Journal of Symbolic Computation 43 (2008) 895–907

Contents lists available at ScienceDirect

Journal of Symbolic Computation

journal homepage: www.elsevier.com/locate/jsc

Proving hypergeometric identities by numerical verifications

Qiang-Hui Guo [1], Qing-Hu Hou, Lisa H. Sun

Center for Combinatorics, LPMC-TJKLC, Nankai University, Tianjin 300071, PR China

ARTICLE INFO

Article history:
Received 14 April 2007
Accepted 16 May 2008
Available online 3 June 2008

Keywords:
Hypergeometric identities
Numerical verification
Degree
Height

ABSTRACT

It is known that proper and q-proper hypergeometric identities can be certified by checking a finite number, say n_1, of initial values. By studying the degree and the height of the determinant of a polynomial matrix, we give a new method to estimate n_1. Examples show that the new estimates are considerably smaller than the previous results.

1. Introduction

It was Zeilberger (1981, 1982) who first realized that it is possible to prove hypergeometric identities by numerical verifications. His original idea was to contend that one can verify an identity by checking it for only a finite number of cases, say for $n \leq n_1$. Here we adopt the notion n_1 as used originally by Yen (1996a). Of course, for a given integer n, the identity is immediately verified, so the question is to find an upper bound of n_1. The existence of n_1 had been established by Yen (1993, 1996a), in which she gave the first *a priori* estimates of n_1 for hypergeometric identities. However her estimates are too large to be practical-sized computations (Petkovsek et al., 1996). She also gave the estimates of n_1 for q-hypergeometric identities in (Yen, 1996b), which was later spectacularly reduced by Zhang and Li (2003) and Zhang (2003). Our goal is to derive sharper bounds so that the verifications become feasible compared with the previous estimations.

First of all, let us recall the definitions of hypergeometric terms and proper hypergeometric terms. As usual, we denote the set of integers and rational numbers by $\mathbb{Z}$ and $\mathbb{Q}$, respectively. Let K be a field of characteristic zero. A function f from $D \subseteq \mathbb{Z}$ to K is called a *hypergeometric term* if there exists a rational function $r(x)$ of x over K such that $f(n+1)/f(n) = r(n)$ for all integers $n \in D$. Similarly, a

E-mail addresses: guo@nankai.edu.cn (Q.-H. Guo), hou@nankai.edu.cn (Q.-H. Hou), sun@cfc.nankai.edu.cn (L.H. Sun).

[1] Tel.: +86 22 23505133; fax: +86 22 23509272.

0747-7171/$ – see front matter
doi:10.1016/j.jsc.2008.05.003

bivariate function F from $D \subseteq \mathbb{Z}^2$ to K is called a (*bivariate*) *hypergeometric term* if $F(n+1,k)/F(n,k)$ and $F(n,k+1)/F(n,k)$ are both bivariate rational functions over K. For our purpose, we focus on the (*bivariate*) *proper hypergeometric terms* over $\mathbb{Q}$ which can be written in the form

$$F(n,k) = P(n,k)\frac{\prod_{i=1}^{uu}(c_i)_{a_in+b_ik}}{\prod_{i=1}^{vv}(w_i)_{u_in+v_ik}}x^k,$$

where $P(n,k)$ is a bivariate polynomial over $\mathbb{Q}$, uu and vv are specific non-negative integers, $a_i, b_i, u_i, v_i \in \mathbb{Z}$, $c_i, w_i, x \in \mathbb{Q}$ and $(a)_k = a(a+1)\cdots(a+k-1)$ denotes the raising factorial. Moreover, we require that $F(n,k)$ does not depend on other parameters.

Now we are ready to summarize the idea of estimating n_1. Suppose that we aim to prove an identity of the form

$$\sum_k F(n,k) = f(n), \quad n \geq n_0, \tag{1}$$

where n_0 is an integer, $F(n,k)$ is a proper hypergeometric term over $\mathbb{Q}$, and $f(n)$ is a hypergeometric term.

It is well known (e.g. Petkovsek et al. (1996)) that any proper hypergeometric term $F(n,k)$ satisfies a non-trivial telescoped recurrence:

$$\sum_{i=0}^{L} a_i(n)F(n+i,k) = R(n,k+1)F(n,k+1) - R(n,k)F(n,k), \tag{2}$$

where L is a non-negative integer, the $a_i(n)$'s are (not all zero) polynomials depending only on n and $R(n,k)$ is a bivariate rational function over K. Suppose that $F(n,k)$ has *finitely supports*, i.e., for each $n \in \mathbb{Z}$, there are only finitely many $k \in \mathbb{Z}$ such that $F(n,k) \neq 0$. Then summing over k on both sides of (2), we obtain that $S(n) = \sum_k F(n,k)$ satisfies a non-trivial linear recurrence with polynomial coefficients $a_i(n)$:

$$\sum_{i=0}^{L} a_i(n)S(n+i) = 0, \quad n \geq n_0. \tag{3}$$

If $a_L(n) \neq 0$ when $n > n_a$, we have

$$S(n+L) = -\frac{1}{a_L(n)}\sum_{i=0}^{L-1} a_i(n)S(n+i), \quad \text{for all } n \geq n_a' = \max\{n_a+1, n_0\}.$$

Then $S(n)$ is completely determined by its initial values: $S(n_0), S(n_0+1), \ldots, S(n_a'+L-1)$. Once proving that $f(n)$ satisfies the same recurrence and $S(n)$ agrees with $f(n)$ on these initial values, we immediately have $S(n) = f(n)$ for all $n \geq n_0$.

To verify that $f(n)$ and $S(n)$ satisfy the same recurrence is to show that $\sum_{i=0}^{L} a_i(n)f(n+i) = 0$, or equally

$$R(n) = \sum_{i=0}^{L} a_i(n)\frac{f(n+i)}{f(n)} = 0.$$

Notice that $R(n)$ is a rational function of n, as $f(n)$ is hypergeometric. Suppose that the degree of the numerator polynomial of $R(n)$ is less than or equal to n_f. Then $R(n)$ is identical to 0 if and only if $R(n) = 0$ holds for $n = n_0, n_0+1, \ldots, n_0+n_f$, which can be verified by checking that $S(n) = f(n)$ for $n = n_0, \ldots, n_0+n_f+L$.

Now let $n_1 = \max\{n_a'+L-1, n_0+n_f+L\}$, then we can safely claim that identity (1) holds for all $n \geq n_0$ if and only if it holds for $n = n_0, \ldots, n_1$.

