On Lattice Path Enumerations

Guoce Xin

March 16, 2004
A New Method on Lattice Path Enumerations

Based on the connection between

Algebra: The theory of iterated Laurent series:

   Unique factorization lemma.

and Combinatorics: The Concept of Gessel pair:

   Unique factorization lemma of paths.
Catalan numbers are the most frequently used numbers in combinatorics other than the binomial coefficients.

It has more than 66 interpretations. (EC2).

**Theorem 1 (Classical Result).** *The number of paths from* $(0,0)$ *to* $(n,n)$ *with either an up step or a right step and never go above the diagonal is the* $n$*-th Catalan number* $C_n$.

The $n$-th Catalan number is:

$$C_n = \frac{1}{n+1} \binom{2n}{n}.$$ 

The generating function of Catalan number is

$$c(x) = \sum_{n\geq0} C_n x^n = \frac{1 - \sqrt{1 - 4x}}{2x}.$$
An example of Catalan path
A Surprising Result

Denote by $\mathcal{H}$ the half line $\{(-k, 0); k \in \mathbb{N}\}$. Given a finite subset $\mathcal{G}$ of $\mathbb{Z}^2$, walks on the slit plane are paths that start at $(0, 0)$ with steps in $\mathcal{G}$ and never hit the half line $\mathcal{H}$ after the starting point.

**Theorem 2 (M. Bousquet-Mélou and G. Schaeffer, 2002).** The number of walks of length $2n + 1$ on the slit plane, with steps in $\{ (\pm 1, \pm 1) \}$ and ending at $(1, 0)$ is the Catalan number $C_{2n+1}$.
Slit Plane
The Field of Iterated Laurent Series

Let $K$ be a field. Then $K\langle x_1 \rangle$ is defined to be the field $K((x_1))$ of Laurent series in $x_1$. Inductively we define $K\langle x_1 \ldots x_m \rangle = K\langle x_1, x_2, \ldots, x_{m-1} \rangle((x_n))$.

**Proposition 3 (Fundamental structure).** A formal Laurent series in $x$ belongs to $K\langle x_1 \ldots x_m \rangle$ if and only if it has a well-ordered support.

**Definition 4.**

$$
\sum_{\substack{(i_1,\ldots,i_m)\in\mathbb{Z}^m \atop (i_1,\ldots,i_m)\in\mathbb{Z}^m,i_j=0}} a_{i_1,\ldots,i_m} x_1^{i_1} \cdots x_m^{i_m}
$$

where $a_{i_1,\ldots,i_m}$ belongs to $K$. 

Unique Factorization Lemma

Lemma 5 (Unique Factorization Lemma, Gessel and Bousquet-Mélou). Let $h(x,t)$ be an element in $K((x))[[t]]$, in which $h(x,0) = 1$. Then $h$ has a unique factorization in $K((x))[[t]]$ such that $h = h_- h_0 h_+$, where $h_- \in K[x^{-1}][[t]]$, $h_0 \in K[[t]]$, and $h_+ \in K[[x,t]]$, more over, $h_-, h_0$, and $h_+$ are all 1 when setting $t = 0$.

Proof. Let $\log h = \sum_{i,j} b_{ij} x^i t^j$. Then

$$h_- = \exp \left( \sum_{i<0,j>0} b_{ij} x^i t^j \right),$$

$$h_0 = \exp \left( \sum_{j>0} b_{0j} t^j \right),$$

$$h_+ = \exp \left( \sum_{i>0,j>0} b_{ij} x^i t^j \right).$$

The uniqueness is obvious. \qed
One Useful Tool in Lattice Path Enumeration.

**Theorem 6.** Let $G(x, t), F(x, t) \in K[[x, t]]$. If $G(x, 0)$ can be written as
$ax +$ higher terms with $a \neq 0$, then

$$
\text{CT} \left( \frac{x}{G(x, t)} \right) F(x, t) = \frac{F(x, t)}{\frac{\partial}{\partial x} G(x, t)} \bigg|_{x=X},
$$

where $X = X(t)$ is the unique element in $tK[[t]]$ such that $G(X, t) = 0$.

