General Randić Index on Trees with a Given Order and Diameter *

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Abstract

The general Randić index $R_\alpha(G)$ of a graph G, which is also called the connectivity index, is defined as the sum of the weights $(d(u)d(v))^\alpha$ of all edges uv of G, where $d(u)$ denotes the degree of a vertex u in G and α is an arbitrary real number. In this paper, we consider the set of trees with a given order and diameter, and determine the extremal trees with the maximum general Randić index for $0 < \alpha < 1$ among this kind of trees. The minimum general Randić index for $-1 \leq \alpha < 0$ is also considered.

1 Introduction

The *Randić index* $R(G)$ of a graph G was introduced by the chemist Milan Randić under the name "*branching index*" in 1975 [7] as the sum of $1/\sqrt{d(u)d(v)}$ over all edges

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uv of G, where $d(u)$ denotes the degree of a vertex u in G, i.e.,

$$R(G) = \sum_{uv \in E} \frac{1}{\sqrt{d(u)d(v)}}.$$

It is well known that $R(G)$ was introduced as one of the many graph-theoretical parameters derived from the graph underlying some molecule. Later, in 1998 Bollobás and Erdös [2] generalized this index by replacing $-\frac{1}{2}$ with any real number α, which is called the general Randić index. The research background of Randić index together with its generalization appears in chemistry or mathematical chemistry and can be found in the literature (see [2], [3], [4], [7]). Recently, finding bounds for the general Randić index of a given class of graphs, as well as related problem of finding the graphs with maximum or minimum general Randić index have attracted much attention (see [10]-[15]). For a comprehensive survey of its mathematical properties, see the survey [8] and the recent book of Li and Gutman [9].

We only consider trees here. For a vertex x of a tree T, we use $N_T(x)$ and $d_T(x)$ to denote the neighborhood and the degree of x, respectively. For two vertices v_i and v_j $(i \neq j)$, the distance between v_i and v_j in T is the number of edges in a shortest path joining v_i and v_j. The diameter of T is the maximum distance between any two vertices of T. We use $T - xy$ to denote the graph obtained from T by deleting the edge $xy \in E(T)$. Similarly, $T + xy$ is the graph that arises from T by adding an edge $xy \notin E(T)$, where $x, y \in E(T)$. Denote by S_n and P_n the star and the path with n vertices, respectively.

Let $\mathcal{T}(n,r)$ be the set of trees with order n and diameter r, and let $T \in \mathcal{T}(n,r)$. A main chain of T is a path in T of length r. Let $P^r = u_0u_1 \dots u_r$ be a main chain of T and assume that $r \geq 2$. We use $T(n, r, k_1, \dots, k_{r-1})$ to denote the tree of order n obtained from P^r by attaching k_i $(k_i \geq 0)$ pendent vertices to u_i, for each $i \in \{1, \dots, r-1\}$. Define

$$\mathcal{T}^*(n,r) = \left\{ T(n, r, k_1, \dots, k_{r-1}) : \sum_{i=1}^{r-1} k_i = n - r - 1 \right\}.$$

It is easy to see that $\mathcal{T}^*(n,r) \subset \mathcal{T}(n,r)$.

Let $T_{i_1,\dots,i_j}(n, r, k_{i_1}, \dots, k_{i_j})$ be the tree in $\mathcal{T}^*(n,r)$ with $k_l \geq 1$ if $l \in \{i_1, \dots, i_j\}$ and $k_l = 0$ otherwise, and let $\mathcal{T}_j^*(n,r) = \{T_{i_1,\dots,i_j}(n, r, k_{i_1}, \dots, k_{i_j}) | 1 \leq i_1, \dots, i_j \leq r-1\}$. In particular, $\mathcal{T}_1^*(n,r) = \{T_i(n, r, n-r-1)$ with $k_i = n - r - 1 \geq 1$ and $k_j = 0$ for $j \neq i | 1 \leq i \leq r-1\}$. Obviously, $\mathcal{T}^*(n,r) = \bigcup_{j=1}^{r-1} \mathcal{T}_j^*(n,r)$.

For terminology and notations not defined here, we refer the readers to [1].

