KINEMATICS OF THE BETHE ANSATZ, RAREFIED BRILLOUIN ZONES, AND DUALITY OF WEYL

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- Prof. James D. Louck presented us in His lectures various aspects of *representation theory of unitary groups* and its applications in physics of atoms, nuclei, and elementary particles

- In this report, we present *an adaptation* of this theory *in condensed matter physics*, within the model of *a finite magnetic Heisenberg chain.*

- The scheme:

$$\boldsymbol{U(n), n = 2s + 1}$$

The chain consists of N nodes, each with the spin s

Representation theory of U(n) is applied to describe KINEMATICS of the one-dim Heisenberg magnet

- An appropriate tool for describing this kinematics is THE WEYL DUALITY between actions of the symmetric group Σ_N and unitary group $U(n)$ in the linear space

$$(\mathbb{C}^n)^{\otimes N},$$

which is
a) physically - the space of all quantum states of the magnet
b)mathematically - the carrier space of the N-th tensor power of the fundamental representation $D^{\{1\}}$ of the unitary group U(n)

- a new aspect of the Weyl duality is related to *rigged string configurations*, introduced by Kerov, Kirillov and Reshetikhin (KKR).

2. Kinematics of the Heisenberg chain

a) The single-node spin

$\tilde{\boldsymbol{n}} = \{\boldsymbol{i} = \mathbf{1}, \mathbf{2}, \dots, \boldsymbol{n}\}$ - the alphabet of spins

$\boldsymbol{m_s} = \boldsymbol{s}, \boldsymbol{s} - \mathbf{1}, \dots, -\boldsymbol{s}$ (orthonormal bases of the single-node space)

$\boldsymbol{h} = \boldsymbol{lc}_{\mathbb{C}}\tilde{\boldsymbol{n}} \cong \mathbb{C}^{\boldsymbol{n}}$ - the single-node space of quantum states or the carrier space of the fundamental (defining) representation $\boldsymbol{D}^{\{1\}}$ of U(n)

$$\boldsymbol{U(n) \ni a = D^{\{1\}}(a)} = \begin{pmatrix} \boldsymbol{a_{11}} & \cdots & \boldsymbol{a_{1n}} \\ -- & -- & -- \\ \boldsymbol{a_{n1}} & \cdots & \boldsymbol{a_{nn}} \end{pmatrix},$$

$$\sum_{i' \in \tilde{n}} a_{ii'} a^{*}_{i'i''} = \delta_{ii''}, \quad i, i'' \in \tilde{n}$$

(unitary condition)

$$D^{\{1\}}(a)|i> = \sum_{i' \in \tilde{n}} a_{i'i}|i'>$$

(notation of action $D^{\{1\}}(a)$ of single-node space)

U(n) - the group of full quantum symmetry of the single-node space h b)The linear chain

$\tilde{N} = \{j = 1, 2, ..., N\}$ - the alphabet of nodes

Remark: It is *essential* in the sequel to distinguish between N and n, and between the alphabets of spins and nodes.

c)The space of all quantum states of the magnet

$$\mathcal{H} = h_1 \otimes h_2 \otimes ... \otimes h_N = \prod_{j \in \tilde{N}} \otimes h_j \cong h^{\otimes N}$$

- $f : \tilde{N} \longrightarrow \tilde{n}$ - a magnetic configuration

or

$$|f> = |i_1, i_2, ..., i_N>, i_j \in \tilde{n}, j \in \tilde{N}$$

or

$i_1 i_2 ... i_N$ - the word of the length N
in the alphabet of spins

- $\tilde{\boldsymbol{n}}^{\tilde{\boldsymbol{N}}} = \{\boldsymbol{f} : \tilde{\boldsymbol{N}} \longrightarrow \tilde{\boldsymbol{n}}\}$ - the set of all magnetic configurations is an orthonormal basis in $\boldsymbol{\mathcal{H}}$,

$$\boldsymbol{\mathcal{H}} = \boldsymbol{lc}_{\mathbb{C}}\, \tilde{\boldsymbol{n}}^{\tilde{\boldsymbol{N}}}, \quad \dim \boldsymbol{\mathcal{H}} = \boldsymbol{n}^{\boldsymbol{N}}$$

Remarks:

(a) The set $\tilde{\boldsymbol{n}}^{\tilde{\boldsymbol{N}}}$ constitutes an initial orthonormal basis in $\boldsymbol{\mathcal{H}}$ for quantum computations. It defines the kinematics of the model, since it spans $\boldsymbol{\mathcal{H}}$ linearly and unitarily,

but

it does not yield itself the stationary states, desired in Physics.

