Chromatic and Flow Polynomials of Graphs

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(This is joint work with Richard P. Stanley)

1 Two arrangements associated with a graph

Let Ω be a finite dimensional vector space. A **subspace arrangement** of Ω is a finite collection $\mathcal{A}$ of flats of Ω. The **semilattice** of a subspace arrangement $\mathcal{A}$ is the poset $L(\mathcal{A})$, whose elements are nonempty intersections of flats in $\mathcal{A}$, including

$$\Omega = \bigcap_{X\in\emptyset} X,$$

and whose partial order is the set inclusion. Associated with a subspace arrangement $\mathcal{A}$ is the **characteristic polynomial**

$$\chi(\mathcal{A},t) = \sum_{X\in L(\mathcal{A})} \mu(X,\Omega)\, t^{\dim X}, \tag{1}$$

where μ is the Möbius function of the poset $L(\mathcal{A})$, defined inductively by $\mu(X, X) = 1$ for any X, and for $X < Y$,

$$\mu(X, Y) = -\sum_{X \leq Z < Y} = -\sum_{X < Z \leq Y} \mu(Z, Y).$$

Let $G = (V, E)$ be a finite graph, loops and multiple edges are allowed. In this talk we consider two arrangements, one in the **vertex space**

$$\mathbb{R}^V = \text{set of all functions from } V \text{ to } \mathbb{R},$$

and the other one in the **edge space**

$$\mathbb{R}^E = \text{set of all functions from } E \text{ to } \mathbb{R}.$$

2 Coloring arrangement

The **coloring arrangement** (usually called the **graph arrangement**) $\mathcal{A}_{\mathrm{CL}}(G;\mathbb{R})$ of G is the collection of the flats

$$H_e = \{\rho \in \mathbb{R}^V \mid \rho(u) = \rho(v) \text{ for } e = uv\}, \quad e \in E.$$

It is known (but seems not well known) that the chromatic polynomial

$$\chi(G, t) = \chi(\mathcal{A}_{\mathrm{CL}}(G;\mathbb{R}), t).$$

For each orientation ε of G, let

$$\begin{aligned}\chi(G,\varepsilon;q) &= \text{number of colorings } \rho : V \to [q] \\ &\quad \text{s.t. } \rho(u) > \rho(v) \text{ for all } u \xrightarrow{\varepsilon} v, \\ \bar{\chi}(G,\varepsilon;q) &= \text{number of colorings } \rho : V \to [q] \\ &\quad \text{s.t. } \rho(u) \geq \rho(v) \text{ for all } u \xrightarrow{\varepsilon} v.\end{aligned}$$

Let $A\mathcal{O}(G)$ be the set of all acyclic orientations of G. Set

$$\begin{aligned}\bar{\chi}(G,q) \;=\;& \text{number of pairs } (\rho,\varepsilon), \text{ where } \varepsilon \in A\mathcal{O}(G),\\ & \rho : V \to \{0,1,\ldots,q-1\} \text{ such that}\\ & \rho(u) \geq \rho(v) \text{ for all } u \stackrel{\varepsilon}{\to} v. \qquad (2)\end{aligned}$$

Theorem 2.1 (Stanley). *Let $G=(V,E)$ be a finite graph without loops. Then*

(a) *For each $\varepsilon \in A\mathcal{O}(G)$, $\chi(G,\varepsilon;q)$ and $\bar{\chi}(G,\varepsilon;q)$ are polynomial functions of degree $|V|$ in the positive integral variable q; and, as polynomials, they satisfy the reciprocity law:*

$$\chi(G,\varepsilon;-t) = (-1)^{|V|}\bar{\chi}(G,\varepsilon;t). \qquad (3)$$

Moreover, $\chi(G,\varepsilon;-1) = (-1)^{|V|}$ and $\bar{\chi}(G,\varepsilon;1) = 1$.

(b) *The polynomials* $\chi(G,t)$ *and* $\bar{\chi}(G,t)$ *can be written as*

$$\chi(G,t) = \sum_{\varepsilon \in A\mathcal{O}(G)} \chi(G,\varepsilon;t), \tag{4}$$

$$\bar{\chi}(G,t) = \sum_{\varepsilon \in A\mathcal{O}(G)} \bar{\chi}(G,\varepsilon;t), \tag{5}$$

and satisfy the reciprocity law:

$$\chi(G,-t) = (-1)^{|V|}\bar{\chi}(G,t).$$

In particular, $|\chi(G,-1)| = |A\mathcal{O}(G)|$.