Among the above three numbers L, n_a and n_f, n_a is the most difficult to estimate and in most cases the largest. Yen used the following method to estimate n_a. First use Sister Celine's method (Fasenmyer,

1945) to obtain a system of linear equations in some unknowns related to the $a_i(n)$'s, then apply Cramer's rule to the system formally and thus get estimates for the degree d and the largest absolute value m of the coefficients of $a_i(n)$ (usually called the *height*), finally take md as n_a by Proposition 3.6 in Yen (1996a).

Different from Yen's method, we use the techniques given by Mohammed and Zeilberger (2005) rather than Sister Celine's method. The resulting system of linear equations is directly in the unknowns $a_0(n), \ldots, a_L(n)$ and is of smaller size. In addition, we dig out more information from the concrete linear equations so that the estimates for the height become more accurate. This method sharply reduces the estimates of n_1 for hypergeometric identities, and is also applicable to the q-hypergeometric cases.

Now we take a simple example to illustrate our method. Let us consider the identity

$$\sum_k k\binom{n}{k} = n2^{n-1}, \quad n \geq 0.$$

By the **Theorem** in Mohammed and Zeilberger (2005), there exist polynomials $a_0(n)$, $a_1(n)$, $x_0(n)$, $x_1(n)$ such that

$$a_0(n)F(n,k) + a_1(n)F(n+1,k) - G(n,k+1) + G(n,k) = 0, \quad n \geq 0,$$

where $G(n,k) = (x_0(n) + x_1(n)k)\binom{n}{k-1}$. Simplifying the above equation and equating to zero the coefficients of each power of k, we get the following system of linear equations after eliminating the common factor $-(n+1)$ in the third equation.

$$\begin{bmatrix} 1 & 0 & 0 & -2 \\ n+1 & n+1 & 2 & -n \\ 0 & 0 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} a_0(n) \\ a_1(n) \\ x_0(n) \\ x_1(n) \end{bmatrix} = 0.$$

By Cramer's rule, we know that

$$[a_0(n), a_1(n), x_0(n), x_1(n)]^{\mathrm{T}}$$
$$= \left[\begin{vmatrix} 2 & 0 & 0 \\ n & n+1 & 2 \\ -1 & 0 & 1 \end{vmatrix}, \begin{vmatrix} 1 & 2 & 0 \\ n+1 & n & 2 \\ 0 & -1 & 1 \end{vmatrix}, \begin{vmatrix} 1 & 0 & 2 \\ n+1 & n+1 & n \\ 0 & 0 & -1 \end{vmatrix}, \begin{vmatrix} 1 & 0 & 0 \\ n+1 & n+1 & 2 \\ 0 & 0 & 1 \end{vmatrix} \right]^{\mathrm{T}}$$

is a non-trivial solution to the above system. Instead of computing these determinants, we estimate their degrees and heights using only the degree and the height of each entry. Theorem 4 tells us that $\deg(a_i(n)) \leq 1, i = 0, 1$ and their heights are less than or equal to 5 (in fact $a_0(n) = 2n+2$ and $a_1(n) = -n$). Noting that the $a_i(n)$'s are polynomials with integer coefficients, Lemma 1 will guarantee that taking $n_a = 5$ is large enough. Noting further that $n_f = 2$ and $L = 1$, we finally get $n_1 = 6$.

This article is organized as follows. We first prove some basic properties on the degree and the height of polynomials and polynomial determinants. Next we provide an algorithm for computing the upper bounds on the degree and the height of the polynomial solution to a system of linear equations. Then we describe the new method of estimating n_1 for hypergeometric identities. Finally, the q-analogue is discussed.

2. The degree and the height of polynomials

For polynomials with integer coefficients, we have the following lemma by the polynomial remainder theorem.

Lemma 1. *Let $P(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_tx^t \in \mathbb{Z}[x]$ be a polynomial in x and $n \geq t$, $a_t \neq 0$. If x_0 is a non-zero integer root of $P(x)$, then $x_0|a_t$. Consequently, for any integer $x > |a_t|$, $P(x) \neq 0$.*

Recall that the largest absolute value of the coefficients of a polynomial P is called the height of P, denoted by $|P|$. Then Lemma 1 states that we may take $n_a = |a_L(n)|$ provided that $a_L(n) \in \mathbb{Z}[n]$. On the other hand, the estimate of n_f depends on the degrees of the $a_i(n)$'s. Therefore, we firstly introduce some properties of the degree and the height of polynomials. We denote the degree of a polynomial $P(x)$ by $\deg(P)$ and define $\deg(0) = -\infty$.

Lemma 2. *Let K be a field of characteristic zero and $P_1, P_2, \ldots, P_n \in K[x]$ be polynomials in x with degrees $d_1, d_2, \ldots, d_n$, respectively. Then*

(1) $\deg(P_1 + P_2 + \cdots + P_n) \le \max\{d_1, d_2, \ldots, d_n\}$;
(2) $\deg(P_1 P_2 \cdots P_n) = d_1 + d_2 + \cdots + d_n$.

Suppose further that K is a subfield of $\mathbb{C}$, the field of complex numbers, and the heights of $P_1, \ldots, P_n$ are $h_1, \ldots, h_n$, respectively. Then

(3) $|P_1 + P_2 + \cdots + P_n| \le h_1 + h_2 + \cdots + h_n$;
(4) $|P_1 P_2 \cdots P_n| \le \prod_{i=1}^{n-1} (\min\{\sum_{j=1}^{i} d_j,\ d_{i+1}\} + 1) \cdot \prod_{i=1}^{n} h_i$.

Proof. The first three assertions are obvious, we thereby only prove the fourth.