(b) The stationary states are eigenstates of a model interaction Hamiltonian

$$\hat{H}|\psi> = E|\psi>$$
$$\hat{H} \in \text{End}\,\mathcal{H}, \quad |\psi> \in \mathcal{H} - \text{an eigenstate},$$
$$\hat{H} = \hat{H}^{+} \Longleftrightarrow E \in \mathbb{R}$$

When $\hat{H}$ is U(n) - invariant, then irreps of U(n) classify the spectrum of $\hat{H}$. But, as a rule, physically interesting Hamiltonians are only U(2) - invariant, $U(2) \subset U(n)$ then still irreps of U(n) provide a convenient *classification scheme* for perturbation calculations within an appropriate scheme of symmetry reduction

(c) Such a convenient classification scheme is provided by the Weyl duality

3. The duality of Weyl

Pairs of dual objects	The symmetric aspect	The unitary aspect
Alphabets	$\tilde{\boldsymbol{N}}$ (nodes)	$\tilde{\boldsymbol{n}}$ (spins)
Groups	$\boldsymbol{\Sigma_N}$	$\boldsymbol{U(n)}$
Actions	$\boldsymbol{A : \Sigma_N \times \mathcal{H} \longrightarrow \mathcal{H}}$	$\boldsymbol{B : U(n) \times \mathcal{H} \longrightarrow \mathcal{H}}$
Irreps	$\boldsymbol{\Delta^\lambda}$	$\boldsymbol{D^\lambda}$
Standard bases	SYT$(\boldsymbol{\lambda})$ (Young tableaux)	WT$(\boldsymbol{\lambda}, \tilde{\boldsymbol{n}})$ (Weyl tableaux)
Robinson-Schensted-Knuth, tableaux RSK(f) $\boldsymbol{f \in \tilde{n}^{\tilde{N}}}$	$\mathcal{Y} =$ Q(f)	$\mathscr{t} =$ P(f)

Definition of actions:

$$A(\sigma) = \begin{pmatrix} f \\ f \circ \sigma^{-1} \end{pmatrix}, \quad f \in \tilde{n}^{\tilde{N}}, \quad \sigma \in \Sigma_N,$$

$$B(a)|f\rangle = \sum_{i'_1,\dots,i'_N \in \tilde{n}} a_{i'_1 i_1} \dots a_{i'_N i_N} |i'_1, \dots, i'_N\rangle$$

for $|f\rangle = |i_1 i_2 \dots i_N > \in \tilde{n}^{\tilde{N}}, \quad a \in U(n)$

with the linear extension from the set $\tilde{n}^{\tilde{N}}$ onto the linear space $\mathcal{H}$

- These two actions commute

$$[\boldsymbol{A}(\sigma), \boldsymbol{B}(a)] = 0, \quad \sigma \in \Sigma_N, \quad \boldsymbol{a} \in \boldsymbol{U}(n),$$

which yields the decomposition of $\mathcal{H}$ into sectors

$$\mathcal{H} = \sum_{\lambda \vdash N, |\lambda| \leq n} \oplus \, \mathcal{H}^{\lambda}$$

so that

$$\boldsymbol{A} \,|_{\mathcal{H}^{\lambda}} = (dim D^{\lambda})\Delta^{\lambda}, \quad \boldsymbol{B} \,|_{\mathcal{H}^{\lambda}} = (dim \Delta^{\lambda}) D^{\lambda}$$

An irreducible basis in the sector $\mathcal{H}^{\lambda}$

$$\{|\lambda t y > |t \in WT(\lambda, \tilde{n}), y \in SYT(\lambda)\}.$$

$\boldsymbol{t}-$ a Weyl tableau, $\boldsymbol{sh\ t} = \boldsymbol{\lambda}$, in the alphabet of spins, i.e. a semistandard Young tableau

it is

1. the label of basis for $\boldsymbol{D}^{\boldsymbol{\lambda}}$ of U(n)
2. the repetition label for $\boldsymbol{\Delta}^{\boldsymbol{\lambda}}$ of $\boldsymbol{\Sigma}_{\boldsymbol{N}}$

$\boldsymbol{y}-$ a Young tableau, $\boldsymbol{sh\ y} = \boldsymbol{\lambda}$, in the alphabet of nodes, i.e. a standard Young tableau

it is

1. the label of basis for $\boldsymbol{\Delta}^{\boldsymbol{\lambda}}$ of $\boldsymbol{\Sigma}_{\boldsymbol{N}}$
2. the repetition label for $\boldsymbol{D}^{\boldsymbol{\lambda}}$ of U(n)

In this way, the duality of Weyl is kept at the level of bases

Physically, such an irreducible basis of the Weyl duality exhibits a maximal set of quantum numbers, which is compatible with the physical quantieties, which "can be measured simultaneously".