Let $T \in \mathcal{T}(n,r)$. In [5], Jiang and Lu gave the sharp upper bound $R_\alpha(T)$ of T for $\alpha = 1$. In [6], Li and Zhao obtained the sharp lower bound $R_\alpha(T)$ of T for $\alpha = -\frac{1}{2}$. In this paper, we will give the sharp upper bound $R_\alpha(T)$ for $0 < \alpha < 1$ and determine the extremal trees. The sharp lower bound $R_\alpha(T)$ of T for $-1 \leq \alpha < 0$ is also considered by using the similar method.

Note that if $r = 2$ or $r = n-1$, then the only trees in $\mathcal{T}(n,r)$ are S_n and P_n, respectively. Since $R_\alpha(S_n) = (n-1)^{\alpha+1}$ and $R_\alpha(P_n) = 4^\alpha(n-3) + 2^{\alpha+1}$, we always assume that $3 \leq r \leq n-2$ in the following sections.

2 Main result

We first give several lemmas which will be used in the proof of our main result.

Lemma 1 *Let $T \in \mathcal{T}(n,r)$, $P^r = u_0u_1\ldots u_r$ be the main chain of T and for some i, $d_T(u_i) = q+1 \geq 3$. Suppose that there exists $v_1 \in V(T) \setminus V(P^r)$ such that $u_iv_1 \in E(T)$ and $d_T(v_1) = p+1 \geq 2$. Denote $N_T(u_i) = \{u_{i-1}, u_{i+1}, v_1, \ldots, v_{q-1}\}$ and $N_T(v_1) = \{u_i, w_1, \ldots, w_p\}$. Let $T' = T - \{v_1w_1, \ldots, v_1w_p\} + \{u_iw_1, \ldots, u_iw_p\}$. Then for $0 < \alpha < 1$, $R_\alpha(T') > R_\alpha(T)$.*

Proof. As shown in Figure 2.1, we have

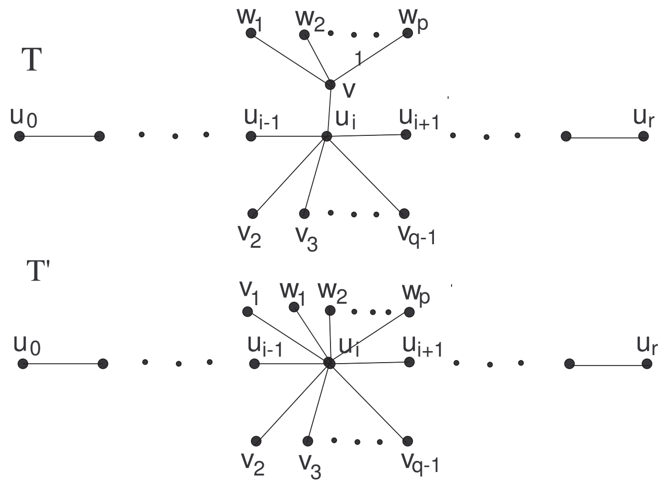

Figure 2.1 T and T'

$$
\begin{aligned}
R_\alpha(T') - R_\alpha(T) &= \left(\sum_{j=1}^{p} d_{T'}^{\alpha}(w_j) + \sum_{k=1}^{q-1} d_{T'}^{\alpha}(v_k) + d_{T'}^{\alpha}(u_{i-1}) + d_{T'}^{\alpha}(u_{i+1})\right)(p+q+1)^\alpha \\
&\quad - \sum_{k=1}^{q-1} d_T^{\alpha}(v_k)(q+1)^\alpha - \sum_{j=1}^{p} d_T^{\alpha}(w_j)(p+1)^\alpha \\
&\quad -[d_T^{\alpha}(u_{i-1}) + d_T^{\alpha}(u_{i+1})](q+1)^\alpha \\
&= \left(\sum_{k=2}^{q-1} d_T^{\alpha}(v_k) + d_T^{\alpha}(u_{i-1}) + d_T^{\alpha}(u_{i+1})\right)[(p+q+1)^\alpha - (q+1)^\alpha] \\
&\quad + \sum_{j=1}^{p} d_T^{\alpha}(w_j)[(p+q+1)^\alpha - (p+1)^\alpha] \\
&\quad +(p+q+1)^\alpha - (p+1)^\alpha(q+1)^\alpha \\
&\geq q[(p+q+1)^\alpha - (q+1)^\alpha] + p[(p+q+1)^\alpha - (p+1)^\alpha] \\
&\quad +(p+q+1)^\alpha - (p+1)^\alpha(q+1)^\alpha \\
&> q[(p+q+1)^\alpha - (q+1)^\alpha] + (p+q+1)^\alpha - (p+1)^\alpha(q+1)^\alpha.
\end{aligned}
$$