First, let us consider the case of $n = 2$. Suppose that $P_1 = \sum_{i=0}^{d_1} a_i x^i$ and $P_2 = \sum_{j=0}^{d_2} b_j x^j$. Then $P_1 P_2 = \sum_{k=0}^{d_1+d_2} c_k x^k$ with $c_k = \sum_{i+j=k} a_i b_j$. Noting that in the expression of c_k, the number of terms in the summation is not greater than $\min\{d_1, d_2\} + 1$, we have, for each k, $|c_k| \le (\min\{d_1, d_2\} + 1)\, h_1 h_2$. Thus,

$$|P_1 P_2| = \max_{0 \le k \le d_1 + d_2} \{|c_k|\} \le (\min\{d_1, d_2\} + 1)\, h_1 h_2.$$

The conclusion follows immediately by induction on n. □

Denote by H the right-hand side of the inequality in assertion (4) of Lemma 2. Note that H depends on the order of P_i's degrees, while $|P_1 P_2 \cdots P_n|$ is free of that order. With $d_1, \ldots, d_n$ fixed, for any permutation π of $\{1, 2, \ldots, n\}$, let $H_\pi = D_\pi \cdot \prod_{i=1}^{n} h_i$, where

$$D_\pi = \prod_{i=1}^{n-1} \left(\min \left\{ \sum_{j=1}^{i} d_{\pi(j)},\ d_{\pi(i+1)} \right\} + 1 \right).$$

Clearly, the minimum of H_π among all the permutations will be large enough to be an upper bound of $|P_1 P_2 \cdots P_n|$. The following lemma tells how to obtain the minimum.

Lemma 3. *Given non-negative integers $d_1, \ldots, d_n$, D_π is minimal when $d_{\pi(1)} \ge d_{\pi(2)} \ge \cdots \ge d_{\pi(n)}$.*

Proof. Suppose that there exist two consecutive terms d_i and d_{i+1} with $d_i < d_{i+1}$. Consider the identical permutation $\mathbf{1}$ and the transposition $\tau = (i, i+1)$. By definition, we have $D_{\mathbf{1}} = \prod_{j=1}^{n-1} b_j$, where

$$\begin{aligned} b_1 &= \min\{d_1,\ d_2\} + 1, \\ &\ \vdots \\ b_{i-1} &= \min\{d_1 + d_2 + \cdots + d_{i-1},\ d_i\} + 1, \\ b_i &= \min\{d_1 + d_2 + \cdots + d_i,\ d_{i+1}\} + 1, \\ &\ \vdots \\ b_{n-1} &= \min\{d_1 + d_2 + \cdots + d_{n-1},\ d_n\} + 1. \end{aligned}$$

Similarly, we have $D_\tau = \prod_{j=1}^{n-1} b'_j$ with $b'_j = b_j$ except for the following two terms:

$$\begin{aligned} b'_{i-1} &= \min\{d_1 + d_2 + \cdots + d_{i-1},\ d_{i+1}\} + 1, \\ b'_i &= \min\{d_1 + d_2 + \cdots + d_{i-1} + d_{i+1},\ d_i\} + 1 = d_i + 1. \end{aligned}$$

There are three cases:
Case 1. If $d_1 + d_2 + \cdots + d_{i-1} \le d_i < d_{i+1}$, then

$$b_{i-1} = b'_{i-1} = d_1 + d_2 + \cdots + d_{i-1} + 1$$

and

$$b_i \ge \min\{d_i,\ d_{i+1}\} + 1 = d_i + 1 = b'_i.$$

Thus we have $D_{\mathbf{1}} \ge D_\tau$.
Case 2. If $d_i < d_1 + d_2 + \cdots + d_{i-1} \le d_{i+1}$, then $b_{i-1} = b'_i = d_i + 1$ and

$$b_i \ge \min\{d_1 + d_2 + \cdots + d_{i-1},\ d_{i+1}\} + 1 = b'_{i-1}.$$

We also have $D_{\mathbf{1}} \ge D_\tau$.
Case 3. If $d_i < d_{i+1} < d_1 + d_2 + \cdots + d_{i-1}$, then

$$b_{i-1} = b'_i = d_i + 1 \quad \text{and} \quad b_i = b'_{i-1} = d_{i+1} + 1,$$

implying that $D_{\mathbf{1}} = D_\tau$.

Summarizing, D_π will not increase if we exchange two consecutive ascend terms. Therefore, D_π is minimal when $d_{\pi(1)} \ge d_{\pi(2)} \ge \cdots \ge d_{\pi(n)}$. □

As a simple example, let $P_1(x) = x + 1,\ P_2(x) = x^2 + 2,\ P_3(x) = x^3 + 1$. Then $d_1 = 1,\ d_2 = 2,\ d_3 = 3$ and hence $D_{123} = D_{213} = 8,\ D_{132} = D_{312} = D_{321} = D_{231} = 6$. We see that D_{321} is minimal in the set $\{D_\pi\}$.

Denote the minimum of H_π by $\operatorname{minh}(P_1, \ldots, P_n)$, where π runs over all permutations. Combining Lemmas 2 and 3, we derive upper bounds on the degree and the height of a determinant whose entries are polynomials.

Theorem 4. *Let K be a field of characteristic zero and $M = [p_{ij}(x)]_{n\times n}$ be a matrix whose entries are polynomials in $K[x]$. Then* $\det(M)$ *is also a polynomial in $K[x]$. Denote the set of permutations of $\{1, 2, \ldots, n\}$ by S_n. Then the degree of* $\det(M)$ *is bounded by*

$$\deg(\det(M)) \le \max\{d_{1,\pi(1)} + d_{2,\pi(2)} + \cdots + d_{n,\pi(n)} \mid \pi \in S_n\} \triangleq \mathcal{D}(M), \tag{4}$$

where $d_{i,j} = \deg(p_{ij}(x))$. Suppose further that $K \subseteq \mathbb{C}$. Then the height of $\det(M)$ *is bounded by*

$$|\det(M)| \le \sum_{\pi\in S_n} \operatorname{minh}\left(p_{1,\pi(1)}(x), p_{2,\pi(2)}(x), \ldots, p_{n,\pi(n)}(x)\right) \triangleq \mathcal{H}(M). \tag{5}$$

For example, let

$$M = [p_{ij}(x)]_{3\times 3} = \begin{bmatrix} 1 & 0 & -2 \\ x+1 & -x & 2 \\ 0 & 1 & 1 \end{bmatrix}.$$

We have

$$\max\{d_{1,\pi(1)} + d_{2,\pi(2)} + d_{3,\pi(3)} \mid \pi \in S_3\} = 1,$$
$$\sum_{\pi\in S_3} \operatorname{minh}\left(p_{1,\pi(1)}(x),\ p_{2,\pi(2)}(x),\ p_{3,\pi(3)}(x)\right) = 5.$$

In fact, we have $\det(M) = -3x - 4$, whose degree and height are 1 and 4, respectively. Clearly, inequalities (4) and (5) hold.

