $s \geq 1$.*

Proof. Let tuv be a P_3-path of G, and $N(t)\backslash\{u, v\} = \{x_1, \cdots, x_r\}$ and $N(v)\backslash\{t, u\} = \{y_1, \cdots, y_s\}$. If both tu and uv are in $E_1(G)$, then u is a common vertex of at least three triangles. Thus, G must be the graph with a K_4 on t, u, v, w, where w is the only vertex joined to the remainder of the graph G. In this case there is only one P_4 of the form $x_itu v$ ($x_1 = w$) and only one P_4 of the form $tuvy_j$ ($y_1 = w$), and then the conclusion of the lemma obviously holds. Next we suppose that there is at least one of tu and uv in $E_2(G)$, without loss of generality, let $tu \in E_2(G)$. So we know that $f(x_1tuv), \cdots, f(x_rtuv)$ have a common middle edge. If $uv \in E_2(G)$, then $f(tuvy_1), \cdots, f(tuvy_s)$ also have a common middle edge. If $uv \in E_1(G)$, then we have that $s = 1$, $N(u) = \{t, v, y_1\}$ and $N(v) = \{t, u, y_1\}$, which is a trivial case. This proves the necessity.

For the sufficiency, let uv be any edge of $E_2(G)$ and let $tuvw$, $t'uvw'$ be two P_4-paths in $S(uv)$. We will distinguish the following four possible cases:

Case 1. The four vertices t, t', w and w' are pairwise distinct.

From the condition we know that $f(tuvw)$ and $f(tuvw')$ have a common middle edge, and $f(tuvw')$ and $f(t'uvw')$ have a common middle edge. Thus $f(tuvw)$ and $f(t'uvw')$ have a common middle edge.

Case 2. $t = t'$ or $w = w'$.

From the condition, we know that $f(tuvw)$ and $f(t'uvw')$ have a common middle edge.

Case 3. $t = w'$ but $t' \neq w$, or $t' = w$ but $t \neq w'$.

By a proof similar to that of Case 1, we can show that $f(tuvw)$ and $f(t^{'}uvw^{'})$ have a common middle edge.

Case 4. $t = w^{'}$ and $t^{'} = w$.

In this case, we see that uv is a common edge of two triangles $C = tuvt$ and $C^{'} = t^{'}uvt^{'}$. Since $uv \in E_2(G)$, we have $\max\{deg(u), deg(v)\} \geq 4$. Without loss of generality, we let $deg(u) \geq 4$. Then there exists a vertex $x \in N(u)\backslash\{t, v, t^{'}\}$. From the condition, we know that $f(tuvw)$ and $f(xuvw)$ have a common middle edge, $f(xuvw)$ and $f(xuvw^{'})$ have a common middle edge, and $f(xuvw^{'})$ and $f(t^{'}uvw^{'})$ have a common middle edge. Thus $f(tuvw)$ and $f(t^{'}uvw^{'})$ have a common middle edge.

To sum up the above cases, we know that $f(S(uv))$ is a double star of $G^{'}$. The proof is complete. ∎

Lemma 2.6 ([11]) *Let $f \in \Gamma_4(G, G^{'})$ and let x_1tuv, x_2tuv, $tuvy_1$ and $tuvy_2$ be four P_4-paths of G. Then $f(x_1tuv)$ and $f(x_2tuv)$ have a common middle edge if and only if $f(tuvy_1)$ and $f(tuvy_2)$ have a common middle edge.*

Lemma 2.7 ([11]) *Let $f \in \Gamma_4(G, G^{'})$ and let x_1tuv, x_2tuv, $tuvy_1$ and $tuvy_2$ be four P_4-paths of G. If $f(x_1tuv)$ and $f(x_2tuv)$ have no common middle edge, then $f(x_1tuv)$, $f(x_2tuv)$, $f(tuvy_1)$ and $f(tuvy_2)$ form a C_4 in $G^{'}$.*

Lemma 2.8 *Let $G, G^{'} \in \mathcal{G}_3$ and $f \in \Gamma_4(G, G^{'})$. If x_1tuv, x_2tuv, $tuvy_1$ and $tuvy_2$ are four P_4-paths of G, then $f(x_1tuv)$ and $f(x_2tuv)$ have a common middle edge, and $f(tuvy_1)$ and $f(tuvy_2)$ have a common middle edge, respectively.*

Proof. Assume, to the contrary, that $f(x_1tuv)$ and $f(x_2tuv)$ have no common middle edge. By Lemma 2.7, we have that $f(x_1tuv)$, $f(x_2tuv)$, $f(tuvy_1)$ and $f(tuvy_2)$ form a C_4 in $G^{'}$ (denoted by $C^{'} = abcda$), say $f(x_1tuv) = abcd$, $f(x_2tuv) = badc$, $f(tuvy_1) = bcda$ and $f(tuvy_2) = dabc$.

We claim that $N(t)\backslash\{u, v, x_1, x_2\} = \emptyset$ and $N(v)\backslash\{t, u, y_1, y_2\} = \emptyset$. Otherwise, there is another member α of $\Pi_4(G)$ with the P_3-path tuv. Without loss of generality, let $\alpha = tuvy_3$ and $y_3 \notin \{t, u, y_1, y_2\}$, then we would have that α' is adjacent to both $f(x_1tuv)$ and $f(x_2tuv)$, which is impossible. Now x_1tuv, x_2tuv, $tuvy_1$ and $tuvy_2$ are all P_4-paths of G with common P_3-path tuv. We will distinguish the following two possible cases:

Case 1. Not all x_1u, x_2u, uy_1 and uy_2 are in $E(G)$.