**Lemma 7.** If $G(x, t) \in R[[x, t]]$ and $G(x, 0)$ can be written as $ax +$ higher terms with $a \neq 0$, then $G(x, t)$ has a unique positive root $X(t)$ for $x$, and this $X = X(t)$ belongs to $tR[[t]]$. 
The plane of $\mathcal{C}(\langle x \rangle)((t))$
Basic Concepts of Lattice Paths

A path $\sigma$ in $\mathbb{Z}^2$ is a finite sequence of lattice points $(a_0, b_0), \ldots, (a_n, b_n)$ in $\mathbb{Z}^2$, in which we call $(a_0, b_0)$ the starting point, $(a_n, b_n)$ the ending point, $(a_i-a_{i-1}, b_i-b_{i-1})$ the steps of $\sigma$, and $n$ the length of $\sigma$.

Given two paths $\sigma_1$ and $\sigma_2$, we define their product $\sigma_1 \sigma_2$ to be the path whose steps are those of $\sigma_1$ followed by those of $\sigma_2$. If $\pi = \sigma_1 \sigma_2$, then we call $\sigma_1$ a head of $\pi$, and $\sigma_2$ a tail of $\pi$. 


Denote by $S^*$ the set of all such paths. Then any $\sigma \in S^*$ can be uniquely factored as $\sigma = s_1s_2 \cdots s_n$ for some $n \geq 0$, and $s_i \in S$ for all $i$. Note that the empty path belongs to $S^*$.

The weight of a step $(a, b) \in S$ is defined to be $\Gamma((a, b)) = x^ay^bt$, and the weight of a path $\sigma = s_1 \cdots s_n$ is defined to be $\Gamma(\sigma) = \Gamma(s_1) \cdots \Gamma(s_n)$. The weight of a path is determined by its length and its end point.

For any two paths $\sigma_1$ and $\sigma_2$, we have $\Gamma(\sigma_1\sigma_2) = \Gamma(\sigma_1)\Gamma(\sigma_2)$.

If $Q \subset S^*$, then the generating function of $P$ is:

$$\Gamma(Q) = \sum_{\sigma \in Q} \Gamma(\sigma).$$
Let $H \subset S^*$. If $H$ is closed under multiplication of paths and contains the empty path, then $H$ is a monoid. We call a nonempty path $\sigma \in H$ a prime of $H$ if it cannot be factored into two nonempty paths in $H$. We say that $H$ is a free monoid if any element in $H$ can be uniquely factored into products of primes in $H$.

If $H$ is a free monoid, then for any $\sigma \in H$ with its factorization into primes as $\sigma = h_1h_2\cdots h_m$, we say that $h_1h_2\cdots h_i$ is an $H$ head of $\sigma$ for $i = 0, 1, \ldots, m$.

If we let $P$ be the set of primes in $H$, then

$$\Gamma(H) = 1/(1 - \Gamma(P)).$$
Example 1. $S^*$ is a free monoid, whose primes are all the elements in $S$.

$$\Gamma(S^*) = \frac{1}{1 - \Gamma(S)}.$$ 

Example 2. The set $S_x$ of all paths in $S^*$ that end on the $x$-axis is a free monoid, whose primes are those paths that only return the $x$-axis at the end point.

$$S_x = [y^0] \Gamma(S^*) := \mathcal{CT}_y \Gamma(S^*).$$
Example 3. The set of all paths in $S^*$ that end on the $x$-axis and never goes below the $x$-axis. What are the prime paths?

An example of the case that $S = \{(1,1), (1,-1)\}$.
One way to prove the classical result

When \( S = \{ (1, 1), (1, -1) \} \), example 3 is just a set of Dyck paths.

Let \( p(x) \) be the generating function for the prime paths. Then

\[
c(x) = \frac{1}{1 - p(x)}.
\]

And in this case, \( p(x) = xc(x) \). Then we get

\[
c(x) = 1 + xc^2(x) \Rightarrow c(x) = \frac{1 \pm \sqrt{1 - 4x}}{2x}.
\]

Since \( c(x) \) is a power series,

\[
c(x) = \frac{1 - \sqrt{1 - 4x}}{2x} = \sum_{n \geq 0} \frac{1}{n + 1} \binom{2n}{n} x^n
\]
Let $\rho$ be a homomorphism from $H$ to $\mathbb{Z}$. The $\rho$ value of a path $\sigma$ is defined as $\rho(\sigma)$.

