$$f(x+1) = x\, f(x), \quad f(1) = 1,$$

$\ln f(x)$ is convex for $x > 0$

$$\Longrightarrow f(x) = \Gamma(x)$$

$$\Gamma(n+1) = n!$$

$$n!_q = 1(1+q)\cdots(1+q+\cdots+q^{n-1})$$

$$= \frac{(1-q)(1-q^2)\cdots(1-q^n)}{(1-q)^n}$$

$0 < q < 1$ $$= \frac{(q;q)_\infty}{(q^{n+1};q)_\infty}(1-q)^{1-(n+1)}$$

$$(a;q)_\infty = \prod_{n=0}^{\infty}(1-aq^n)$$

$$\Gamma_q(x) = \frac{(q;q)_\infty}{(q^x;q)_\infty}(1-q)^{1-x}$$

$$\Gamma_q(x+1) = \frac{1-q^x}{1-q}\,\Gamma_q(x)$$

$$\ln \Gamma_q(x) = \sum_{n=0}^{\infty} \ln(1-q^{n+1}) - \ln(1-q^{n+x})$$

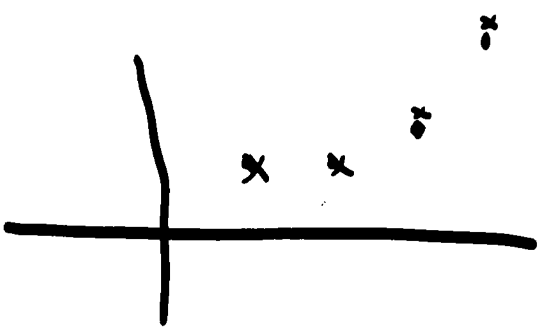
$$+ (1-x)\ln(1-q)$$

$$\frac{d^2}{dx^2}\ln \Gamma_q(x) = \frac{d}{dx}\sum_{n=0}^{\infty} \frac{q^{n+x}\ln q}{1-q^{n+x}}$$

$$= \sum_{n=0}^{\infty} \frac{q^{n+x}(\ln q)^2}{(1-q^{n+x})^2} > 0, \quad x > 0$$

$n!_q$ is an increasing function of q,

$0 < q < 1$, $n = 2, 3, \ldots$ $\quad 0!_q = 1!_q = 1$.



black < blue

$$h(q) = \frac{(\ln q)^2 q^a}{(1-q^a)^2}$$

$$h'(q) = \frac{a q^{a-1} (\ln q)(1+q^a)}{(1-q^a)^3} \cdot \left[\frac{2(1-q^a)}{a(1+q^a)} + \ln q\right]$$

Since $\ln q < 0$, $a = p + x > 0$, we need to show $\frac{2(1-q^a)}{a(1+q^a)} + \ln q < 0$ to have $\frac{\partial}{\partial q}\frac{\partial^2}{\partial x^2} \ln \Gamma_q(x) > 0$, or $h(q)$ increases.

$$k(q) = \frac{2(1-q^a)}{a(1+q^a)} + \ln q.$$

$$k'(q) = \frac{(1-q^a)^2}{q(1+q^a)^2} \geq 0, \quad q>0$$

$k(1)=0$, so $k(q)<0$, $0<q<1$.

Thus

$$\frac{d^2}{dx^2} \ln \Gamma_r(x) \leq \frac{d^2}{dx^2} \ln \Gamma_q(x), \quad 0<r<q<1.$$

$$\frac{d^2}{dx^2} \ln \frac{\Gamma_q(x)}{\Gamma_r(x)} \geq 0, \quad x>0, \quad 0<r<q<1.$$