Without loss of generality, we let $x_1u \notin E(G)$. Since $G \in \mathcal{G}_3$, there are two vertices $p, q \in N(x_1)$ and a vertex $z \in N(u)\backslash\{v\}$ such that px_1tu, qx_1tu and x_1tuz are P_4-paths in G. If $f(x_1tuv)$ and $f(x_1tuz)$ have a common middle edge, then, as both $f(x_1tuv)$ and $f(x_1tuz)$ are adjacent to $f(px_1tu)$, we know that $f(x_1tuv)$ and $f(x_1tuz)$ have a common P_3-path, say abc. Now, let $f(x_1tuz) = abcd'$, so $f(x_1tuz)$ is adjacent to $f(tuvy_2)$, but x_1tuz is not adjacent to $tuvy_2$ in $P_4(G)$, a contradiction to the fact that $f \in \Gamma_4(G, G')$. If $f(x_1tuv)$ and $f(x_1tuz)$ have no common middle edge, then by Lemma 2.7 $f(px_1tu)$, $f(qx_1tu)$, $f(x_1tuv)$ and $f(x_1tuz)$ form a C_4 in G' (denoted by C''). Obviously, $C' = C''$, and so we have $f(x_2tuv) = f(x_1tuz)$, a contradiction to the fact that $f : \Pi_4(G) \to \Pi_4(G')$ is a bijective mapping. Thus $f(x_1tuv)$ and $f(x_2tuv)$ have a common middle edge. By Lemma 2.6, we have that $f(tuvy_1)$ and $f(tuvy_2)$ have a common middle edge.

Case 2. All x_1u, x_2u, uy_1 and uy_2 are in $E(G)$.

Subcase 2.1 $\max\{deg(x_1), deg(x_2), deg(y_1), deg(y_2)\} \geq 4$.

Without loss of generality, we let $deg(x_1) \geq 4$. Then there are two vertices $p, q \in N(x_1)$ such that px_1tu and qx_1tu are P_4-paths in G. Now we consider $f(x_1tuv)$ and $f(x_1tux_2)$ with the following two possible cases: One is that $f(x_1tuv)$ and $f(x_1tux_2)$ have a common middle edge, the other is that $f(x_1tuv)$ and $f(x_1tux_2)$ have no common middle edge. By a proof similar to that of Case 1, it is easy to see that $f(x_1tuv)$ and $f(x_2tuv)$ have a common middle edge. By Lemma 2.6, $f(tuvy_1)$ and $f(tuvy_2)$ have a common middle edge.

Subcase 2.2 $deg(x_1) = deg(x_2) = deg(y_1) = deg(y_2) = 3$.

Let $\alpha_i = x_ituv$ and $\beta_i = tuvy_i$ for $i = 1, 2$. Then, as x_1tuv, x_2tuv, $tuvy_1$ and $tuvy_2$ are all P_4-paths of G with common P_3-path tuv, we have that $deg(\alpha_i) = deg(\alpha_i') = deg(\beta_i) = deg(\beta_i') = 3$ for $i = 1, 2$. We can assume that there exist a vertex $p \in N(x_1)$ and a vertex $q \in N(x_2)$ such that px_1tu and qx_2tu are P_4-paths in G. Let $\beta_3 = px_1tu$ and $\beta_4 = qx_2tu$. Since $\beta_3 \sim \alpha_1$, without loss of generality, we have $\beta_3^{'} = gabc$ and $g \notin \{a, b, c, d\}$. Suppose that $\alpha_3 \in \Pi_4(G)$ with $\alpha_3^{'} = gadc$. Now $\alpha_3^{'} \sim \beta_1^{'}$, so α_3 has a common P_3-path uvy_1 with β_1 by $N(t)\backslash\{u, v, x_1, x_2\} = \emptyset$, and then let $\alpha_3 = uvy_1r$. Next we note that $N(b) = \{a, c, d\}$ and $N(d) = \{a, b, c\}$. Otherwise, we would have a fourth P_4 in $\Pi_4(G^{'})$ which is adjacent to $\alpha_1^{'}$ or $\beta_1^{'}$, a contradiction to $deg(\alpha_1^{'}) = deg(\beta_1^{'}) = 3$. Since $\beta_4 \sim \alpha_2$, then we know that $\beta_4^{'}$ has a common P_3-path adc with $\alpha_2^{'}$. So let $\beta_4^{'} = adch$. If $\alpha_4 \in \Pi_4(G)$ with $\alpha_4^{'} = abch$, then $\alpha_4{'} \sim \beta_2^{'}$. By a similar argument as $\alpha_3^{'}$, we can let $\alpha_4 = uvy_2s$. Now $\alpha_3^{'} \sim \beta_4^{'}$ and $\alpha_4^{'} \sim \beta_3^{'}$, then we conclude that $x_1 = y_2$, $x_2 = y_1$, $p = q = v$ and $r = s = t$ in G.