By the Lagrange Mean-Value Theorem, there exist $\xi \in (q+1, p+q+1)$ and $\zeta \in (p+q+1, pq+p+q+1)$ such that $(p+q+1)^\alpha - (q+1)^\alpha = \alpha p \xi^{\alpha-1}$ and $(p+q+1)^\alpha - (p+1)^\alpha(q+1)^\alpha = -\alpha pq\zeta^{\alpha-1}$, respectively. Hence

$$
\begin{aligned}
R_\alpha(T') - R_\alpha(T) &> q[(p+q+1)^\alpha - (q+1)^\alpha] + (p+q+1)^\alpha - (p+1)^\alpha(q+1)^\alpha \\
&= \alpha pq(\xi^{\alpha-1} - \zeta^{\alpha-1}) > 0,
\end{aligned}
$$

since $0 < \alpha < 1$ and $\xi < \zeta$. ∎

Corollary 1 *Let $T \in \mathcal{T}(n,r)$. Then for $0 < \alpha < 1$, there exists $T' \in \mathcal{T}^*(n,r)$ such that $R_\alpha(T') > R_\alpha(T)$.*

Proof. Let $P^r = u_0u_1 \dots u_r$ be the main chain of T. Denote $m(T) = |\{v | d_T(v) \geq 2, v \in V(T) \setminus V(P^r)\}|$. Clearly, if $m(T) = 0$, then $T \in \mathcal{T}^*(n,r)$.

Claim : If $m(T) \neq 0$, then for some $u_i \in V(P^r)$, there exists $v_0 \notin V(P^r)$ such that $u_iv_0 \in E(T)$ and $d_T(v_0) \geq 2$.

By the definition of $m(T)$, if $m(T) \neq 0$, there exists $v \in V(T) \setminus V(P^r)$ with $d_T(v) \geq 2$. Since T is a tree, there exists a path $P = u_iv_0, \dots, v_{s-1}v_s$ $(v_s = v)$ joining u_i and v with $v_j \in V(T) \setminus V(P^r)$ $(0 \leq j \leq s)$. Then v_0 is the vertex as required and hence the claim holds.

If $m(T) = 1$, then by Lemma 1 and the claim, there exists $T' \in \mathcal{T}^*(n,r)$ such that $R_\alpha(T') > R_\alpha(T)$ and $m(T') = 0$.

If $m(T) = k \geq 2$, then from Lemma 1 and our claim, we have trees T^k, $T^{k-1}, \dots,$ T^1 such that $R_\alpha(T^k) > R_\alpha(T^{k-1}) > \cdots > R_\alpha(T^1) > R_\alpha(T)$ and $m(T^{k-j}) = j$ $(0 \leq j \leq k-1)$. So $m(T^k) = 0$, i.e., $T^k \in \mathcal{T}^*(n,r)$. This proves the lemma. ∎

Lemma 2 *Let A be a positive constant and $0 \leq x \leq A$. For $0 < \alpha < 1$, denote $G(x) = (A-x+2)^{\alpha-1}[\alpha 2^\alpha + \alpha(x+2)^\alpha + (A-x+2) + \alpha(A-x)] - (x+2)^{\alpha-1}[\alpha 2^\alpha + \alpha(A-x+2)^\alpha + x + 2 + \alpha x]$, then*

$$G(x) \begin{cases} > 0 & \text{if } 0 \leq x < \frac{A}{2}; \\ = 0 & \text{if } x = \frac{A}{2}; \\ < 0 & \text{if } \frac{A}{2} < x \leq A. \end{cases}$$