Note that $\mathcal{D}(M)$ depends only on the degrees of the entries and $\mathcal{H}(M)$ depends on the degrees and the heights of the entries. Therefore, we need only play with numbers to obtain these two upper bounds, which makes the computation faster than calculating $\det(M)$ explicitly. However, since we need to run over all permutations in the computation, it becomes slow when the order of M is large. To get rid of that, we may consider a larger degree bound

$$\mathcal{D}'(M) = \sum_{i=1}^{n} \max\{d_{i,1}, \ldots, d_{i,n}\}.$$

3. The degree-height bound algorithm

Let D be an integral domain and $M \in D^{l\times m}$ an $l \times m$ $(l < m)$ matrix of rank ρ. We refine Yen's idea in Yen (1996a) to find a non-trivial solution to the system of linear equations $M\mathbf{x} = 0$, where $\mathbf{x}$ is the column vector $[x_1, x_2, \ldots, x_m]^{\mathrm{T}}$.

1. Permute the rows and columns of M, so that the submatrix $\widetilde{M}$ formed by the first ρ rows and the first ρ columns is of rank ρ. Meanwhile, permute the unknowns x_j's accordingly to the column permutation. Denote the new system by $M'\mathbf{x}' = 0$.
2. Let $\mathbf{y}$ be the column vector formed by the first ρ rows of the $(\rho + 1)$th column of M' and let $\tilde{\mathbf{x}} = [x'_1, \ldots, x'_\rho]^{\mathrm{T}}$. We now consider the inhomogeneous linear system $\widetilde{M}\tilde{\mathbf{x}} = -\mathbf{y}$.
3. Let $\widetilde{M}_i$ be the matrix obtained by replacing the ith column of $\widetilde{M}$ with $-\mathbf{y}$. By Cramer's rule, we have that

$$[\det \widetilde{M}_1 / \det \widetilde{M}, \det \widetilde{M}_2 / \det \widetilde{M}, \ldots, \det \widetilde{M}_\rho / \det \widetilde{M}]^{\mathrm{T}}$$

is a solution to $\widetilde{M}\tilde{\mathbf{x}} = -\mathbf{y}$. Hence,

$$[\det \widetilde{M}_1, \det \widetilde{M}_2, \ldots, \det \widetilde{M}_\rho, \det \widetilde{M}, 0, \ldots, 0]^{\mathrm{T}}$$

is a non-trivial solution to $M'\mathbf{x}' = 0$.
4. Corresponding to the column permutation in step 1, rearrange the elements of $\mathbf{x}'$. Then we finally obtain a non-trivial solution to the system $M\mathbf{x} = 0$.

From the above construction, we see that each entry of the solution is the determinant of a $\rho \times \rho$ submatrix of M up to a sign. Thus, when M is a polynomial matrix, the degrees and the heights of these determinants are bounded by Theorem 4. From the definition of $\mathcal{D}$ and $\mathcal{H}$, we immediately get the following two lemmas, which enable us to avoid testing all $\rho \times \rho$ submatrices of M.

Lemma 5. *Let K be a field of characteristic zero and $A = [p_{ij}(n)]$ a square matrix whose entries are non-zero polynomials in $K[n]$. Then for any square submatrix A' of A, we have $\mathcal{D}(A') \le \mathcal{D}(A)$. Suppose further that $p_{ij}(x) \in \mathbb{Z}[x]$ for all i, j. Then $\mathcal{H}(A') \le \mathcal{H}(A)$.*

Lemma 6. *Let K be a field of characteristic zero and $A = [p_{ij}(n)]$, $A' = [p'_{ij}(n)]$ be two square matrices of the same order whose entries are polynomials in $K[n]$. Suppose that $\deg(p'_{ij}) \le \deg(p_{ij})$ for all i, j. Then $\mathcal{D}(A') \le \mathcal{D}(A)$. Suppose further that $K \subseteq \mathbb{C}$ and $|p'_{ij}| \le |p_{ij}|$ for all i, j. Then $\mathcal{H}(A') \le \mathcal{H}(A)$.*

Lemma 5 converts the computation of $\rho \times \rho$ submatrices to $l \times l$ submatrices and enables us to avoid computing the rank ρ, which is equivalent to solving the system. Proper use of Lemma 6 further reduces the computation to one special $l \times l$ matrix. To this end, we define two kinds of transformations on a matrix.

Definition 7. Let K be a field of characteristic zero and $A = [p_{ij}(n)]$ be a matrix whose entries are polynomials in $K[n]$.

A 0-1 *augment* is the transformation of replacing each zero entry of A with 1. The resulting matrix is denoted by $\overline{A}$.

Suppose that $K \subseteq \mathbb{C}$. A *DH augment* is the transformation as follows: Choose a column, say the kth column, of the matrix A, replace those entries $p_{ij}(n)$ that satisfy $|p_{ij}| < |p_{ik}|$ or $\deg(p_{ij}) < \deg(p_{ik})$ with hn^d where $h = \max\{|p_{ij}|, |p_{ik}|\}$ and $d = \max\{\deg(p_{ij}), \deg(p_{ik})\}$, and finally delete the kth column from A. The resulting matrix is denoted by $\widehat{A}$.

Now we are ready to present the degree-height bound algorithm (**DHB algorithm** in short).
Input: An $l \times m$ $(l < m)$ matrix $M = [p_{ij}(n)]$ whose entries are polynomials in $\mathbb{Z}[n]$.
Output: Two integers d_a and h_a.

1. Repeat DH augment on M and set $M = \widehat{M}$ until M becomes an $l \times l$ square matrix.
2. Do 0-1 augment on M and set $M = \overline{M}$.
3. Return $d_a = \mathcal{D}(M)$, $h_a = \mathcal{H}(M)$.

Remark 8. Notice that the DH augment involves the selection of a special column. To obtain smaller bounds, we may take an *as small as possible* column, e.g., the one with minimum height sum or degree sum.

The following theorem shows that d_a and h_a are upper bounds on the degree and the height of a non-trivial solution to the system $M\mathbf{x} = 0$.