$\ln \dfrac{\Gamma_q(x)}{\Gamma_r(x)}$ is convex

$$\ln \frac{\Gamma_q(1)}{\Gamma_r(1)} = \ln \frac{\Gamma_q(2)}{\Gamma_r(2)} = \ln 1 = 0$$

$$(1-x)f(x) = (1-ax)f(qx)$$

$$\cancel{f(x)=}\ \frac{(1-ax)}{(1-x)} f(qx)$$

$$(x;q)_n = (1-x)(1-qx)\cdots(1-xq^{n-1})$$

$$= \frac{(1-ax)(1-aqx)}{(1-x)(1-qx)} f(q^2x)$$

$$= \frac{(ax;q)_n}{(x;q)_n} f(q^n x)$$

$$|q|<1$$

$$= \frac{(ax;q)_\infty}{(x;q)_\infty} f(0)$$

$$f(x) = \sum_{n=0}^{\infty} c_n x^n$$

$$\sum_{n=0}^{\infty} c_n x^n - \sum_{n=0}^{\infty} c_n x^{n+1}$$

$$= \sum_{n=0}^{\infty} c_n q^n x^n - \sum_{n=0}^{\infty} a c_n q^n x^{n+1}$$

$$\sum_{n=0}^{\infty} c_n(1-q^n)x^n = \sum_{n=0}^{\infty} c_n(1-aq^n)x^{n+1}$$

$$\parallel$$

$$\sum_{n=0}^{\infty} c_{n+1}(1-q^{n+1})x^{n+1}$$

$$c_{n+1} = \frac{(1-aq^n)}{(1-q^{n+1})}c_n$$

$$c_n = \frac{(a;q)_n}{(q;q)_n}c_0 \qquad c_0 = f(0)$$

$$\sum_{n=0}^{\infty}\frac{(a;q)_n}{(q;q)_n}x^n = \frac{(ax;q)_\infty}{(x;q)_\infty}$$

$$(1-q^{-N}x)\cdots(1-q^{-1}x) = \sum_{n=0}^{N}\frac{(q^{-N};q)_n}{(q;q)_n}x^n$$

$$\frac{(q^{-N};q)_n}{(q;q)_n} = \frac{(1-q^{-N})\cdots(1-q^{-N+n-1})}{(1-q)\cdots(1-q^n)} = \frac{(-1)^n q^{-Nn+\binom{n}{2}}(q;q)_N}{(q;q)_n(q;q)_{N-n}}$$

$$\begin{bmatrix} N \\ k \end{bmatrix}_q = \frac{(q;q)_N}{(q;q)_k (q;q)_{N-k}} = \frac{N!_q}{k!_q (N-k)!_q}$$

$$(a+b)(a+bq)\cdots(a+bq^{N-1})$$

$$= \sum_{k=0}^{N} \begin{bmatrix} N \\ k \end{bmatrix}_q q^{\binom{k}{2}} a^{N-k} b^k$$

$$\frac{1}{(1-x)(1-xq)\cdots(1-xq^n)}$$

$$= \sum_{k=0}^{\infty} \begin{bmatrix} n+k \\ k \end{bmatrix}_q x^k$$

$$(a_1 - b_1 x) f(x) = (a_2 - b_2 x) f(qx)$$

$$f(x) = \sum_{-\infty}^{\infty} c_n x^n$$

$$\sum a_1 c_n x^n - b_1 c_n x^{n+1}$$

$$= \sum a_2 c_n q^n x^n - b_2 c_n q^n x^{n+1}$$

$$\sum (a_1 - a_2 q^n) c_n x^n = \sum (b_1 - b_2 q^n) c_n x^{n+1}$$

$$(a_1 - a_2 q^{n+1}) c_{n+1} = (b_1 - b_2 q^n) c_n$$

$$c_{n+1} = \frac{b_1 - b_2 q^n}{a_1 - a_2 q^{n+1}} c_n = \frac{b_1}{a_1} \frac{\left(1 - \frac{b_2}{b_1} q^n\right)}{\left(1 - \frac{a_2}{a_1} q^{n+1}\right)} c_n$$

$$c_n = \left(\frac{b_1}{a_1}\right)^n \frac{\left(\frac{b_2}{b_1};q\right)_n}{\left(\frac{a_2}{a_1}q;q\right)_n} c_0$$