If $H$ is a free monoid, then any map from $H$ to $\mathbb{Z}$ defined on the primes of $H$ induces a homomorphism. If in addition, $H$ is a subset of $S^*$, then the natural map to the end point of a path is a homomorphism from $H$ to $\mathbb{Z}^2$. Therefore, any homomorphism from $\mathbb{Z}^2$ to $\mathbb{Z}$ induces a homomorphism from $H$ to $\mathbb{Z}$ through that natural map. The following two homomorphisms are useful. Define $\rho_x(\sigma)$ to be the $x$ coordinate of the ending point of $\sigma$, then $\rho_x$ is clearly a homomorphism. Similarly we can define $\rho_y$. 
Gessel Pair

If $H$ is a free monoid, and $\rho$ is a homomorphism from $H$ to $\mathbb{Z}$, then we call $(H, \rho)$ a Gessel pair. For a Gessel pair $(H, \rho)$, we define:

A *minus-path* is either the empty path or a path whose $\rho$ value is negative and less than the $\rho$ values of all the other $H$ heads.

A *zero-path* is a path with $\rho$ value 0 and all of whose $H$ heads have nonnegative $\rho$ values.

A *plus-path* is a path all of whose $H$ heads (except $\epsilon$) have positive $\rho$ values.
Unique Factorization Lemma of Paths

**Proposition 8.** If \((H, \rho)\) is a Gessel pair, then \(H_-, H_0, \) and \(H_+\) are all free monoids. The map from \(H\) to \(H_- \times H_0 \times H_+\) defined by \(\pi \rightarrow (\pi_-, \pi_0, \pi_+)\) is a bijection.

In a Gessel pair \((H, \rho)\), the weight of an element \(\pi \in H\) is defined to be \(\Gamma(\pi)z^{\rho(\pi)}\), where \(z\) is a new variable. When \(H\) is also a subset of \(S^*\) and we are considering the Gessel pair \((H, \rho_x)\), the power in \(z\) is always the same as the power in \(x\) for any \(\pi\) in \(H\). So we can replace \(z\) by 1 and let \(x\) play the same role as \(z\).
An example: \( H = \{(1, \pm 1)\}^* \), \( \rho = \rho_y \).
Since the factorization in $H$ is with respect to $\rho$, the factorization of the generating function is with respect to $z$.

**Theorem 9.** For any Gessel pair $(H, \rho)$, we have $\Gamma(H_{-}) = [\Gamma(H)]_{-}$, $\Gamma(H_{0}) = [\Gamma(H)]_{0}$, and $\Gamma(H_{+}) = [\Gamma(H)]_{+}$.

*Proof.* We have $\Gamma(H) = \Gamma(H_{-})\Gamma(H_{0})\Gamma(H_{+})$. It is easy to check this is the unique factorization of $\Gamma(H)$ with respect to $z$.  

□
Example 10. Let $S$ be $\{(1,r),(1,-1)\}$ with $r \geq 1$, and $H = S^\ast$. Consider the Gessel pair $(H, \rho_y)$.

Note that in this case the length of a path equals the $x$ coordinate of its end point. Replacing $x$ by 1 will not lose any information.

Clearly we have

$$\Gamma(H) = \Gamma(S^\ast) = \frac{1}{1 - t(y^r + 1/y)}.$$  

We see that $H_\perp$ is the set of paths in $S^\ast$ that never go below level 1 after the starting point. The set $H_0$ contains all paths in $S^\ast$ that end on level 0 and never go below level 0. When $r = 1$, this becomes Dyck paths.
An example of $H_0$: of the case $r = 2$. 
The case $r = 2$: To compute $\Gamma(H_0) := F(t)$, we let $Y_1(t)$ be the unique positive root of $y - t(1 + y^3)$ and let $Y_2$ and $Y_3$ be the other roots. Then

$$\frac{y - t(1 + y^3)}{y} = (1 - Y_1/y) \cdot A(t) \cdot (1 - y/Y_2)(1 - y/Y_3)$$

(2)

is the desired unique factorization. Thus $F(t) = 1/A(t)$.

Equate coefficients of $y^{-1}$ on both sides of (2). Then

$-t = -A(t)Y_1(t)$.

So $F(t) = Y_1(t)/t$, and $F(t) = 1 + t^{r+1}F(t)^{r+1}$. 
More Examples

**Example 11.** Let $S$ be $\{(1,1), (1,-1)\}$, and let $H = S^*$. Let $\rho$ be determined by $\rho(1,1) = r$ and $\rho(1,-1) = -1$.

It is easy to see that this example is isomorphic to the previous one.

Or let $H = (a,b)^*$, the free monoid generated by $a$ and $b$. And let $\rho$ be generated by $\rho(a) = 1$, $\rho(b) = -1$. 
Example 12. In general if $H = S^*$, then $(H, \rho_y)$ is a Gessel pair.