Let $\alpha_5 = tx_1vx_2$ and $\beta_5 = x_2tx_1v$. Now α_4 and α_5 have a common P_3-path tx_1v with β_3 and β_5 in G, then we will show that $\alpha_4^{'}$, $\alpha_5^{'}$, $\beta_3^{'}$ and $\beta_5^{'}$ form a C_4 in $G^{'}$. If $\alpha_4^{'}$ and $\alpha_5^{'}$ have a common middle edge in $G^{'}$, then, as both $\alpha_4^{'}$ and $\alpha_5^{'}$ are adjacent to $\beta_3^{'}$, we know that $\alpha_4^{'}$ and $\alpha_5^{'}$ have a P_3-path abc in common. So let $\alpha_5^{'} = abch^{'}$, now $\alpha_5^{'} \sim \beta_2^{'}$, but $\alpha_5 \not\sim \beta_2$, a contradiction. Thus $\alpha_4^{'}$ and $\alpha_5^{'}$ have no common middle edge. By Lemma 2.7, we have that $\alpha_4^{'}$, $\alpha_5^{'}$, $\beta_3^{'}$ and $\beta_5^{'}$ form a C_4 in $G^{'}$, and then $g = h$. Since $\alpha_5 \sim \beta_3$ and $\beta_5 \sim \alpha_4$, we conclude that $\alpha_5^{'} = cgab$ and $\beta_5^{'} = bcga$. Let $\alpha_6 = tx_2vx_1$ and $\beta_6 = x_1tx_2v$. Now α_3 and α_6 have a common P_3-path tx_2v with β_4 and β_6. Then, by a similar argument as above, we can get that $\alpha_3^{'}$, $\alpha_6^{'}$, $\beta_4^{'}$ and $\beta_6^{'}$ form a $C_4 = gadcg$ in $G^{'}$ with $\alpha_6^{'} = dcga$ and $\beta_6^{'} = cgad$.

Now, let $\alpha_7 = ux_1tx_2$ and $\beta_7 = x_1tx_2u$. Since $N(v)\backslash\{t, u, x_1, x_2\} = \emptyset$ and $deg(x_2) = 3$, we know that $deg(\beta_5) = 4$ and $N(\beta_5) = \{\alpha_4, \alpha_5, \beta_6, \beta_7\}$. So, $deg(\beta_5^{'}) = 4$, and then $\alpha_4^{'}$, $\alpha_5^{'}$, $\beta_6^{'}$, and $dbcg$ are exactly the four neighbors of $\beta_5^{'}$. Since f is a P_4-isomorphism from G to $G^{'}$, we have $\beta_7^{'} = dbcg$. By the same argument, we can have $deg(\beta_6^{'}) = deg(\beta_6) = 4$, and then $\alpha_7^{'} = gadb$. Now $\alpha_7 \sim \beta_7$, but $\alpha_7^{'} \not\sim \beta_7^{'}$, a contradiction. Then $f(x_1tuv)$ and $f(x_2tuv)$ have a common middle edge. By Lemma 2.6, $f(tuvy_1)$ and $f(tuvy_2)$ have a common middle edge. ∎

Remark. Let $G, G' \in \mathcal{G}_3$ and $f \in \Gamma_4(G, G')$. If x_1tuv, x_2tuv, $tuvy_1$ and $tuvy_2$ are four P_4-paths of G, then, without loss of generality, we can let $f(x_1tuv) = x_1't'u'v'$, $f(x_2tuv) = x_2't'u'v'$, $f(tuvy_1) = t'u'v'y_1'$, and $f(tuvy_2) = t'u'v'y_2'$.

By Lemma 2.8, we have $f(x_1tuv)$ and $f(x_2tuv)$ have a common middle edge. Then, as both $f(x_1tuv)$ and $f(x_2tuv)$ are adjacent to $f(tuvy_1)$, we have that $f(x_1tuv)$ and $f(x_2tuv)$ have a common P_3-path with $f(tuvy_1)$, say $t'u'v'$. So we can let $f(x_1tuv) = x_1't'u'v'$, $f(x_2tuv) = x_2't'u'v'$, and $f(tuvy_1) = t'u'v'y_1'$. Since $f(tuvy_2)$ is adjacent to both $f(x_1tuv)$ and $f(x_2tuv)$, then $f(tuvy_2)$ has a common P_3-path $t'u'v'$ with $f(x_1tuv)$ and $f(x_2tuv)$. So let $f(tuvy_2) = t'u'v'y_2'$ and $y_2' \notin \{t', u', v', y_1'\}$ but possibly $y_2' = x_1'$ or $y_2' = x_2'$.

Lemma 2.9 *Let $G, G' \in \mathcal{G}_3$ and $f \in \Gamma_4(G, G')$. If there is a $C_4 = xtuvx$ in G, then $f(xtuv)$, $f(tuvx)$, $f(uvxt)$ and $f(vxtu)$ form a C_4 in G'.*

Proof. Since $f(xtuv)$ is adjacent to $f(tuvx)$, let $f(xtuv) = habc$ and $f(tuvx) = abcd$. If $f(uvxt)$ and $f(xtuv)$ have a common middle edge, then, as $f(uvxt)$ is adjacent to $f(tuvx)$, we have that $f(uvxt)$ has a common P_3-path abc with $f(tuvx)$, say $f(uvxt) = h'abc$. Since $f(vxtu)$ is adjacent to both $f(xtuv)$ and $f(uvxt)$, we know that $f(vxtu)$ has a common P_3-path abc with $f(xtuv)$ and $f(uvxt)$, and then let $f(vxtu) = abcd'$. Apply Lemma 2.8 for f^{-1} and the four P_4-paths $habc$, $h'abc$, $abcd$, and $abcd'$ of G', we have that $xtuv$ and $uvxt$ have a common middle edge in G, a contradiction. Thus $f(uvxt)$ and $f(xtuv)$ have no common middle edge. Then, as $f(uvxt)$ is adjacent to $f(tuvx)$, we have that $f(uvxt) = bcdg$. Since $f(vxtu)$ is adjacent to both $f(xtuv)$ and $f(uvxt)$, the middle edge of $f(vxtu)$ must be a common edge of $f(xtuv)$ and $f(uvxt)$. However, the edge bc is not the middle edge of $f(vxtu)$, so $ha = dg$ is the middle edge of $f(vxtu)$, i.e., $g = a$, $h = d$. Thus, $f(xtuv)$, $f(tuvx)$, $f(uvxt)$ and $f(vxtu)$ form a C_4 in G' with $f(vxtu) = cdab$.
∎

3 Main result

Denote by $\mathcal{H}$ the set of all graphs obtained by taking one or more copies of the complete graph K_4, choosing one edge in each copy, and identifying together all chosen edges into a single edge, or $\mathcal{H}$ is the class of graphs $K_2 + (mK_2)$, where '+' denotes *join* and mG means the union of m disjoint copies of G.