What is $(a;q)_n$ when $n=-1,-2,\dots$?

$$(a;q)_n = (1-a)(1-aq)\cdots(1-aq^{n-1})$$

$$= \frac{(a;q)_\infty}{(aq^n;q)_\infty}$$

Let $b_1 = a_1$, $\frac{b_2}{b_1} = a$, $\frac{a_2}{a_1}q = b$.

$$\sum_{-\infty}^{\infty} \frac{(a;q)_n}{(b;q)_n} x^n = f(x;a,b)$$

$b = q^N$ $\qquad \frac{1}{(q^N;q)_n} = \frac{(q^{N+n};q)_\infty}{(q^N;q)_\infty}$, $= 0$ when $n=-N,-N-1,\dots$

$b \to q^{N+1}$

$$\sum_{-\infty}^{\infty} \frac{(a;q)_n}{(q^{N+1};q)_n} x^n = \sum_{n=-N}^{\infty} \frac{(a;q)_n}{(q^{N+1};q)_n} x^n$$

$$= \sum_{n=0}^{\infty} \frac{(a;q)_{n-N}}{(q^{N+1};q)_{n-N}} x^{n-N}$$

$$= \frac{(a;q)_{-N}}{(q^{N+1};q)_{-N}} x^{-N} \sum_{n=0}^{\infty} \frac{(aq^{-N};q)_n}{(q;q)_n} x^n$$

$$= \frac{(a;q)_\infty (q;q)_\infty}{(aq^{-N};q)_\infty (q^{N+1};q)_\infty} x^{-N} \frac{(axq^{-N};q)_\infty}{(x;q)_\infty}$$

$$= \frac{(q;q)_\infty (axq^{-N};q)_N (ax;q)_\infty (a;q)_\infty}{(q^{N+1};q)_\infty (aq^{-N};q)_N x^N (a;q)_\infty (x;q)_\infty}$$

$$= \frac{(q;q)_\infty \left(\frac{q}{ax};q\right)_N (ax;q)_\infty}{(q^{N+1};q)_\infty \left(\frac{q}{a};q\right)_N (x;q)_\infty}$$

$$= \frac{(ax;q)_\infty \left(\frac{q}{ax};q\right)_\infty (q;q)_\infty \left(\frac{q^{N+1}}{a};q\right)_\infty}{(x;q)_\infty \left(\frac{q^{N+1}}{ax};q\right)_\infty (q^{N+1};q)_\infty \left(\frac{q}{a};q\right)_\infty}$$

$$= \sum_{-\infty}^{\infty} \frac{(a;q)_n}{(q^{N+1};q)_n} x^n \qquad q^{N+1} = b.$$

$$\sum_{-\infty}^{\infty} \frac{(a;q)_n}{(b;q)_n} x^n = \frac{(ax;q)_\infty \left(\frac{q}{ax};q\right)_\infty (q;q)_\infty \left(\frac{b}{a};q\right)_\infty}{(x;q)_\infty \left(\frac{b}{ax};q\right)_\infty (b;q)_\infty \left(\frac{q}{a};q\right)_\infty}$$

$$\left|\frac{b}{a}\right| < |x| < 1$$

$b = 0 \quad x \to \frac{x}{a}, \; a \to \infty$

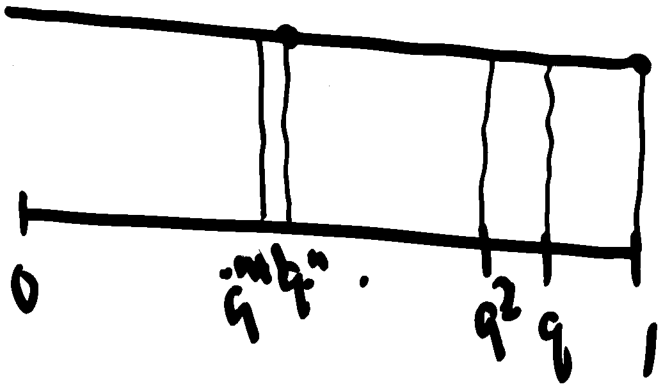
$$\sum_{-\infty}^{\infty} (-1)^n q^{\binom{n}{2}} x^n = (x;q)_\infty \left(\frac{q}{x};q\right)_\infty (q;q)_\infty$$