Proof. Let $A - x + 2 = c(x+2)$, then we have

$$\begin{aligned} G(x) &= c^{\alpha-1}(x+2)^{\alpha-1}[\alpha 2^\alpha + \alpha(x+2)^\alpha + c(x+2) + \alpha c(x+2) - 2\alpha] \\ &\quad -(x+2)^{\alpha-1}[\alpha 2^\alpha + \alpha c^\alpha (x+2)^\alpha + (x+2) + \alpha(x+2) - 2\alpha] \\ &= (x+2)^{\alpha-1}[-\alpha(x+2)^\alpha(c^\alpha - c^{\alpha-1}) + (\alpha+1)(x+2)(c^\alpha - 1)] \\ &\quad +(x+2)^{\alpha-1}[\alpha(1-c^{\alpha-1})(2-2^\alpha)]. \end{aligned}$$

For $0 \leq x < \frac{A}{2}$, we have $c > 1$. So $\alpha(1-c^{\alpha-1})(2-2^\alpha) \geq 0$, and $(\alpha+1)(x+2)(c^\alpha - 1) - \alpha(x+2)^\alpha(c^\alpha - c^{\alpha-1}) > (x+2)^\alpha[(\alpha+1)(c^\alpha - 1) - \alpha(c^\alpha - c^{\alpha-1})] > 0$, since the function $g(y) := (\alpha+1)(y^\alpha - 1) - \alpha(y^\alpha - y^{\alpha-1})$ is strictly increasing for $y > 1$ ($\frac{dg(y)}{dy} = \alpha y^{\alpha-2}(y-1+\alpha) > 0$), and hence $G(x) > 0$.

On the other hand, since $G(x) = -G(A-x)$, we have $G(x) < 0$ when $\frac{A}{2} < x \leq A$.

It is easy to check that $G(x) = 0$ when $x = \frac{A}{2}$. Thus the lemma holds. ∎

Lemma 3 *Let A be a positive constant and $0 \leq x \leq A$. For $0 < \alpha < 1$, denote* $H(x) = (A-x+2)^{\alpha-1}[(A-x+2)+\alpha(A-x+2^{\alpha+1})] - (x+2)^{\alpha-1}[(x+2)+\alpha(x+2^{\alpha+1})]$, *then*

$$H(x) \begin{cases} > 0 & \text{if } 0 \leq x < \frac{A}{2}; \\ = 0 & \text{if } x = \frac{A}{2}; \\ < 0 & \text{if } \frac{A}{2} < x \leq A. \end{cases}$$

Proof. Let $h(x) = (x+2)^{\alpha-1}[(x+2)+\alpha(x+2^{\alpha+1})]$, then

$$\frac{dh(x)}{dx} = \alpha(x+2)^{\alpha-2}(x+\alpha x+\alpha 2^{\alpha+1}+4-2^{\alpha+1}) > 0.$$

Thus if $A-x+2 > x+2$, $H(x) > 0$; if $A-x+2 = x+2$, $H(x) = 0$; if $A-x+2 < x+2$, $H(x) < 0$. This proves the lemma. ∎

Lemma 4 *Let $r \geq 3$ and $T \in \mathcal{T}_k^*(n,r)$ $(k \geq 2)$. Then for $0 < \alpha < 1$, there exists $T' \in \mathcal{T}_{k-1}^*(n,r)$ such that $R_\alpha(T') > R_\alpha(T)$.*

Proof. Since $T \in \mathcal{T}_k^*(n,r)$ $(k \geq 2)$, let $P^r = u_0u_1\ldots u_r$ be the main chain of T and u_i, u_j $(1 \leq i < j \leq r-1)$ the first and second vertex in $V(P^r)$ such that $d_T(u_i)$, $d_T(u_j) \geq 3$. Then $d_T(u_{i-1}) \leq 2$ and $d_T(u_{j+1}) \geq 1$. If $d_T(u_{i-1}) > d_T(u_{j+1})$, then we have $d_T(u_{i-1}) = 2$ and $d_T(u_{j+1}) = 1$, which implies u_i and u_j are the only two vertices with degree greater than two in $V(P^r)$. In this case, we exchange the order of u_i and u_j in the main chain P^r. So we can always assume that $d_T(u_{i-1}) \leq d_T(u_{j+1})$.

Let $N_T(u_i) = \{u_{i-1}, u_{i+1}, v_1, \ldots, v_p\}$ and $N_T(u_j) = \{u_{j-1}, u_{j+1}, w_1, \ldots, w_{q-p}\}$ $(p \geq 1, q-p \geq 1)$. Set $T' = T - \{v_1u_i, \ldots, v_pu_i\} + \{v_1u_j, \ldots, v_pu_j\}$, then $T' \in T_{k-1}^*(n,r)$ (See Figure 2.2). Next we show that $R_\alpha(T') > R_\alpha(T)$. We consider two cases.