Theorem 9. *Suppose that $M = [p_{ij}(n)]$ is a matrix whose entries are polynomials in $\mathbb{Z}[n]$. Let d_a and h_a be the integers obtained by the DHB algorithm. Then there exists a non-trivial solution $\mathbf{x} = [x_1(n), x_2(n), \ldots, x_m(n)]^{\mathrm{T}}$ to the system $M\mathbf{x} = 0$ satisfying $\deg(x_i(n)) \le d_a$ and $|x_i(n)| \le h_a$ for all $i = 1, 2, \ldots, m$.*

Proof. For convenience, we denote the resulting matrices after step 1 and step 2 in the DHB algorithm by M_1 and M_2, respectively. Let A be any $l \times m$ $(l < m)$ matrix whose entries are polynomials in n. From the definition of DH augment, for each square submatrix $B = [p_{ij}(n)]$ of A, there exists a square submatrix $C = [q_{ij}(n)]$ of $\widehat{A}$ such that $\deg(p_{ij}) \le \deg(q_{ij})$ and $|p_{ij}| \le |q_{ij}|$. By Lemma 6, we have $\mathcal{D}(B) \le \mathcal{D}(C)$ and $\mathcal{H}(B) \le \mathcal{H}(C)$. Repeating the discussion, we derive that for each square submatrix B of M, there exists a square submatrix C of M_1 such that $\mathcal{D}(B) \le \mathcal{D}(C)$ and $\mathcal{H}(B) \le \mathcal{H}(C)$. One more step shows that this also holds for M_2.

Now by Lemma 5, for each square submatrix C of M_2, we have $\mathcal{D}(C) \le \mathcal{D}(M_2)$ and $\mathcal{H}(C) \le \mathcal{H}(M_2)$. Therefore, for each square submatrix B of M, we finally have

$$\mathcal{D}(B) \le \mathcal{D}(M_2) \quad \text{and} \quad \mathcal{H}(B) \le \mathcal{H}(M_2).$$

Since there exists a non-trivial solution to the system $M\mathbf{x} = 0$ represented by the determinants of submatrices of M up to a sign, the theorem follows. □

Sometimes we may need only the bound on part of the unknowns, say x_j's, $j \in S \subseteq \{1, 2, \ldots, m\}$. Then in the DHB algorithm we may use the following *partial DH augment* instead: Choose a column, say the kth $(k \in S)$ column, of the matrix A, replace those entries $p_{ij}(n)$ $(j \in S)$ that satisfy $|p_{ij}| < |p_{ik}|$ or $\deg(p_{ij}) < \deg(p_{ik})$ with $h \cdot n^d$ where $h = \max\{|p_{ij}|, |p_{ik}|\}$ and $d = \max\{\deg(p_{ij}), \deg(p_{ik})\}$, delete the kth column from A, and finally also denote the resulting matrix by $\widehat{A}$. Consequently, we call the resulting algorithm the *partial DHB algorithm*.

Notice that d_a and h_a obtained by the partial DHB algorithm are the upper bounds for the degree and the height of the x_j's with $j \in S$ which is a part of a non-trivial solution to the system $M\mathbf{x} = 0$. It can be proved by the same discussion as in Theorem 9.

4. Estimating n_1 for hypergeometric identities

We adopt the notation in Mohammed and Zeilberger (2005) to write a proper hypergeometric term over $\mathbb{Q}$ as

$$F(n, k) = POL(n, k) \cdot H(n, k), \tag{6}$$

with $POL(n, k)$ being a polynomial in n and k, and

$$H(n, k) = \frac{\prod_{j=1}^{A} (a''_j)_{a'_j n + a_j k} \prod_{j=1}^{B} (b''_j)_{b'_j n - b_j k}}{\prod_{j=1}^{C} (c''_j)_{c'_j n + c_j k} \prod_{j=1}^{D} (d''_j)_{d'_j n - d_j k}} z^k,$$

where $a_j, a'_j, b_j, b'_j, c_j, c'_j, d_j, d'_j$ are non-negative integers, and $z, a''_j, b''_j, c''_j, d''_j \in \mathbb{Q}$.

Let

$$L = \max\left\{ \sum_{j=1}^{A} a_j + \sum_{j=1}^{D} d_j, \quad \sum_{j=1}^{B} b_j + \sum_{j=1}^{C} c_j \right\}. \tag{7}$$

Mohammed and Zeilberger (2005) showed that there exist polynomials $e_0(n), \ldots, e_L(n)$ in n and a rational function $R(n, k)$ such that $G(n, k) = R(n, k)F(n, k)$ satisfies

$$\sum_{i=0}^{L} e_i(n)F(n+i, k) = G(n, k+1) - G(n, k). \tag{8}$$

More precisely, let

$$\overline{H}(n, k) = \frac{\prod_{j=1}^{A}(a_j'')_{a_j'n+a_jk} \prod_{j=1}^{B}(b_j'')_{b_j'n-b_jk}}{\prod_{j=1}^{C}(c_j'')_{c_j'(n+L)+c_jk} \prod_{j=1}^{D}(d_j'')_{d_j'(n+L)-d_jk}} z^k, \tag{9}$$

$$u(k) = z\prod_{j=1}^{A}(a_j'n + a_jk + a_j'')_{a_j} \prod_{j=1}^{D}(d_j'(n+L) - d_jk + d_j'' - d_j)_{d_j}, \tag{10}$$

$$v(k) = \prod_{j=1}^{B}(b_j'n - b_jk + b_j'' - b_j)_{b_j} \prod_{j=1}^{C}(c_j'(n+L) + c_jk + c_j'')_{c_j}. \tag{11}$$

Let further

$$h(k) = \sum_{i=0}^{L} e_i(n) POL(n+i, k) \cdot \frac{H(n+i, k)}{\overline{H}(n, k)} \tag{12}$$

and $X(k) = \sum_{i=0}^{m} x_i(n)k^i$, where $m = \deg h - \max\{\deg u, \deg v\}$, $x_i(n)$ and $e_i(n)$ are unknown polynomial expressions in n that have to be determined. Note that $h(k)$ is a polynomial in k. Then there is a non-trivial solution $e_0(n), \ldots, e_L(n), x_0(n), \ldots, x_m(n)$ to the equation

$$u(k)X(k+1) - v(k-1)X(k) - h(k) = 0. \tag{13}$$

Moreover, Eq. (8) holds for $G(n, k) = v(k-1)X(k)\overline{H}(n, k)$.