$$\int_{-\infty}^{\infty} e^{-t^2}\,dt = \sqrt{\pi}$$

$$= \lim_{h\to 0} h \sum_{-\infty}^{\infty} e^{-(nh+c)^2} \qquad = \lim_{h \to 0} h e^{-c^2} \sum_{-\infty}^{\infty} \left(e^{-h^2}\right)^{n^2} \cdot \left(e^{-2hc}\right)^n$$

$$\sum_{-\infty}^{\infty} q^{n^2} x^n = (q^2;q^2)_\infty (-qx;q^2)_\infty \left(-\frac{q}{x};q^2\right)_\infty$$

$$\int_0^1 x^k \, dx$$



$$\|$$

$$\lim_{q \uparrow 1} (1-q) \sum_{n=0}^{\infty} (q^n)^k q^n = \lim_{q \to 1^-} \frac{(1-q)}{(1-q^{n+1})}$$

$$= \lim_{q \to 1^-} \frac{1}{1+q+\cdots+q^n} = \frac{1}{n+1}$$

$$\int_0^a f(x) \, d_q x = a(1-q) \sum_{n=0}^{\infty} f(aq^n) \, q^n$$

$$\int_0^1 x^{a-1} \frac{(qx;q)_\infty}{(q^b x;q)_\infty} \, d_q x = (1-q) \sum_{n=0}^{\infty} q^{an} \frac{(q^{n+1};q)_\infty}{(q^{n+b};q)_\infty}$$

$$= \frac{(1-q)(q;q)_\infty}{(q^b;q)_\infty} \sum_{n=0}^{\infty} \frac{(q^b;q)_n}{(q;q)_n} q^{an}$$

$$\int_0^1 x^{a-1} \frac{(qx;q)_\infty}{(q^b x;q)_\infty}\, d_q x = \frac{(1-q)(q;q)_\infty (q^{a+b};q)_\infty}{(q^a;q)_\infty (q^b;q)_\infty}$$

$$= \frac{\Gamma_q(a)\Gamma_q(b)}{\Gamma_q(a+b)}$$

$$\int_0^1 x^{a-1}(1-x)^{b-1}\,dx = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$$

$$\int_0^\infty f(x)\,d_q x = \lim_{N\to\infty} \int_0^{q^{-N}} f(x)\,d_q x$$

$$= \lim_{N\to\infty} q^{-N}(1-q)\sum_{n=0}^{\infty} f(q^{-N+n})\,q^n$$

$$\left(x = \frac{t}{1+t}\right)$$

$$= \lim_{N\to\infty} (1-q)\sum_{n=-N}^{\infty} f(q^n)\,q^n$$

$$= \sum_{-\infty}^{\infty} f(q^n)\,q^n.$$

$$\int_0^\infty \frac{t^{a-1}}{(1+t)^{a+b}}\,dt = \frac{1}{c^a}\int_0^\infty \frac{t^{a-1}\,dt}{(1+ct)^{a+b}}$$

$$\sum_{-\infty}^{\infty} \frac{(bq^n;q)_\infty}{(aq^n;q)_\infty} x^n = \frac{(ax)_\infty \left(\frac{q}{ax}\right)_\infty (q)_\infty \left(\frac{b}{a}\right)_\infty}{(x)_\infty \left(\frac{b}{ax}\right)_\infty (a)_\infty \left(\frac{q}{a}\right)_\infty}$$