Case 1. $j = i+1$.

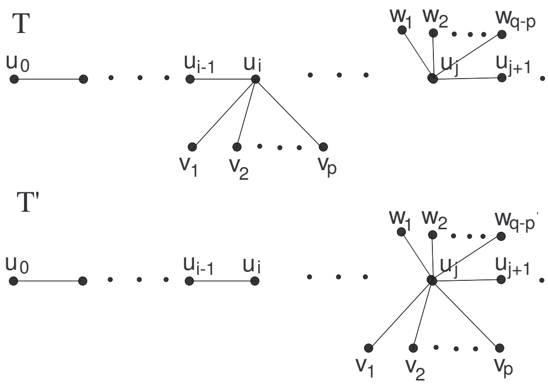

Figure 2.2 T and T'

In this case, we get

$$\begin{aligned}
R_\alpha(T') - R_\alpha(T) &= q(q+2)^\alpha + [d^\alpha_{T'}(u_{j+1}) + 2^\alpha](q+2)^\alpha + 2^\alpha d^\alpha_{T'}(u_{i-1}) \\
&\quad -p(p+2)^\alpha - d^\alpha_T(u_{i-1})(p+2)^\alpha - (p+2)^\alpha(q-p+2)^\alpha \\
&\quad -(q-p)(q-p+2)^\alpha - d^\alpha_T(u_{j+1})(q-p+2)^\alpha \\
&= d^\alpha_T(u_{j+1})[(q+2)^\alpha - (q-p+2)^\alpha] - d^\alpha_T(u_{i-1})[(p+2)^\alpha - 2^\alpha] \\
&\quad +(q+2^\alpha)(q+2)^\alpha - [(q-p+2)^\alpha + p](p+2)^\alpha \\
&\quad -(q-p)(q-p+2)^\alpha \\
&\geq d^\alpha_T(u_{i-1})[(q+2)^\alpha - (p+2)^\alpha - (q-p+2)^\alpha + 2^\alpha] \\
&\quad +(q+2^\alpha)(q+2)^\alpha - [(q-p+2)^\alpha + p](p+2)^\alpha \\
&\quad -(q-p)(q-p+2)^\alpha \\
&\geq 2^\alpha[(q+2)^\alpha - (p+2)^\alpha - (q-p+2)^\alpha + 2^\alpha] + (q+2^\alpha)(q+2)^\alpha \\
&\quad -[(q-p+2)^\alpha + p](p+2)^\alpha - (q-p)(q-p+2)^\alpha \\
&= (q+2^{\alpha+1})(q+2)^\alpha - [(p+2)^\alpha + 2^\alpha + q - p](q-p+2)^\alpha \\
&\quad -(p+2^\alpha)(p+2)^\alpha + 4^\alpha.
\end{aligned}$$

The last inequality holds since $d_T(u_{i-1}) \leq 2$ and the function $f(x) := (x+q-p)^\alpha - x^\alpha$ is strictly decreasing for $x \geq 0$ when $0 < \alpha < 1$.

Let $F(x) = (q+2^{\alpha+1})(q+2)^\alpha - [(x+2)^\alpha + 2^\alpha + q - x](q-x+2)^\alpha - (x+2^\alpha)(x+2)^\alpha + 4^\alpha$, then we have

$$\begin{aligned}
\frac{dF(x)}{dx} &= (q-x+2)^{\alpha-1}[\alpha 2^\alpha + \alpha(x+2)^\alpha + \alpha(q-x) + q - x + 2] \\
&\quad -(x+2)^{\alpha-1}[\alpha 2^\alpha + \alpha(q-x+2)^\alpha + \alpha x + x + 2].
\end{aligned}$$

Then by Lemma 2, for $0 \le x \le q$, $\frac{dF(x)}{dx}$ reaches its minimum only at $x = 0$ or $x = q$ where $\frac{dF(x)}{dx} = 0$, hence $\frac{dF(x)}{dx} \ge 0$ and equality holds if and only if $x = 0$ or $x = q$. Therefore $F(p) > \max\{F(0), F(q)\} = 0$ for $0 < p < q$, i.e., $R_\alpha(T') > R_\alpha(T)$.

