FROM THE "LOST" NOTEBOOK

p. 26

$$\sum_{n=0}^{\infty} q^{n^2} a^n = \prod_{n=1}^{\infty} \left(1 + a q^{2n-1} \left(1 + y_1(n) + y_2(n) \cdots\right)\right)$$

where

$$y_1(n) = \frac{\sum_{j=n}^{\infty} (-1)^j q^{j(j+1)}}{\sum_{j\geq 0} (-1)^j (2j+1) q^{j(j+1)}},$$

$$y_2(n) = \frac{y_1(n) \sum_{j=n}^{\infty} (j+1)(-1)^j q^{j(j+1)}}{\sum_{j\geq 0} (-1)^j (2j+1) q^{j(j+1)}}$$

Define

$$P_n(a) = \prod_{j=1}^{\infty}(1-q^{2j}) \sum_{h=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix}_{q^2} q^{2j} \prod_{i=h+1}^{n}(1+aq^{2i-1})$$

where

$$\begin{bmatrix} n \\ j \end{bmatrix}_q = \frac{(1-q^n)(1-q^{n-1})\cdots(1-q^{n-j+1})}{(1-q^j)(1-q^{j-1})\cdots(1-q)}$$

THEN

$$\lim_{n\to\infty} P_n(a) = \sum_{j=0}^{\infty} q^{j^2} a^j .$$

THEOREM. FOR $0 \leqq m < n$ AND $0 < q < 1$

$$p_n(-q^{-2m-1}) > 0.$$

THEOREM. FOR $0 \leqq m \leqq \frac{n-1}{2}$, $0 < q < \frac{1}{4}$

$$p_n(-q^{-4m-2}) < 0$$

Define

$$\overline{P_n}(a)=\prod_{j=1}^{\infty}(1-q^{2j})\sum_{j=0}^{n}\begin{bmatrix} n \\ j \end{bmatrix}_{q^2}\prod_{i=j+1}^{n}(1+aq^{2i-1})$$

THEN

$$P_{n+1}(a)-(1+aq^{2n+1})P_n(a)$$

$$=q^{2n+2}\overline{P}_n(aq^2)$$

SO IF $0 < q < \frac{1}{4}$

$$p_\infty(a) = \sum_{n=0}^{\infty} q^{n^2} a^n$$

HAS SIMPLE, NEG., REAL ZEROS x_i

AND

$$-q^{-1} > x_1 > -q^{-2} > x_2 > -q^{-3}$$

$$> -q^{-5} > x_3 > -q^{-6} > x_4$$

$$> -q^{-7} > -q^{-9} \cdots$$

COROLLARY. THE SEQUENCE $\{x_{n,i}\}_{n \geq i}$ IS DECREASING FOR FIXED i ODD, AND INCREASING FOR FIXED i EVEN.

IF $0 < q < \frac{1}{4}$

AND $x_{n,i}$ denotes

the i^{th} zero of

$p_n(x)$, then

$$-q^{-1} > x_{n,1} > -q^{-2} > x_{n,2} > -q^{-3}$$

$$> -q^{-5} > x_{n,3} > -q^{-6} > x_{n,4}$$

$$> -q^{-7} > -q^{-9} > x_{n,5} > \cdots$$

FROM OUR KNOWLEDGE OF THE ZEROS x_i of $\rho_\infty(a)$, WE DEDUCE THAT

$$x_N = \frac{-q^{1-2N}}{1+Y_1(N)}$$

where

$$Y_1(N) = O(q)$$

FROM JACOBI'S TRIPLE PRODUCT

$$\prod_{n=1}^{\infty}(1-q^{2n})(1+aq^{2n-1})(1+a^{-1}q^{2n-1}) = \sum_{n=-\infty}^{\infty} q^{n^2} a^n,$$

WE MAY DEDUCE

$$\prod_{n=1}^{\infty}(1-q^{2n})(1+aq^{2n-1})(1+a^{-1}q^{2n-1}) = w \sum_{j=0}^{\infty} (-1)^n (2j+1) q^{j(j+1)} + O(w^3)$$

where

$$w = 1 + \frac{q}{a}.$$

HENCE EVENTUALLY

$$\sum_{n=1}^{\infty} (-1)^{n-N} q^{n^2+2Nn-n+N^2-N} (1+Y_1(N))^{n+N}$$

$$= -Y_1(N) \sum_{n=0}^{\infty} (-1)^n (2n+1) q^{n^2+n}$$

$$+ O(Y_1(N)^3)$$

FROM HERE IT IS A SHORT DISTANCE TO

$$Y_1(N) = \frac{\sum_{n=N}^{\infty} (-1)^n q^{n^2+n}}{\sum_{n=0}^{\infty} (-1)^n (2n+1) q^{n^2+n}} \pmod{q^{3N^2+3N}}$$

which is $y_1(N)$.

$y_2(N)$ is extracted similarly.