Lemma 3.1 *Let $G, G^{'} \in \mathcal{G}_3$, and let f be a P_4-isomorphism from G to $G^{'}$. Then f is double star-preserving at every edge of $E_2(G)$, or both G and G' belong to $\mathcal{H}$.*

Proof. We only need to show that f satisfies the condition of Lemma 2.5. Let tuv be a P_3-path in G, where $N(t)\backslash\{u, v\} = \{x_1, \cdots, x_m\}$, $N(v)\backslash\{t, u\} = \{y_1, \cdots, y_n\}$, and $m \geq 1$, $n \geq 1$. Then we will distinguish the following three cases:

Case 1. $m \geq 2$ and $n \geq 2$.

The lemma is obvious by Lemma 2.8.

Case 2. $m = 1$ and $n \geq 2$, or $m \geq 2$ and $n = 1$.

If $m = 1$, then the edge tv must belong to $E(G)$, i.e., $N(t) = \{x_1, u, v\}$. So we only need to show that $f(tuvy_1), \cdots, f(tuvy_n)$ have a common middle edge.

Subcase 2.1 $x_1u \notin E(G)$.

Since $G \in \mathcal{G}_3$, there are two vertices $p, q \in N(x_1)$ and a vertex $z \in N(u)\backslash\{v\}$ such that px_1tu, qx_1tu and x_1tuz are P_4-paths in G. Then, from the above Remark, we can let $f(x_1tuv) = x_1^{'}t^{'}u^{'}v^{'}$ and $f(x_1tuz) = x_1^{'}t^{'}u^{'}z^{'}$. Since $f(tuvy_1), \cdots, f(tuvy_n)$ are adjacent to $f(x_1tuv)$, but none of $tuvy_1, \cdots, tuvy_n$ is adjacent to x_1tuz, we have that $f(tuvy_1), \cdots, f(tuvy_n)$ have a common P_3-path $t^{'}u^{'}v^{'}$ with $f(x_1tuv)$. Thus, $f(tuvy_1), \cdots, f(tuvy_n)$ have a common middle edge.

Subcase 2.2 $x_1u \in E(G)$.

(1) There is a vertex $p \in N(x_1)\backslash\{t, u, v\}$.

Since $G \in \mathcal{G}_3$, there are two vertices $r, s \in N(p)$ such that rpx_1t and spx_1t are P_4-paths in G. From the Remark, we can let $f(rpx_1t) = r^{'}p^{'}x_1^{'}t^{'}$, $f(spx_1t) = s^{'}p^{'}x_1^{'}t^{'}$, $f(px_1tu) = p^{'}x_1^{'}t^{'}u^{'}$ and $f(px_1tv) = p^{'}x_1^{'}t^{'}v^{'}$. Since $f(x_1tuv)$ is adjacent to $f(px_1tu)$, but x_1tuv is not adjacent to px_1tv, then $f(x_1tuv)$ has a common P_3-path $x_1^{'}t^{'}u^{'}$ with $f(px_1tu)$. So let $f(x_1tuv) = x_1^{'}t^{'}u^{'}v_1$.

(1.1) $deg(u) \geq 4$.

Then there exists a vertex $z \in N(u)\backslash\{v\}$ such that x_1tuz is a P_4-path in G. Since $f(x_1tuz)$ is adjacent to $f(px_1tu)$, but x_1tuz is not adjacent to px_1tv, we have that $f(x_1tuz)$ has a common P_3-path $x_1^{'}t^{'}u^{'}$ with $f(px_1tu)$, say $f(x_1tuz) = x_1^{'}t^{'}u^{'}z^{'}$. Similarly, since $f(tuvy_1), \cdots, f(tuvy_n)$ are adjacent to $f(x_1tuv)$, then $f(tuvy_1), \cdots, f(tuvy_n)$ must have a common P_3-path $t^{'}u^{'}v_1$ with $f(x_1tuv)$. Otherwise, $tuvy_1, \cdots, tuvy_n$ are adjacent to x_1tuz, a contradiction. Hence $f(tuvy_1), \cdots, f(tuvy_n)$ have a common middle edge.

(1.2) $deg(u) = 3$, i.e., $N(u) = \{x_1, t, v\}$.