The results of Mohammed and Zeilberger do not ensure that $e_0(n), \ldots, e_L(n)$ are not all zeros, however, this bad situation rarely happens and can be prejudged. Suppose that $e_0(n), \ldots, e_L(n)$ are all zeros, then $h(k) = 0$ but $X(k) \neq 0$. By (13), we have

$$\frac{v(k-1)}{u(k)} = \frac{X(k+1)}{X(k)}. \tag{14}$$

It is well known that the rational function $v(k-1)/u(k)$ has a unique Gopser–Petkovšek representation (Petkovšek, 1992) (GP representation, in short):

$$\frac{v(k-1)}{u(k)} = \frac{a(k)}{b(k)} \frac{c(k+1)}{c(k)}.$$

In most cases, $a(k)/b(k) \neq 1$, which implies that Eq. (14) does not hold. Under this situation, we must have that $e_0(n), \ldots, e_L(n)$ are non-trivial, i.e., not all zeros.

Up to now, we have already L. The rest task is to estimate n_a and n_f.

Notice that Eq. (13) is actually a system of linear equations in the unknowns $e_0(n), \ldots, e_L(n)$ and $x_0(n), \ldots, x_m(n)$, which can be written as $M\mathbf{x} = 0$. By multiplying the common denominators, we may assume that each entry of M is a polynomial in $\mathbb{Z}[n]$. Thus there exists a non-trivial solution whose elements are all polynomials in $\mathbb{Z}[n]$. Now apply the partial DHB algorithm to the matrix M. The output integers d_a and h_a are the upper bounds of the degrees and the heights of the $e_i(n)$'s.

Now suppose that $F(n, k)$ has finitely supports. Summing over k on both sides of Eq. (8) leads to a recurrence satisfied by $S(n) = \sum_k F(n, k)$:

$$\sum_{i=0}^{L} e_i(n)S(n+i) = 0. \tag{15}$$

Let $e_{L'}(n)$ be the last non-zero polynomial reading from e_0 to e_L. Since $e_{L'}(n)$ is a polynomial with integer coefficients, we have $e_{L'}(n) \neq 0$ for $n > h_a$ by Lemma 1. Therefore, we can take $n_a = h_a$.

Finally, n_f is given by an upper bound of the degree of the numerator polynomial of

$$R(n) = \sum_{i=0}^{L} e_i(n) \frac{f(n+i)}{f(n)}.$$

Let $D(n)$ be the common denominator of $f(n+i)/f(n)$ for $i = 0, \ldots, L$. Then the degree of the numerator polynomial of $R(n)$ is bounded by the largest degree of $e_i(n)$ (bounded by d_a computed above) plus the largest degree of $f(n+i)/f(n) \cdot D(n)$ for $i = 0, \ldots, L$.

In conclusion, we get the following algorithm on estimating n_1 for proper hypergeometric identities.

Input: A proper hypergeometric term $F(n,k)$ over $\mathbb{Q}$ with finitely supports, a hypergeometric term $f(n)$, and n_0.

Output: An integer n_1 such that $\sum_k F(n,k) = f(n)$ holds for $n \geq n_0$ if and only if it holds for $n = n_0, \ldots, n_1$.

1. Write $F(n,k)$ in the form of (6).
2. Compute L by (7).
3. Compute $\overline{H}(n,k)$, $u(k)$, $v(k)$ and $h(k)$ by (9)–(12).
4. Compute the GP representation of

   $$\frac{v(k-1)}{u(k)} = \frac{a(k)}{b(k)} \frac{c(k+1)}{c(k)}.$$

   If $a(k)/b(k) = 1$, the algorithm fails. Otherwise continue the following procedures.
5. Equate to zero the coefficient of each power of k in (13) to get a system of linear equations $M\mathbf{x} = 0$ in the unknowns $e_i(n)$, $0 \leq i \leq L$ and $x_j(n)$, $0 \leq j \leq m$.
6. Apply the partial DHB algorithm to M to get d_a and h_a.
7. Compute the common denominator $D(n)$ of $f(n+i)/f(n)$, $i = 0, \ldots, L$ and then find the largest degree d_f of $f(n+i)/f(n) \cdot D(n)$. Set $n_f = d_a + d_f$.
8. Return $n_1 = \max\{n'_a + L - 1, n_0 + n_f + L\}$, where $n'_a = \max\{h_a + 1, n_0\}$.

Let us look at two examples.

Example 10. Estimate n_1 for the identity

$$\sum_k \binom{n}{k} = 2^n, \quad \forall n \geq 0. \tag{16}$$

First write the summand $F(n,k) = \binom{n}{k}$ in the form of (6):

$$F(n,k) = \frac{(1)_n}{(1)_{n-k}(1)_k}.$$

A straightforward computation gives that $L = 1$, $u(k) = n - k + 1$ and $v(k) = k + 1$. Since the GP representation of $v(k-1)/u(k)$ is $k/(n-k+1) \neq 1$, we are ensured that $e_0(n)$, $e_1(n)$ are not both zeros. Now Eq. (13) becomes

$$(e_0(n) - 2x_0(n))k + (n+1)(x_0(n) - e_0(n) - e_1(n)) = 0.$$

Equating the coefficients of each power of k to 0 yields

$$\begin{bmatrix} 1 & 0 & -2 \\ -1 & -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} e_0(n) \\ e_1(n) \\ x_0(n) \end{bmatrix} = 0.$$

By the partial DHB algorithm in which we choose the second column in the DH augment, we get $d_a = 0$ and $h_a = 3$. Noting that $2^{n+j}/2^n = 2^j$ for $j = 0, 1$, we have $d_f = 0$. Finally, $n_1 = \max\{4, 1\} = 4$. It is a trivial task for a computer to check the four initial values.

Example 11. Estimate n_1 for

$$\sum_k \binom{n}{k}^2 = \binom{2n}{n}. \tag{17}$$

In this case we arrive at the following homogeneous linear system

$$\begin{bmatrix} n^2+2n+1 & n^2+2n+1 & n^2+2n+1 & -1 & -1 & -1 \\ 4n^2+10n+6 & 2n^2+4n+2 & 0 & -2 & n & 2n+2 \\ 6n^2+18n+13 & n^2+2n+1 & 0 & 0 & 2n+3 & -n^2+3 \\ 2n+3 & 0 & 0 & 0 & 0 & -n-1 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} e_0(n) \\ e_1(n) \\ e_2(n) \\ x_0(n) \\ x_1(n) \\ x_2(n) \end{bmatrix} = \mathbf{0}.$$

By the partial DHB algorithm in which the third column is chosen for the DH augment, we get $d_a = 6$ and $n'_a = 12\,089 + 1 = 12\,090$. Moreover, we have $d_f = 2$. Therefore $n_1 = \max\{n'_a + L - 1, n_0 + d_a + d_f + L\} = 12\,091$. Notice that it is still a hard task for a computer to check all this initial values.