Case 2. $j \ge i + 2$.

In this case, we have

$$\begin{aligned} R_\alpha(T') - R_\alpha(T) &= q(q+2)^\alpha - p(p+2)^\alpha - [2^\alpha + d_T^\alpha(u_{i-1})][(p+2)^\alpha - 2^\alpha] \\ &\quad -(q-p)(q-p+2)^\alpha + [2^\alpha + d_T^\alpha(u_{j+1})][(q+2)^\alpha - (q-p+2)^\alpha] \\ &\ge q(q+2)^\alpha - p(p+2)^\alpha - (q-p)(q-p+2)^\alpha \\ &\quad -[2^\alpha + d_T^\alpha(u_{i-1})]\{[(q-p+2)^\alpha - 2^\alpha] - [(q+2)^\alpha - (p+2)^\alpha]\} \\ &\ge q(q+2)^\alpha - p(p+2)^\alpha - (q-p)(q-p+2)^\alpha \\ &\quad -(2^\alpha + 2^\alpha)\{[(q-p+2)^\alpha - 2^\alpha] - [(q+2)^\alpha - (p+2)^\alpha]\} \\ &= (q + 2^{\alpha+1})(q+2)^\alpha - (p + 2^{\alpha+1})(p+2)^\alpha \\ &\quad -(q - p + 2^{\alpha+1})(q-p+2)^\alpha + 2 \cdot 4^\alpha. \end{aligned}$$

The last inequality holds since $d_T(u_{i-1}) \le 2$ and the function $f(x) := (x+q-p)^\alpha - x^\alpha$ is strictly decreasing for $x \ge 0$ when $0 < \alpha < 1$.

Let $F(x) = (q+2^{\alpha+1})(q+2)^\alpha - (x+2^{\alpha+1})(x+2)^\alpha - (q-x+2^{\alpha+1})(q-x+2)^\alpha + 2\cdot 4^\alpha$, then

$$\begin{aligned} \frac{dF(x)}{dx} &= (q-x+2)^{\alpha-1}[(q-x+2) + \alpha(q-x+2^{\alpha+1})] \\ &\quad -(x+2)^{\alpha-1}[(x+2) + \alpha(x+2^{\alpha+1})]. \end{aligned}$$

By Lemma 3, we have

$$\frac{dF(x)}{dx} \begin{cases} > 0 & \text{if } 0 \le x < \frac{q}{2}; \\ = 0 & \text{if } x = \frac{q}{2}; \\ < 0 & \text{if } \frac{q}{2} < x \le q. \end{cases}$$

Hence for $0 \le x \le q$, $F(x)$ reaches its maximum at $x = \frac{q}{2}$ and its minimum at $x = 0$ or $x = q$. Thus, for $0 < p < q$, $F(p) > \max\{F(0), F(q)\} = 0$.

This completes the proof of the lemma. ∎

Corollary 2 *Let $r \ge 3$ and $T \in \mathcal{T}_k^*(n, r)(k \ge 2)$. Then for $0 < \alpha < 1$, there exists $T' \in \mathcal{T}_1^*(n, r)$ such that $R_\alpha(T') > R_\alpha(T)$.*

Proof. It follows immediately from Lemma 4. ∎

We now can prove the main result.

Theorem 1 *Let $r \geq 3$ and $T \in \mathcal{T}(n, r)$. Then for $0 < \alpha < 1$, we have*

$$R_\alpha(T) \leq \begin{cases} (n + 2^\alpha - 3)(n-2)^\alpha + 2^\alpha & \text{if } r = 3; \quad (1) \\ (n - r - 1 + 2^{\alpha+1})(n - r + 1)^\alpha + 4^\alpha(r-4) + 2^{\alpha+1} & \text{if } r \geq 4. \quad (2) \end{cases}$$

Equalities hold in (1) and (2) if and only if $T \cong T_i(n, 3, n-4)$, $i = 1, 2$ and $T \in \mathcal{T}_1^*(n, r) \setminus \{T_1(n, r, n-r-1), T_{r-1}(n, r, n-r-1)\}$, respectively.