Let $\alpha_1 = rpx_1t$, $\alpha_2 = spx_1t$, $\beta_1 = px_1tu$, $\beta_2 = px_1tv$ and $\gamma = x_1tuv$. Since $n \geq 2$, there must exist some i such that $y_i \neq x_1$. Without loss of generality, say $y_1 \neq x_1$. Let $\alpha_3 = x_1tvy_1$, $\alpha_4 = x_1tvu$ and $\beta_3 = ux_1tv$. Now β_2 and β_3 have a common P_3-path x_1tv with α_3 and α_4. Since $\alpha_3, \alpha_4 \sim \beta_2$, but $\alpha_3, \alpha_4 \not\sim \beta_1$, we know that $\alpha_3^{'}$ and $\alpha_4^{'}$ have a common P_3-path $x_1^{'}t^{'}v^{'}$ with $\beta_2^{'}$. Then, from the Remark, we can let $\alpha_3^{'} = x_1^{'}t^{'}v^{'}y_1^{''}$, $\alpha_4^{'} = x_1^{'}t^{'}v^{'}u_1$, and $\beta_3^{'} = u_2x_1^{'}t^{'}v^{'}$. Now, let $\alpha_5 = vux_1t$ and $\beta_4 = tvux_1$. By Lemma 2.9, we get that $\beta_3^{'}$, $\alpha_4^{'}$, $\beta_4^{'}$ and $\alpha_5^{'}$ form a C_4 in $G^{'}$, and then $u_1 = u_2$. Since $\beta_4 \sim \alpha_4$ and $\alpha_5 \sim \beta_3$, we have that $\beta_4^{'} = t^{'}v^{'}u_1x_1^{'}$ and $\alpha_5^{'} = v^{'}u_1x_1^{'}t^{'}$. Next we note that $u_1 = u^{'}$ and $v_1 = v^{'}$. If $u_1 \neq u^{'}$, then let $\theta_1 \in \Pi_4(G)$ with $\theta_1^{'} = u_1x_1^{'}t^{'}u^{'}$. Now, $\theta_1^{'} \sim \alpha_5^{'}, \gamma^{'}$, but there is no P_4 of $\Pi_4(G)$ which is adjacent to both α_5 and γ, a contradiction. Thus, $u_1 = u^{'}$. Similarly, if $v_1 \neq v^{'}$, let $\theta_2 \in \Pi_4(G)$ with $\theta_2^{'} = x_1^{'}t^{'}u^{'}v^{'}$. Now $\theta_2^{'} \sim \beta_1^{'}$, and so px_1t is the common P_3-path of θ_2 and β_1, otherwise $\theta_2 = \gamma$, a contradiction. So let $\theta_2 = r_1px_1t$. Then we have $\theta_2 \sim \beta_2$, but $\theta_2^{'} \not\sim \beta_2^{'}$, a contradiction.

Next we will show that $f(tuvy_1), \cdots, f(tuvy_n)$ have a common P_3-path $t^{'}u^{'}v^{'}$ with $f(x_1tuv)$, unless $G, G' \in \mathcal{H}$. Let $\gamma_i = tuvy_i$ for $1 \leq i \leq n$.

First, we consider $x_1v \notin E(G)$. Assume, to the contrary, that $\gamma_1^{'}$ has a common P_3-path $x_1^{'}t^{'}u^{'}$ with $\gamma^{'}$, say $\gamma_1^{'} = ax_1^{'}t^{'}u^{'}$. Since $deg(y_1) \geq 3$, there is a vertex $z_1 \in N(y_1)$ such that uvy_1z_1 is a P_4-path in G. Let $\eta_1 = uvy_1z_1$, now $\eta_1 \sim \gamma_1$, but $\eta_1 \not\sim \beta_1$, then $\eta_1^{'}$ has a common P_3-path $ax_1^{'}t^{'}$ with $\gamma_1^{'}$. So let $\eta_1^{'} = bax_1^{'}t^{'}$. If $a \neq v^{'}$, let $\theta_3 \in \Pi_4(G)$ with $\theta_3^{'} = ax_1^{'}t^{'}v^{'}$. Now, $\theta_3^{'} \sim \alpha_4^{'}, \eta_1^{'}$, but there is no P_4 of $\Pi_4(G)$ which is adjacent to both α_4 and η_1, a contradiction. If $a = v^{'}$, then $\gamma_1^{'}$ is in a $C_4 = u^{'}v^{'}x_1^{'}t^{'}u^{'}$ of $G^{'}$. Apply Lemma 2.9 for f^{-1}, we have that γ_1 is in a C_4 of G, which is impossible. Thus $\gamma_1^{'}$ has a common P_3-path $t^{'}u^{'}v^{'}$ with $\gamma^{'}$. By the same argument, we would have that $\gamma_1^{'}, \cdots, \gamma_n^{'}$ have a common P_3-path $t^{'}u^{'}v^{'}$ with $\gamma^{'}$. Thus $f(tuvy_1), \cdots, f(tuvy_n)$ have a common middle edge.

At last, we will consider $x_1v \in E(G)$. Without loss of generality, we let $\gamma_n = tuvx_1$, i.e., $y_n = x_1$. By the above argument we have that $\gamma_1^{'}, \cdots, \gamma_{n-1}^{'}$ have a common P_3-path $t^{'}u^{'}v^{'}$ with $\gamma^{'}$, and then let $\gamma_i^{'} = t^{'}u^{'}v^{'}y_i^{'}$ for $1 \leq i \leq n-1$. Now, suppose that $\gamma_n^{'}$ and $\gamma^{'}$ have a P_3-path $x_1^{'}t^{'}u^{'}$ in common. Let $\beta_5 = uvx_1t$ and $\beta_6 = vx_1tu$. By Lemma 2.9, we have that $\gamma^{'}$, $\gamma_n^{'}$, $\beta_5^{'}$ and $\beta_6^{'}$ form a C_4 in $G^{'}$, and then $x_1^{'} \sim v^{'}$. So $\gamma_n^{'} = v^{'}x_1^{'}t^{'}u^{'}$. Since $\beta_5 \sim \gamma_n$ and $\beta_6 \sim \gamma$, we have $\beta_5^{'} = u^{'}v^{'}x_1^{'}t^{'}$ and $\beta_6^{'} = t^{'}u^{'}v^{'}x_1^{'}$. In fact, $deg(\beta_5) = 2$ by $N(t) = \{x_1, u, v\}$ and $N(u) = \{x_1, t, v\}$. So $\deg(\beta_5^{'}) = 2$, and then $N(t^{'}) = \{x_1^{'}, u^{'}, v^{'}\}$ and $N(u^{'}) = \{x_1^{'}, t^{'}, v^{'}\}$.