Note that for the above two examples, the bounds given by Yen (1993) were 10^{11} and 10^{115} respectively.

5. Estimating n_1 for q-hypergeometric identities

In this section, K is a field of characteristic zero and q is transcendental over K. A function f from $D \subseteq \mathbb{Z}$ to the rational function field $K(q)$ is called a *q-hypergeometric term* if there exists a rational function $r(x)$ of x over $K(q)$ such that $f(n+1)/f(n) = r(q^n)$ for all integers $n \in D$. A bivariate function F from $D \subseteq \mathbb{Z}^2$ to $K(q)$ is called a (*bivariate*) *q-hypergeometric term* if there are bivariate rational functions $r(x, y)$ and $s(x, y)$ over $K(q)$ such that $F(n+1, k)/F(n, k) = r(q^n, q^k)$ and $F(n, k+1)/F(n, k) = s(q^n, q^k)$. Analogous to the ordinary case, we focus on the (*bivariate*) *q-proper hypergeometric terms* which can be written in the form

$$F(n,k) = P(q^n, q^k)\frac{\prod_{i=1}^{uu}[c_i]_{a_i n + b_i k}}{\prod_{i=1}^{vv}[w_i]_{u_i n + v_i k}} q^{Jk(k-1)/2} z^k,$$

where $P(q^n, q^k)$ is a bivariate Laurent polynomial in q^n, q^k over $K(q)$, uu and vv are specific non-negative integers, $a_i, b_i, u_i, v_i, J \in \mathbb{Z}$, $c_i, w_i, z \in K(q)$ and $[a]_k = (1-a)(1-aq)\cdots(1-aq^{k-1})$ denotes the q-shifted factorial.

Estimating n_1 for q-hypergeometric identities can be done just as the q-analogue of the ordinary case. In this case, we have a linear recurrence of the form

$$\sum_{i=0}^{L} a_i(q^n, q) S(n+i) = 0, \quad n \ge n_0,$$

where the $a_i(q^n, q)$'s are polynomials in q^n and q over K. The following lemma shows that n_a can be set to be the degree of $a_L(q^n, q)$ in q.

Lemma 12 (Yen (1996b)). *Let $P(q^n, q)$ be a non-zero polynomial in q^n and q over K. Suppose that the degree of $P(q^n, q)$ in q is m. Then $P(q^n, q) \neq 0$ for all $n \ge m+1$.*

Write a q-proper hypergeometric term $F(n, k)$ in the form

$$F(n,k) = POL(n,k)\frac{\prod_{j=1}^{A}[a''_j]_{a'_j n + a_j k}\prod_{j=1}^{B}[b''_j]_{b'_j n - b_j k}}{\prod_{j=1}^{C}[c''_j]_{c'_j n + c_j k}\prod_{j=1}^{D}[d''_j]_{d'_j n - d_j k}} q^{Jk(k-1)/2} z^k, \tag{18}$$

where $POL(n, k)$ is a Laurent polynomial in q^n and q^k over $K(q)$, $a_j, a'_j, b_j, b'_j, c_j, c'_j, d_j, d'_j$ are non-negative integers, $z, a''_j, b''_j, c''_j, d''_j \in K(q)$ and $J \in \mathbb{Z}$. Mohammed and Zeilberger (2005) also provided the q-**Theorem** which states that $F(n, k)$ has a telescoped recurrence

$$\sum_{i=0}^{L} e_i(q^n, q)F(n+i, k) = G(n, k+1) - G(n, k) \tag{19}$$

of order

$$L = \max\left\{J + \sum_{j=1}^{A} a_j^2, \quad \sum_{j=1}^{C} c_j^2\right\} + \max\left\{-J + \sum_{j=1}^{D} d_j^2, \quad \sum_{j=1}^{B} b_j^2\right\}. \tag{20}$$

Moreover, the coefficients $e_0(q^n, q), \ldots, e_L(q^n, q)$ and some extra unknowns $x_{-m_1}(q^n, q), \ldots, x_{m_2}(q^n, q)$ satisfy a system of linear equations $M\mathbf{x} = 0$, where the entries of M are polynomials in q^n and q.

Different from the ordinary case, in the q-case, we need the degree bound in q of the e_i's for the estimation of n_a by Lemma 12, and their degree bound in q^n for the estimation of n_f, while the height does not make sense in this case. So we are now in place to introduce the counterpart of the augment and DHB algorithm for the q-case.

Definition 13. Let $A = [p_{ij}(x)]$ be a matrix whose entries are polynomials in $K[x]$.

A *degree augment* is the transformation as follows: Choose a column, say the kth column, of the matrix A, replace those entries p_{ij} that satisfy $\deg(p_{ij}) < \deg(p_{ik})$ with x^d where $d = \deg(p_{ik})$, and finally delete the kth column from A. The resulting matrix is denoted by $\widehat{A}$.

q-DB Algorithm

Input: An $l \times m$ $(l < m)$ matrix $M = [p_{ij}(q^n, q)]$ whose entries are polynomials in q^n and q.
Output: Two integers n_a and d_a.

1. Let $M' = M$, regard the entries of M' as polynomials in q, then repeat degree augment on M' and set $M' = \widehat{M'}$ until M' becomes an $l \times l$ square matrix. Denote the resulting matrix by M_q.
2. Let $M' = M$, regard the entries of M' as polynomials in q^n, then repeat degree augment on M' and set $M' = \widehat{M'}$ until M' becomes an $l \times l$ square matrix. Denote the resulting matrix by M_{q^n}.
3. Do 0-1 augment on M_q and M_{q^n} respectively, and set $M_q = \overline{M_q}$, $M_{q^n} = \overline{M_{q^n}}$.
4. Return $n_a = \mathcal{D}(M_q)$, $d_a = \mathcal{D}(M_{q^n})$.

The counterpart of Theorem 9 for the q-case, which guarantees that the output n_a and d_a are degree bounds in q and q^n, can be deduced in a thorough similar way. So is the *partial q-DB algorithm*, the counterpart of partial DHB algorithm.

Now we give the following algorithm to estimate n_1 for the q-case.

Input: A q-proper hypergeometric term $F(n, k)$ with finitely supports, a q-hypergeometric term $f(n)$, and n_0.
Output: An integer n_1 such that $\sum_k F(n, k) = f(n)$ holds for $n \geq n_0$ if and only if it holds for $n = n_0, \ldots, n_1$.