$a = -c$, $b = -c q^{\alpha+\beta}$, $x = q^\alpha$

$$\int_0^\infty \frac{(-cq^{\alpha+\beta}x;q)_\infty}{(-cx;q)_\infty} x^{\alpha-1} d_q x$$

$$= \frac{(-cq^{\alpha})_\infty \left(-\frac{q^{1-\alpha}}{c}\right)_\infty}{(-c)_\infty \left(-\frac{q}{c}\right)_\infty} \frac{\Gamma_q(\alpha)\Gamma_q(\beta)}{\Gamma_q(\alpha+\beta)}$$

$$\frac{(-cq^\alpha)_\infty}{(-c)_\infty} \sim \frac{1}{(1+c)^\alpha}$$

$$\frac{\left(-\frac{q^{1-\alpha}}{c}\right)_\infty}{\left(-\frac{q}{c}\right)_\infty} \sim \left(1+\frac{1}{c}\right)^\alpha$$

$$f(x) = c\, f(qx)$$

$$f(x) = x^{\alpha} \qquad \frac{x^{\alpha}}{q^{\alpha} x^{\alpha}} = c$$

$$e^{\alpha \ln q} = q^{\alpha} = \frac{1}{c} = e^{-\ln c} \qquad \alpha = -\frac{\ln c}{\ln q}.$$

$$f(x) = (ax;q)_{\infty}\left(\frac{q}{ax};q\right)_{\infty}$$

$$\frac{f(x)}{f(qx)} = \frac{(ax;q)_{\infty}\left(\frac{q}{ax};q\right)_{\infty}}{(aqx;q)_{\infty}\left(\frac{1}{ax};q\right)_{\infty}}$$

$$= \frac{(1-ax)}{\left(1-\frac{1}{ax}\right)} = -ax$$

$$f(x) = \frac{(ax;q)_{\infty}\left(\frac{q}{ax};q\right)_{\infty}}{(bx;q)_{\infty}\left(\frac{q}{bx};q\right)_{\infty}}$$

$$\frac{f(x)}{f(qx)} = \frac{-ax}{-bx} = \frac{a}{b} = c$$

$$\int_0^\infty \frac{t^{\alpha-1}}{1+t}\,dt = \Gamma(\alpha)\Gamma(1-\alpha) = \frac{\pi}{\sin \pi\alpha}$$

$$\int_0^\infty \frac{t^{\alpha-1}}{1+ct}\,d_qt = \frac{\Gamma_q(\alpha)\Gamma_q(1-\alpha)\,(-cq^\alpha;q)_\infty(-\frac{q^{1-\alpha}}{c};q)_\infty}{(-c;q)_\infty(-\frac{q}{c};q)_\infty}$$

$$= \frac{(1-q)\,(q;q)_\infty^2}{(-c;q)_\infty(-\frac{q}{c};q)_\infty}\;\frac{(-cq^\alpha;q)_\infty(-\frac{q^{1-\alpha}}{c};q)_\infty}{(-q^\alpha;q)_\infty(-q^{1-\alpha};q)_\infty}$$

$$(1-q)\sum_{-\infty}^{\infty}\frac{q^{\alpha n}}{1+cq^n}$$

$$(1-bx)f(x)=(1-ax)f(qx)$$

$$f(x)=\sum_{-\infty}^{\infty} B_{n+\lambda}(x;q)_{n+\lambda}$$

$$B_{n+\lambda}=\frac{(aq^{\lambda}/b;q)_n\, q^n}{(q^{\lambda+1};q)_n\left(\frac{q^{\lambda+1}}{b};q\right)_n}B_\lambda$$

when $|x|>\left|\frac{q}{a}\right|$

$$f(x)=\sum_{-\infty}^{\infty}\frac{A_{n+\lambda}}{(x;q)_{n+\lambda}}$$

$$A_{n+\lambda}=\frac{(q^{\lambda};q)_n\left(\frac{q^{\lambda}}{a};q\right)_n q^n}{\left(\frac{b}{a}q^{\lambda+1};q\right)_n}$$

when $|x|<\frac{1}{|b|}$.