Claim 1. $N(v^{'}) = \{t^{'}, u^{'}, y_1^{'}, \cdots, y_{n-1}^{'}, x_1^{'}\}$, i.e., $deg(v^{'}) = deg(v)$.

Let $S = \{t^{'}, u^{'}, y_1^{'}, \cdots, y_{n-1}^{'}, x_1^{'}\}$. If there exists a vertex $w^{'} \in N(v^{'}) \backslash S$, then $t^{'}u^{'}v^{'}w^{'}$ is a P_4-path in $G^{'}$. Let $\gamma_0 \in \Pi_4(G)$ with $\gamma_0^{'} = t^{'}u^{'}v^{'}w^{'}$, now $\gamma_0^{'} \sim \gamma^{'}$, then γ_0 must have a common P_3-path x_1tu with γ. Otherwise, $\gamma_0 = \gamma_i$ for some $i \in \{1, \cdots, n\}$, a contradiction. So let $\gamma_0 = wx_1tu$, and $w \neq v$. Since $deg(w) \geq 3$, there is a vertex $q \in N(w)$ such that qwx_1t is a P_4-path in G. Let $\alpha_0 = qwx_1t$, now $\alpha_0 \sim \gamma_0$, so $\alpha_0^{'}$ has a common P_3-path $u^{'}v^{'}w^{'}$ with $\gamma_0^{'}$. Otherwise, $\alpha_0^{'} = \gamma^{'}$, a contradiction. So let $\alpha_0^{'} = u^{'}v^{'}w^{'}q^{'}$. Let $\beta_0 = wx_1tv$, now $\beta_0 \sim \alpha_0, \alpha_4$. But there is no P_4 of $\Pi_4(G^{'})$ which is adjacent to both $\alpha_0^{'}$ and $\alpha_4^{'}$, a contradiction. Hence, $N(v^{'}) = S$.

Let $\alpha_6 = px_1vu$, now $\alpha_6 \sim \gamma_n$, then $\alpha_6^{'}$ has a common P_3-path $t^{'}x_1^{'}v^{'}$ with $\gamma_n^{'}$, otherwise, $\alpha_6^{'} = \gamma^{'}$, a contradiction. By Claim 1, without loss of generality, we have $\alpha_6^{'} = t^{'}x_1^{'}v^{'}y_1^{'}$. There is at least one of r and s which is not equal to v, say $r \neq v$, and then let $\beta_7 = rpx_1v$. Since $\beta_7 \sim \alpha_6$, then $\beta_7^{'}$ has a common P_3-path $x_1^{'}v^{'}y_1^{'}$ with $\alpha_6^{'}$. Otherwise, $\beta_7^{'} = \gamma_n^{'}$, a contradiction. So let $\beta_7^{'} = x_1^{'}v^{'}y_1^{'}z^{'}$. For $deg(r) \geq 3$, there must be a new member α_7 of $\Pi_4(G)$ that is adjacent to both α_1 and β_7, then we require that $y_1^{'} = r^{'}$, $z^{'} = p^{'}$ and $\alpha_7^{'} = v^{'}y_1^{'}p^{'}x_1^{'}$. Let $\eta \in \Pi_4(G)$ with $\eta^{'} = u^{'}v^{'}y_1^{'}p^{'}$, now $\eta^{'} \sim \gamma_1^{'}$, then η has a common P_3-path uvy_1 with γ_1. Otherwise, $\eta = \gamma$, a contradiction. So let $\eta = uvy_1z$. Now $\alpha_7^{'} \sim \beta_7^{'}, \eta^{'}$, then we require that $y_1 = r$, $z = p$ and $\alpha_7 = vy_1px_1$. Let $\alpha_8 = px_1vy_1$ and $\beta_8 = x_1vy_1p$. By Lemma 2.9, we know that $\alpha_7^{'}$, $\beta_7^{'}$, $\alpha_8^{'}$ and $\beta_8^{'}$ form a C_4 in $G^{'}$. Since $\alpha_8 \sim \beta_7$ and $\beta_8 \sim \alpha_7$, then we have $\alpha_8^{'} = p^{'}x_1^{'}v^{'}y_1^{'}$ and $\beta_8^{'} = y_1^{'}p^{'}x_1^{'}v^{'}$.

Claim 2. $N(p) = \{v, x_1, y_1\}$, $N(y_1) = \{p, v, x_1\}$, $N(p^{'}) = \{v^{'}, x_1^{'}, y_1^{'}\}$, and $N(y_1^{'}) = \{p^{'}, v^{'}, x_1^{'}\}$.

We see that η and β_8 have a common P_3-path py_1v with α_7. Since $\eta^{'}$ and $\beta_8^{'}$ have no common middle edge, then we have that $N(p) = \{v, x_1, y_1\}$. Otherwise, there is a new P_4-path apy_1v with $a \in N(p)\backslash\{v, x_1, y_1\}$. Then, by Lemma 2.8, we know that $\eta^{'}$ and $\beta_8^{'}$ have a common middle edge, a contradiction. Similarly, we see that α_1 and β_7 have a common P_3-path y_1px_1 with α_7, then, as $\alpha_1^{'}$ and $\beta_7^{'}$ have no common middle edge, we have that $N(y_1) = \{p, v, x_1\}$. Now, $\eta^{'}$ and $\beta_7^{'}$ have a common P_3-path $p^{'}y_1^{'}v^{'}$ with $\alpha_7^{'}$, but η and β_7 have no common middle edge. Since f^{-1} is also a P_4-isomorphism, then, by the same argument, we have that $N(p^{'}) = \{v^{'}, x_1^{'}, y_1^{'}\}$. Similarly, $\alpha_1^{'}$ and $\beta_8^{'}$ have a common P_3-path $y_1^{'}p^{'}x_1^{'}$ with $\alpha_7^{'}$, and as α_1 and β_8 have no common middle edge, we know that $N(y_1^{'}) = \{p^{'}, v^{'}, x_1^{'}\}$.