We see that $H_+$ is the set of paths in $S^*$ that never go below the line $y = 1$ after the starting point.

If we let $J = H_+$, then $J$ is also a free monoid. The primes of $J$ are paths that start at $(0,0)$, end at some positive level $d$, and never hit level $d - 1$ or lower.

The set $H_0$ contains all paths in $S^*$ that end on the line $y = 0$, and never go below the line $y = 0$. In other words, $H_0$ contains all paths in $S^*$ that stays in the upper half plane and end on the $x$-axis.
If we let \( J = H_0 \), then \((J, \rho_x)\) is a Gessel pair. The set \( J_+ \) contains all paths in \( J \) that avoiding the half line \( \mathcal{H} \) after the starting point. This is the same as walks on the half plane avoiding half line. (Bousquet-Mélou).

The set \( J_0 \) contains all paths in \( J \) that ending at \((0, 0)\) and never touch the half line \( \mathcal{H} \) except \((0, 0)\).
Walks avoiding half plane and half line
Example 13. For any $S$, let $H$ be the set of paths that end on the $x$-axis. Then $(H, \rho_x)$ is a Gessel pair.

The set $H_+$ contains all paths that end on the $x$ axis and never hit the half line $H = \{(-k,0) | k \geq 0\}$ after the starting point. This is exactly the walks on the slit plane that end on the $x$-axis.

The set $H_0$ contains all paths that end at $(0,0)$, and never touch $(-k,0)$ for $k = 1, 2, \ldots$. This was called the set of loops by Bousquet-Mélou.
Example 14. For any $S$, let $H$ be the set of paths that end on the $x$-axis and never go below the line $y = -d$ for some given $d > 0$. Then it is easy to check that $(H, \rho_x)$ is a Gessel pair.

The set $H_+$ contains all paths that end on the $x$-axis, and never hit the half line $\mathcal{H}$ after the starting point, and never go below the line $y = -d$.

The set $H_0$ can be similarly described.
**Example 15.** About walks on the half plane avoiding half line. More precisely, walks that never touch the half line $\mathcal{H}$ and never hit a point $(i,j)$ with $j < 0$. This is a continuation of Example 12. We denote by $H_S(x,y;t)$ the generating function for such paths.

We obtain the following result, which includes

**Theorem 16.** For any well-ordered set $\mathfrak{S}$. Let $p$ be the smallest positive number such that there is an $\mathfrak{S}$-path end at $(p,0)$. Then the number of walks on the half plane avoiding half line that end at $(p,0)$ and is of length $n$ is equal to one $n$th of the number of paths that end at $(p,0)$ and is of length $n$. 
Proof. We use the notations of Example 12. From the Gessel pair \((S^*, \rho_y)\), we have \(\Gamma(H_0) = (\Gamma(S^*))_0\) and

\[
\log \Gamma(H_0) = \sum_{y} \log \Gamma(S^*).
\]

Now let \(J = H_0\) and consider the Gessel pair \((J, \rho_x)\). Then

\[
\log \Gamma(J_0J_+) = \sum_{x} \log \Gamma(J).
\]

In particular, we have

\[
[x^p] \Gamma(J_+) = [x^p] \log \Gamma(J) = [x^p] \log \Gamma(H_0) = [x^p] \sum_{y} \log \Gamma(S^*).
\]

Therefore,

\[
[x^p t^n] \Gamma(J_+) = [x^p y^0 t^n] \Gamma(S)^n.
\]

This prove the theorem. \(\square\)
Example 17. If $S = \{(1,0), (-1,0), (0,1), (0,-1)\}$, then $\Gamma(S) = t(x + y + x^{-1} + y^{-1})$.

$$
\text{CT}_y \Gamma(S^*) = \text{CT}_y \frac{y}{y - t(x + x^{-1})y - ty^2 - t} = \frac{1}{1 - t(x + x^{-1}) - 2tY},
$$

where $Y = Y(x,t)$ is the unique positive root of $y - t(x + x^{-1})y - ty^2 - t$.

$$
Y = \frac{1 - t (x + x^{-1}) - \sqrt{(1 - t (x + x^{-1}))^2 - 4t^2}}{2t}.
$$

After simplifying, the desired generating function can be written as

$$
\text{CT}_y \Gamma(S^*) = \left[ (1 - t (x + x^{-1}))^2 - 4t^2 \right]^{-1/2}.
$$