3 Flow arrangement

Fix an orientation ε of G. For each vertex $v \in V$, let

$$\begin{aligned} E^+(v,\varepsilon) &= \text{set of edges with arrows from } v, \\ E^-(v,\varepsilon) &= \text{set of edges with arrows to } v. \end{aligned}$$

A function $\phi \in \mathbb{R}^E$ is called a **flow** of the digraph (G,ε) if

$$\sum_{e \in E^+(v,\varepsilon)} \phi(e) = \sum_{e \in E^-(v,\varepsilon)} \phi(e)$$

at each vertex $v \in V$. A flow ϕ is called **nowhere-zero** if $\phi(e) \neq 0$ for all $e \in E$. Let

$$\mathrm{FL}(G,\varepsilon;\mathbb{R}) = \text{set of all flows of } (G,\varepsilon).$$

The **flow arrangement** $\mathcal{A}_{\text{FL}}(G, \varepsilon; \mathbb{R})$ of the digraph (G, ε) is the collection of the flats

$$F_e = \{\phi \in \text{FL}(G, \varepsilon; \mathbb{R}) \mid \phi(e) = 0\}, \quad e \in E.$$

Coincidentally, the flow polynomial

$$f(G, t) = \chi(\mathcal{A}_{\text{FL}}(G, \varepsilon; \mathbb{R}), t).$$

The function $f(G, q)$ is defined as the number of nowhere-zero flows of (G, ε) with values in $\mathbb{Z}/q\mathbb{Z}$. Note that $f(G, q)$ is independent of the orientation ε and the structure of the abelian group $\mathbb{Z}/q\mathbb{Z}$.

A **cut** of G is a partition $\{S, T\}$ of the vertex set $V(G)$ such that $S \neq \emptyset \neq T$; $\{S, T\}$ is called a **directed cut** of (G, ε) if all edges between S and T have the same direction from S to T or from T to S. A minimal cut is called a **bond**. When the edge set $E(G)$ is linearly ordered, a subset of $E(G)$ is called a **broken bond** if it is obtained from a bond by removing the maximum edge in the bond.

Two orientations ε_1 and ε_2 of G are called **Eulerian equivalent**, denoted $\varepsilon_1 \sim \varepsilon_2$, if the spanning subgraph induced by

$$\{e \in E \mid \varepsilon_1(e) \neq \varepsilon_2(e)\}$$

is a directed Eulerian graph with the orientation either ε_1 or ε_2, i.e., the in-degree is the same as the out-degree at every vertex.

Let $\mathcal{O}(G)$ be the set of all orientations on G. Then $\sim$ is an equivalence relation on $\mathcal{O}(G)$; it is also an equivalence relation on

$$B\mathcal{O}(G) \;=\; \text{set of orientations of } G \text{ without directed cut.}$$

We denote by $E\mathcal{O}(G)$ set of equivalence classes of $\sim$ on $B\mathcal{O}(G)$. For each $\varepsilon \in B\mathcal{O}(G)$, i.e., (G, ε) has no directed cut, let

$$\begin{aligned} f_{\mathbb{Z}}(G, q) =& \text{number of } \mathbb{Z}\text{-flows } \phi \text{ of } (G, \varepsilon) \\ & \text{s.t. } 0 < |\phi(e)| < q \text{ for } e \in E, \end{aligned}$$

$$\begin{aligned} \bar{f}_{\mathbb{Z}}(G, q) =& \text{number of pairs } (\phi, \varepsilon) \text{, where} \\ & \varepsilon \in B\mathcal{O}(G),\ \phi \text{ is a } \mathbb{Z}\text{-flow,} \\ & 0 \le f(e) \le q \text{ for } e \in E. \end{aligned}$$

4 Main results

Theorem 4.1. *Let $G = (V, E)$ be a bridgeless finite graph, loops and multiple edges are allowed. Then*

(a) *For each $\varepsilon \in B\mathcal{O}(G)$, $f_{\mathbb{Z}}(G, \varepsilon; q)$ and $\bar{f}_{\mathbb{Z}}(G, \varepsilon; q)$ are polynomial functions of degree $r(G)$ in the positive in-*

tegral variable q, and, as polynomials, satisfy the reciprocity law:

$$f_{\mathbb{Z}}(G,\varepsilon;-t) = (-1)^{r(G)} \bar{f}_{\mathbb{Z}}(G,\varepsilon;t) \tag{7}$$

Moreover, $f_{\mathbb{Z}}(G,\varepsilon;0) = (-1)^{r(G)}$, $\bar{f}_{\mathbb{Z}}(G,\varepsilon;0) = 1$.