Claim 3. $N(x_1) = \{t, u, y_1, \cdots, y_{n-1}, v\}$ and $N(x_1^{'}) = \{t^{'}, u^{'}, y_1^{'}, \cdots, y_{n-1}^{'}, v^{'}\}$, and n is odd.

Since $deg(t) + deg(x_1) - 4 = deg(\gamma_n) = deg(\gamma_n^{'}) = deg(u^{'}) + deg(v^{'}) - 4$, and $deg(t) = deg(u^{'}) = 3$, then $deg(x_1) = deg(v^{'})$. Similarly, we can have $deg(v) = deg(x_1^{'})$ for $deg(\beta_6) = deg(\beta_6^{'})$. By Claim 1, we know that $deg(v) = deg(v^{'})$. Thus, $deg(x_1) = deg(v)$ and $deg(x_1^{'}) = deg(v^{'})$. Now, suppose that $N(x_1) = \{t, u, p, p_2, \cdots, p_{n-1}, v\}$. By Claim 2, we know that $p \in N(v)\backslash\{x_1, y_1\}$ and $y_1 \in N(x_1)\backslash\{p, v\}$. Thus, without loss of generality, let $p = y_{n-1}$ and $y_1 = p_{n-1}$. Similarly, $p^{'} \in N(v^{'})\backslash\{x_1^{'}, y_1^{'}\}$ and $y_1^{'} \in N(x_1^{'})\backslash\{p^{'}, v^{'}\}$, then we can let $p^{'} = y_{n-1}^{'}$ and $y_1^{'} = p_{n-1}^{'}$. In fact we would see that x_1, v, y_1 and y_{n-1} form a K_4 in G, and $x_1^{'}$, $v^{'}$, $y_1^{'}$ and $y_{n-1}^{'}$ form a K_4 in $G^{'}$. By a similar proof as above, without loss of generality, we can show that $N(p_i) = \{v, x_1, y_i\}$ and $N(y_i) = \{p_i, v, x_1\}$ for $2 \leq i \leq n-2$. Then $p_i \in N(v)\backslash\{y_1, y_i, y_{n-1}, x_1\}$ and $y_i \in N(x_1)\backslash\{p, p_i, p_{n-1}, v\}$, without loss of generality, we can let $p_i = y_{n-i}$ and $y_i = p_{n-i}$ for $2 \leq i \leq \lfloor \frac{n}{2} \rfloor$. So, if n is even, then $p_{\frac{n}{2}} = y_{\frac{n}{2}}$ and $N(p_{\frac{n}{2}}) = \{v, x_1\}$, which is impossible. Thus n is odd, and $N(x_1) = \{t, u, y_1, \cdots, y_{n-1}, v\}$. By the same argument, we can show that $N(x_1^{'}) = \{t^{'}, u^{'}, y_1^{'}, \cdots, y_{n-1}^{'}, v^{'}\}$ and $N(y_i^{'}) = \{v^{'}, x_1^{'}, y_{n-i}^{'}\}$ for $2 \leq i \leq n-2$.

Now, we conclude that the four vertices x_1, v, y_i and y_{n-i} form a K_4 in G, and $x_1^{'}$, $v^{'}$, $y_i^{'}$ and $y_{n-i}^{'}$ form a K_4 in $G^{'}$, for $2 \leq i \leq \frac{n-1}{2}$. Hence, $G, G^{'} \in \mathcal{H}$.

(2) $N(x_1) = \{t, u, v\}$.

Without loss of generality, let $y_n = x_1$. By Lemma 2.9, we have that $f(x_1tuv)$, $f(tuvx_1)$, $f(uvx_1t)$ and $f(vx_1tu)$ form a C_4 in $G^{'}$, say $f(x_1tuv) = abcd$, $f(tuvx_1) = bcda$, $f(uvx_1t) = cdab$ and $f(vx_1tu) = dabc$. In fact $deg(tuvx_1) = 2$ by $N(t) = \{x_1, u, v\}$ and $N(x_1) = \{t, u, v\}$. So $deg(bcda) = 2$, and then $N(a) = \{b, c, d\}$ and $N(b) = \{a, c, d\}$. Then, as $f(tuvy_1), \cdots, f(tuvy_{n-1})$ are adjacent to $f(x_1tuv)$, we know that $f(tuvy_1), \cdots, f(tuvy_{n-1})$ have a common P_3-path bcd with $f(x_1tuv)$. Hence, $f(tuvy_1), \cdots, f(tuvy_{n-1})$ and $f(tuvx_1)$ have a common middle edge.

Case 3. $m = 1$ and $n = 1$.

This case is trivial.

To sum up the above cases, we have proved that f is double star-preserving at each edge e of $E_2(G)$. ∎

Lemma 3.2 *Let $G, G' \in \mathcal{G}_3 \backslash \mathcal{H}$ and let f be a P_4-isomorphism from G to G'. Then f is double star-preserving at the edge uv, where $uv \in E_1(G)$.*

Proof. Let uv be a common edge of two triangles $C = xuvx$ and $C' = yuvy$ with $deg(u) = deg(v) = 3$. We will prove that $f(xuvy)$ and $f(yuvx)$ have a common middle edge.