Proof. For any $T \in \mathcal{T}(n, 3)$, we have $T \in \mathcal{T}^*(n, 3)$. Since $R_\alpha(T_1(n, 3, n-4)) = R_\alpha(T_2(n, 3, n-4)) = (n + 2^\alpha - 3)(n-2)^\alpha + 2^\alpha$, it follows from Corollary 2 that (1) holds and the equality holds if and only if $T \cong T_i(n, 3, n-4)$, $i = 1, 2$.

Now we consider the case $r \geq 4$. By Corollaries 1 and 2, for any $T \in \mathcal{T}(n, r)$, there exists $T' \in \mathcal{T}_1^*(n, r)$ such that $R_\alpha(T') \geq R_\alpha(T)$. By an elementary calculation, we have $R_\alpha(T_i(n, r, n-r-1)) = R_\alpha(T_j(n, r, n-r-1)) = (n - r - 1 + 2^{\alpha+1})(n - r + 1)^\alpha + 4^\alpha(r-4) + 2^{\alpha+1} > (n - r + 2^\alpha)(n - r + 1)^\alpha + (r-3)4^\alpha + 2^\alpha = R_\alpha(T_1(n, r, n-r-1)) = R_\alpha(T_{r-1}(n, r, n-r-1))$ if $2 \leq i, j \leq r-2$. Therefore (2) holds and the equality holds if and only if $T \in \mathcal{T}_1^*(n, r) \setminus \{T_1(n, r, n-r-1), T_{r-1}(n, r, n-r-1)\}$. ∎

3 Remarks

In Section 2, we have considered the maximum general Randić index on trees with a given order and diameter for $0 < \alpha < 1$. We remark in this section that by using the similar method, we can also get the minimum general Randić index on trees in $\mathcal{T}(n, r)$ for $-1 \leq \alpha < 0$. (We omit the proofs of Lemmas 5 and 6.)

Lemma 5 *Let $T \in \mathcal{T}(n, r)$. Then for $-1 \leq \alpha < 0$, there exists $T' \in \mathcal{T}^*(n, r)$ such that $R_\alpha(T') < R_\alpha(T)$.*

Lemma 6 *Let $r \geq 3$ and $T \in \mathcal{T}_k^*(n, r) (k \geq 2)$. Then for $-1 \leq \alpha < 0$, there exists $T' \in \mathcal{T}_1^*(n, r)$ such that $R_\alpha(T') < R_\alpha(T)$.*

Theorem 2 *Let $r \geq 3$ and $T \in \mathcal{T}(n,r)$. Then for $-1 \leq \alpha < 0$,*

$$R_\alpha(T) \geq (n-r+2^\alpha)(n-r+1)^\alpha + 4^\alpha(r-3) + 2^\alpha.$$

Equality holds if and only if $T \cong T_1(n,r,n-r-1)$ or $T \cong T_{r-1}(n,r,n-r-1)$.

Proof. It follows from Lemmas 5 and 6 that for any $T \in \mathcal{T}(n,r)$, there exists $T' \in \mathcal{T}_1^*(n,r)$ such that $R_\alpha(T') \leq R_\alpha(T)$. If $r = 3$, then $R_\alpha(T_1(n,3,n-4)) = R_\alpha(T_2(n,3,n-4)) = (n-3+2^\alpha)(n-2)^\alpha + 2^\alpha$. So we may assume that $r \geq 4$.

Now by an elementary calculation, $R_\alpha(T_i(n,r,n-r-1)) = R_\alpha(T_j(n,r,n-r-1)) = (n-r+2^\alpha)(n-r+1)^\alpha + 4^\alpha(r-3) + 2^\alpha < (n-r+2^{\alpha+1}-1)(n-r+1)^\alpha + (r-4)4^\alpha + 2^{\alpha+1} = R_\alpha(T_1(n,r,n-r-1)) = R_\alpha(T_{r-1}(n,r,n-r-1))$ if $2 \leq i,j \leq r-2$. Hence $R_\alpha(T) \geq (n-r+2^\alpha)(n-r+1)^\alpha + 4^\alpha(r-3) + 2^\alpha$ holds with equality if and only if $T \cong T_1(n,r,n-r-1)$ or $T \cong T_{r-1}(n,r,n-r-1)$. ∎

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