$0<q<1$

$$B_q(x,y) = \frac{\Gamma_q(x)\Gamma_q(y)}{\Gamma_q(x+y)} \qquad B_q(1,1)=1.$$

$$B_q(x,1) = \frac{\Gamma_q(x)}{\Gamma_q(x+1)} = \frac{(1-q)}{(1-q^x)}$$

$$\frac{d}{dq}\frac{1-q}{1-q^x} = \frac{-(1-q^x)+x(1-q)q^{x-1}}{(1-q^x)^2}$$

$$= \frac{-1+xq^{x-1}-(x-1)q^x}{(1-q^x)^2}$$

$$h(q) = -1+xq^{x-1}-(x-1)q^x, \quad h(1)=0$$

$$h'(q) = x(x-1)q^{x-2} - x(x-1)q^{x-1}$$

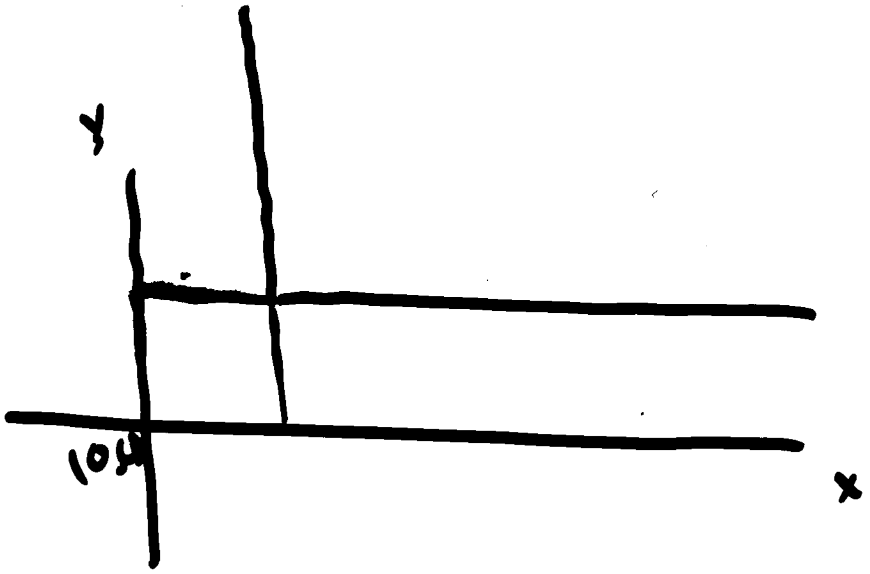
$$= x(x-1)q^{x-2}(1-q) > 0 \text{ when}$$

$x>1$

< 0 when

$0<x<1$. $\therefore$ $h(q)$ <0 when $x>1$, >0 when $0<x<1$.

$\therefore$ $B_q(x,1)$ increases in q when $0<x<1$ and decreases in q when $x>1$.



— increases

— decreases

$$B_q(x,y) = \frac{\Gamma_q(x)\Gamma_q(y)}{\Gamma_q(x+y)}$$

$0<x,y<1, \quad 1 \le x+y<2$

$\Gamma_q(x), \Gamma_q(y)$ increase in q.