1. Write $F(n, k)$ in the form of (18).
2. Compute L by (20).
3. Compute $\overline{H}(n, k)$, $u(q^k)$, $v(q^k)$ and $h(q^k)$ as in Mohammed and Zeilberger (2005).
4. Compute the q-analogue of the GP representation (Koornwinder, 1993) of

$$\frac{v(q^{k-1})}{u(q^k)} = \frac{a(q^k)}{b(q^k)} \frac{c(q^{k+1})}{c(q^k)}.$$

   If $a(q^k)/b(q^k) = 1$, the algorithm fails. Otherwise continue the following procedures.
5. By the Eq. (19), get a system of linear equations $M\mathbf{x} = 0$ in the unknowns $e_i(q^n, q)$, $0 \leq i \leq L$ and $x_j(q^n, q)$, $-m_1 \leq j \leq m_2$.

6. Apply the partial q-DB algorithm on the e_i's to M to get the bounds n_a and d_a.
7. Compute the common denominator $D(q^n)$ of $f(n+i)/f(n), i = 0, \ldots, L$ and then find the largest degree d_f of $f(n+i)/f(n) \cdot D(q^n)$. Set $n_f = d_a + d_f$.
8. Return $n_1 = \max\{n'_a + L - 1, n_0 + n_f + L\}$, where $n'_a = \max\{n_a + 1, n_0\}$.

Here are some examples.

Example 14. Estimate n_1 for a finite version of Jacobi's triple product identity (Andrews, 1976)

$$\sum_k \begin{bmatrix} 2n \\ n+k \end{bmatrix} q^{\binom{k}{2}} z^k = (-z^{-1}q; q)_n (-z; q)_n, \quad n \geq 0.$$

The summand can be expressed in the form of

$$F(n, k) = \frac{[1]_{2n}}{[1]_{n+k}[1]_{n-k}} q^{k(k-1)/2} z^k.$$

By the corresponding equation of (19), we get the following homogeneous linear system:

$$\begin{bmatrix} qq^n & 0 & zq^n + 1 & 0 \\ -q\left(1 + q^2 q^{2n}\right) & -q(q^3 q^{4n} - q^2 q^{2n} - qq^{2n} + 1) & -z - q^2 q^n & q\left(zqq^n + 1\right) \\ qq^n & 0 & 0 & -z - qq^n \end{bmatrix} \times \begin{bmatrix} e_0(q^n, q) \\ e_1(q^n, q) \\ x_{-1}(q^n, q) \\ x_0(q^n, q) \end{bmatrix} = 0.$$

Applying the partial q-DB algorithm in which we choose the second column for the degree augment, we get $n_a = 5$ and $d_a = 6$. Thus $n'_a = n_a + 1 = 6$. Furthermore, since $f(n+1)/f(n) = (1 + z^{-1}q^{n+1})(1 + zq^n)$, we get $d_f = 2$. Finally, we derive that $n_1 = 9$.

Note that in this example, $n_a = 5 < 6 = n_f$.

Example 15. Compute n_1 for the q-Chu-Vandermonde identity (cf. Gasper and Rahman (2004))

$$\sum_k q^{k^2} \begin{bmatrix} n \\ k \end{bmatrix}^2 = \begin{bmatrix} 2n \\ n \end{bmatrix}, \quad n \geq 0.$$

By the corresponding equation of (19), we get a system of linear equations $M\mathbf{x} = 0$, where M is a 5 by 6 matrix. Applying the partial q-DB algorithm, we get $n'_a = n_a + 1 = 24$ and $d_a = 13$. Simplifying $f(n+i)/f(n)$ leads to $d_f = 6$. Finally, we derive that $n_1 = 25$.

Example 16. Estimate n_1 for a finite form of Euler's pentagonal number theorem due to L. J. Rogers (e.g. Andrews (1976))

$$\sum_k \frac{(-1)^k (q; q)_n q^{k(3k-1)/2}}{(q; q)_{n+k} (q; q)_{n-k}} = 1, \quad n \geq 0.$$

The algorithm returns $n_1 = 42$.

For the above three examples, the bounds given by Zhang (2003) are 74, 191, 209, respectively.

Acknowledgements

We would like to thank professor W.Y.C. Chen for helpful discussions and suggestions, and the anonymous referees for valuable comments. This work was supported by the 973 Project, the PCSIRT Project of the Ministry of Education, the Ministry of Science and Technology, and the National Science Foundation of China.

References

Andrews, G.E., 1976. The Theory of Partitions. In: Encyclopedia of Mathematics and its Application, vol. 2. Addison-Wesley, Reading.
Fasenmyer, M.C., 1945. Some generalized hypergeometric polynomials. Ph.D. Thesis, University of Michigan.
Gasper, G., Rahman, M., 2004. Basic Hypergeometric Series, second ed. Cambridge University Press, Cambridge, MA.
Koornwinder, T.H., 1993. On Zeilberger's algorithm and its q-analogue. J. Comput. Appl. Math. 48, 91–111.
Mohammed, M., Zeilberger, D., 2005. Sharp upper bounds for the orders of the recurrences outputted by the Zeilberger and q-Zeilberger algorithms. J. Symbolic Comp. 39, 201–207.
Petkovšek, M., 1992. Hypergeometric solutions of linear recurrences with polynomial coefficients. J. Symbolic Comput. 14, 243–264.
Petkovsek, M., Wilf, H.S., Zeilberger, D., 1996. A = B. Wellesley, Massachusetts.
Yen, L., 1993. Contributions to the proof theory of hypergeometric identities. Ph.D. Thesis, University of Pennsylvania.
Yen, L., 1996a. A two-line algorithm for proving terminating hypergeometric identities. J. Math. Anal. Appl. 198, 856–878.
Yen, L., 1996b. A two-line algorithm for proving q-hypergeometric identities. J. Math. Anal. Appl. 213, 1–14.
Zeilberger, D., 1981. All binomial identities are verifiable. Proc. Natl. Acad. Sci. USA 78, 4000.
Zeilberger, D., 1982. Sister Celine's technique and its generalizations. J. Math. Anal. Appl. 85, 114–145.
Zhang, B.-Y., 2003. A new elementary algorithm for proving q-hypergeometric identities. J. Symbolic Comp. 35, 293–303.
Zhang, B.-Y., Li, J., 2003. An improvement of the two-line algorithm for proving q-hypergeometric identities. Discrete Math. 268, 273–286.