(b) *The polynomials* $f_{\mathbb{Z}}(G,q)$ *and* $\bar{f}_{\mathbb{Z}}(G,q)$ *can be written as*

$$f_{\mathbb{Z}}(G,t) = \sum_{\varepsilon\in B\mathcal{O}(G)} f_{\mathbb{Z}}(G,\varepsilon;t), \tag{8}$$

$$\bar{f}_{\mathbb{Z}}(G,t) = \sum_{\varepsilon\in B\mathcal{O}(G)} \bar{f}_{\mathbb{Z}}(G,\varepsilon;t), \tag{9}$$

and satisfy the reciprocity law:

$$f_{\mathbb{Z}}(G,-t) = (-1)^{r(G)} \bar{f}_{\mathbb{Z}}(G,t). \tag{10}$$

Moreover, $|f_{\mathbb{Z}}(G,0)| = |B\mathcal{O}(G)|$.

Remark. The polynomial property of the function $f_{\mathbb{Z}}(G,q)$ was independently obtained by Kochol [**?**]; and the reciprocity law was independently obtained by Beck and Zaslavsky [**?**]. However, we obtained slightly more explicit expression for the integral flow polynomial $f_{\mathbb{Z}}(G,q)$.

Theorem 4.2. *Let $G=(V,E)$ be a bridgeless finite graph, loops and multiple edges are allowed. Then $\bar{f}(G,q)$ is a polynomial function of degree $r(G)$ in the positive integral variable q, and satisfy the reciprocity law:*

$$f(G,-q)=(-1)^{r(G)}\bar{f}(G,q).$$

Moreover, $f(G,q)$ and $\bar{f}(G,q)$ can be written as

$$f(G,q) = \sum_{\varepsilon \in E\mathcal{O}(G)} f_{\mathbb{Z}}(G,\varepsilon;q),$$

$$\bar{f}(G,q) = \sum_{\varepsilon \in E\mathcal{O}(G)} \bar{f}_{\mathbb{Z}}(G,\varepsilon;q).$$

In particular, $|f(G,-1)| = |B\mathcal{O}(G)|$, $|f(G,0)| = |E\mathcal{O}(G)|$.

Theorem 4.3 (Broken Bond Theorem). *Let $G = (V,E)$ be a bridgeless graph. Then*

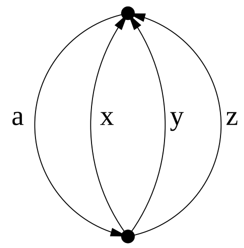
$$f(G,q) = \sum_{i=0}^{r(G)} (-1)^i b_i q^{r(G)-i},$$

where b_i is the number of i-subsets of G containing no broken bonds.

Remark. The Broken Bond Theorem seems not explicitly stated in the literature, to the best of our knowledge. However, it can be derived directly from the Broken Circuit Theorem for geometric matroid, once the matroid for the flow polynomial is realized as the characteristic polynomial of the matroid.

Example.

$$f(G,q) = \sum_{i=0}^{r(G)} (-1)^i b_i q^{r(G)-i},$$



$$x + y + z = a \;\;\Rightarrow\;\; z = a - (x+y).$$

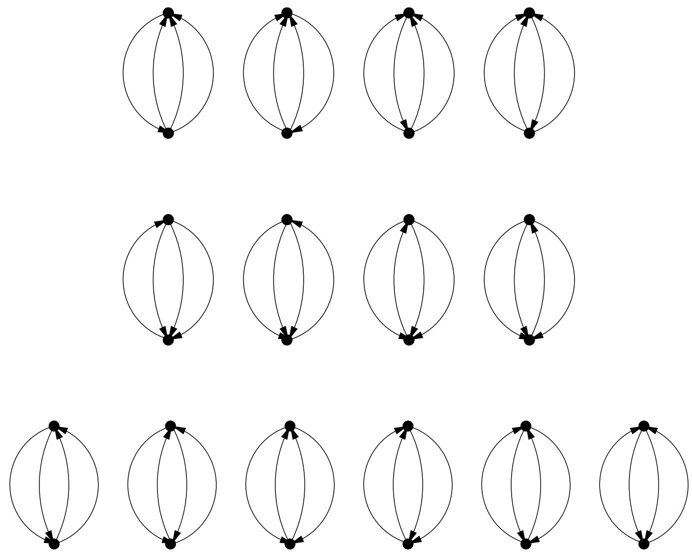
$$\begin{aligned} f(G,q) &= (q-1)[(q-1)^2 - (q-2)] \\ &= q^3 - 4q^2 + 6q - 3. \end{aligned}$$

$$\#\{\text{0-set}\} = 1,\;\; \#\{\text{1-set}\} = 4,$$

$$\#\{\text{2-set}\} = 6,\;\; \#\{\text{3-set}\} = 3;$$

$$|f(G,-1)| = \#\{\text{orientations without directed cut}\} = 14,$$

$$|f(G,0)| = \#\{\text{Eulerian equiv. classes of o.w.o.d.c.}\} = 3.$$



Thank you!

Happy Birthday to James!

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