$\Gamma_q(x+y)$ decreases

$\therefore B_q(x,y)$ increases in q when

$0<x,y<1, 1 \le x+y$

Let $x, y \geq 1$

$$\frac{B_q(x,y+1)}{B_q(x,y)} = \frac{\Gamma_q(y+1)\,\Gamma_q(x+y)}{\Gamma_q(y)\,\Gamma_q(x+y+1)}$$

$$= \frac{(1-q^y)}{(1-q^{x+y})}$$

$$q^y = r$$

$$q^{x+y} = r^z, \quad z > 1$$

$$B_q(x,y+1) = \frac{(1-q^y)}{(1-q^{x+y})} B_q(x,y)$$

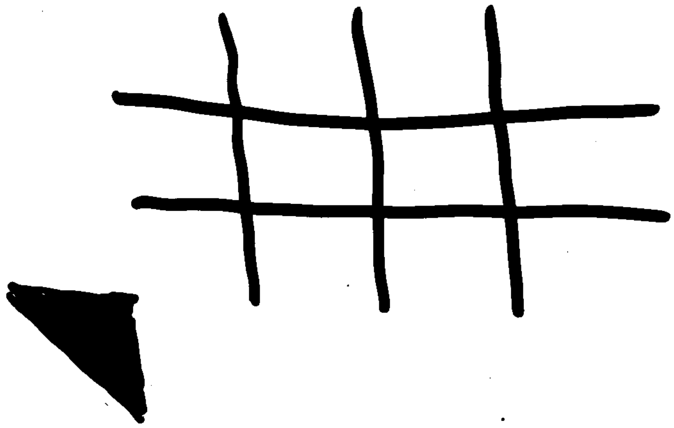
and both $\frac{1-q^y}{1-q^{x+y}}$ and $B_q(x,y)$ decrease,

the first when $x > 0$, $y > 0$, the second when

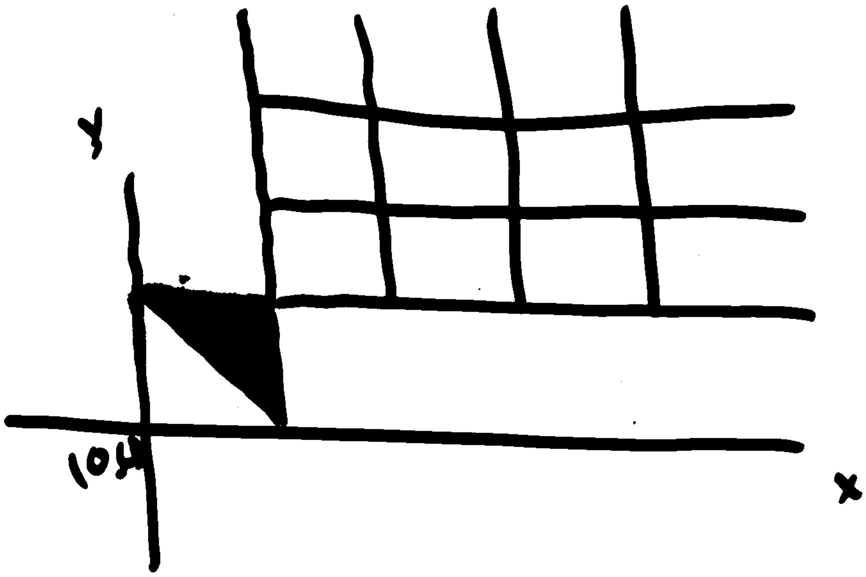
$x \geq 1$, $y = 1$, and then by induction when $x \geq 1$,

$y =$ positive integer. $B_q(x,y) = B(x,y)$ so

we have the following picture.



The obvious conjecture is that the region when $0<x,y\leq 1$ is blue and when $x,y\geq 1$ is black.



— increases

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$$B_q(x,y) = \frac{\Gamma_q(x)\Gamma_q(y)}{\Gamma_q(x+y)}$$

$0<x,y<1, \quad 1 \le x+y<2$

$\Gamma_q(x), \Gamma_q(y)$ increase in q.

$\Gamma_q(x+y)$ decreases

$\therefore B_q(x,y)$ increases in q when

$0<x,y<1, 1\le x+y$

The obvious conjecture is that the region when $0<x,y \le 1$ is blue and when $x,y \ge 1$ is black.