Since $G \neq K_4$, G has at least five vertices. Without loss of generality, let $t \in N(x)\backslash\{u, v, y\}$. For $G \in \mathcal{G}_3$, there exist a vertex $s \in N(t)\backslash\{x, y\}$ and a vertex $p \in N(s)\backslash\{t, x\}$ such that $stxu$, $stxv$ and $pstx$ are P_4-paths in G. Let $\alpha_1 = stxu$, $\alpha_2 = stxv$ and $\beta_0 = pstx$. By Lemma 3.1, we know that α_1' and α_2' have a common middle edge. Since $\alpha_1, \alpha_2 \sim \beta_0$, we have that α_1' and α_2' have a common P_3-path, say $s't'x'$. Then, let $\alpha_1' = s't'x'u'$ and $\alpha_2' = s't'x'v'$. Let $\beta_1 = txuv$, $\beta_2 = txuy$, $\gamma_1 = xuvy$ and $\gamma_2 = xuyv$. Since $\beta_1, \beta_2 \sim \alpha_1$, but $\beta_1, \beta_2 \not\sim \alpha_2$, we have that β_1' and β_2' have a common P_3-path $t'x'u'$ with α_1'. So let $\beta_1' = t'x'u'v_1$ and $\beta_2' = t'x'u'y_1$. Similarly, $\gamma_1 \sim \beta_1$ but $\gamma_1 \not\sim \beta_2$, and $\gamma_2 \sim \beta_2$ but $\gamma_2 \not\sim \beta_1$, then we would let $\gamma_1' = x'u'v_1y_2$ and $\gamma_2' = x'u'y_1v_2$. Let $\beta_3 = txvu$, $\beta_4 = txvy$, $\gamma_3 = xvuy$ and $\gamma_4 = xvyu$. By symmetry, we can let $\beta_3' = t'x'v'u_1$, $\beta_4' = t'x'v'y_3$, $\gamma_3' = x'v'u_1y_4$ and $\gamma_4' = x'v'y_3u_2$. Since $\gamma_2 \sim \gamma_4$, it requires that $y_1 = y_3$, $u' = u_2$ and $v' = v_2$.

Now we suppose that $f(xuvy)$ and $f(yuvx)$ have no common middle edge, i.e., $u'v_1 \neq v'u_1$. We will distinguish the following two possible cases:

Case 1. $u' \neq u_1$ and $v' \neq v_1$.

Suppose that $\alpha_3, \alpha_4 \in \Pi_4(G)$ with $\alpha_3' = v_1u'x'v'$ and $\alpha_4' = u'x'v'u_1$. Now, $\alpha_3' \sim \gamma_1'$ and $\alpha_4' \sim \gamma_3'$. Hence, $\alpha_3 = a_1xuv$ or $uvyb_1$, and $\alpha_4 = a_2xvu$ or $vuyb_2$. In any cases, $\alpha_3 \not\sim \alpha_4$, but in fact $\alpha_3' \sim \alpha_4'$, a contradiction.

Case 2. $u' \neq u_1$ and $v' = v_1$ (or $u' = u_1$ and $v' \neq v_1$).

We claim that $y_2 = y_1 = u_1$. If $y_2 \neq y_1$, let $\alpha_5 \in \Pi_4(G)$ with $\alpha_5' = x'u'v'y_1$. Now $\alpha_5' \sim \beta_1'$, then α_5 has a common P_3-path txu with β_1. Otherwise, $\alpha_5 = \gamma_1$, a contradiction. So let $\alpha_5 = rtxu$, now $\alpha_5 \sim \beta_2$, but $\alpha_5' \not\sim \beta_2'$, a contradiction. Thus $y_2 = y_1$. If $u_1 \neq y_1$, let $\alpha_6 \in \Pi_4(G)$ with $\alpha_6' = x'u'v'u_1$. By a similar argument as α_5', we conclude $u_1 = y_1$, and then $\beta_3' = \beta_4'$, a contradiction.

Thus $f(xuvy)$ and $f(yuvx)$ have a common middle edge. The proof is now complete. ∎

Theorem 3.3 *Let $G, G' \in \mathcal{G}_3 \backslash \mathcal{H}$. Then $f \in \Gamma_4(G, G')$ if and only if f is induced by an edge-isomorphism from G to G', i.e., $P_4(G)$ is isomorphic to $P_4(G')$ if and only if the line graph $L(G)$ is isomorphic to $L(G')$.*

Proof. The "if" part is obvious. For the "only if" part, let $f \in \Gamma_4(G, G')$. By Theorem 2.4, we only need to prove that both f and f^{-1} are double star-preserving. Then, from Lemmas 3.1 and 3.2, we have that f is double star-preserving. Since G' has the same property as G, we have that f^{-1} is also double star-preserving, which completes the proof. ∎

From Theorems 2.2 and 3.3, the following result is immediate.

Theorem 3.4 *Let $G, G' \in \mathcal{G}_3 \backslash \mathcal{H}$. Then $f \in \Gamma_4(G, G')$ if and only if f is induced by an isomorphism from G to G', i.e., $P_4(G)$ is isomorphic to $P_4(G')$ if and only if G is isomorphic to G'.*

By Lemma 3.1, it is easy to get a result as follows.

Corollary 3.5 *Let $G \in \mathcal{H}$ and $G' \in \mathcal{G}_3$. If f is a P_4-isomorphism from G to G', then G is isomorphic to G'.*

Corollary 3.6 *The P_4-transformation is one-to-one on $\mathcal{G}_3$.*

Proof. Let f be a P_4-isomorphism from G to G'. If $G \in \mathcal{G}_3 \backslash \mathcal{H}$ and $G' \in \mathcal{H}$, apply Corollary 3.5 for f^{-1}, then $G \in \mathcal{H}$, a contradiction. So from Theorems 2.1, 2.3 and 3.4, and Corollary 3.5, we have this result immediately. ∎

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