In this basis

$$\boldsymbol{A(\sigma)|\lambda t\, y\rangle = \sum_{y'\in SYT(\lambda)} \Delta^{\lambda}_{y'y}(\sigma)|\lambda t y'\rangle \ \ in\, \Sigma_N}$$

$$\boldsymbol{B(a)|\lambda t y\rangle = \sum_{t'\in WT(\lambda,\tilde{n})} D^{\lambda}_{t't}(a)|\lambda t' y\rangle \ \ in\, U(n}$$

Remarks:

(a) There is an interrelation between assumed standard bases of irreps and the corresponding matrices

$$|\boldsymbol{\lambda t y} > \ \sim \ \boldsymbol{D^{\lambda}_{t't}(a),\ \Delta^{\lambda}_{y'y}(\sigma)}$$

(b) In the literature, for the case of the symmetric group Σ_N, the same set $\boldsymbol{SYT(\lambda)}$ is used to distinct bases of $\boldsymbol{\Delta^{\lambda}}$, e.g. the normal basis, with $\boldsymbol{\Delta^{\lambda}_{y'y}(\sigma) \in \mathbb{Z},\ \sigma \in \Sigma_N}$, clearly differs from the orthogonal Young basis, with $\boldsymbol{\Delta^{\lambda}_{y'y}(\sigma) \in \mathbb{R}}$. Here we refer to this latter basis, for reason of the probabilistic interpretation of quantum states of the magnet.

4. Kostka matrices at the level of bases

The action $\boldsymbol{A} : \Sigma_{\boldsymbol{N}} \times \mathcal{H} \to \mathcal{H}$ decomposes the set $\tilde{\boldsymbol{n}}^{\tilde{\boldsymbol{N}}}$ into orbits

$$\tilde{\boldsymbol{n}}^{\tilde{\boldsymbol{N}}} = \bigcup_{\mu \models N} \mathcal{O}_{\mu}, \quad \mu = (\mu_1, \mu_2, \ldots, \mu_n)$$

$$\mu_i = |\{f^{-1}(j) = i \mid j \in \tilde{\boldsymbol{N}}\}|$$ - number of nodes with the spin i in the magnetic configuration f

$$\mathcal{O}_{\mu} = \{f \circ \sigma^{-1} \mid \sigma \in \Sigma_{\boldsymbol{N}}\}$$ - an orbit of $\Sigma_{\boldsymbol{N}}$ on $\tilde{\boldsymbol{n}}^{\tilde{\boldsymbol{N}}}$

$$\boldsymbol{A}|_{lc\mathcal{O}_{\mu}} = \sum_{\lambda \vdash N (\lambda \trianglerighteq \mu)} \boldsymbol{K}_{\lambda\mu} \boldsymbol{\Delta}^{\lambda}$$ - Kostka decomposition

$$= Q^{\Sigma_N : \Sigma^{\mu}} =$$

At the level of bases

$$|\boldsymbol{\mu}\,\boldsymbol{\lambda}\,\boldsymbol{t}\,\boldsymbol{y}\rangle = \sum_{f\in\mathcal{O}_\mu} \begin{bmatrix} \boldsymbol{\mu} & \boldsymbol{\lambda} & \boldsymbol{t} \\ \boldsymbol{f} & \boldsymbol{y} & \end{bmatrix} |\boldsymbol{f}\rangle,$$

$\begin{bmatrix} \boldsymbol{\mu} & \boldsymbol{\lambda} & \boldsymbol{t} \\ \boldsymbol{f} & \boldsymbol{y} & \end{bmatrix}$ - elements of Kostka matrices

at the level of bases

the matrix
is labelled by $\boldsymbol{\mu}$
rows by f $\in \boldsymbol{\mathcal{O}_\mu}$
columns by $(\boldsymbol{\lambda t y}) = \boldsymbol{RSK}(\boldsymbol{f'}), \boldsymbol{f'} \in \boldsymbol{\mathcal{O}_\mu}$

$\boldsymbol{t} \in \boldsymbol{WT}(\boldsymbol{\lambda}, \boldsymbol{\mu}, \boldsymbol{\tilde{n}}),\ \boldsymbol{y} \in \boldsymbol{SYT}(\boldsymbol{\lambda})$
$|\boldsymbol{WT}(\boldsymbol{\lambda}, \boldsymbol{\mu}, \boldsymbol{\tilde{n}})| = \boldsymbol{K_{\lambda\mu}}$

Young and Weyl tableaux play the role of basis and repetition labels, respectively, for $\boldsymbol{\Delta^\lambda}$ in $\boldsymbol{A}|_{\boldsymbol{\mathcal{O}_\mu}}$, in accordance with the famous Robinson-Schensted-Knut algorithm (RSK)

$$(\lambda\, t\, y) \equiv (\lambda(f'), P(f'), Q(f')) = \mathrm{RSK}(f'),$$
$$f' \in \mathcal{O}_{\mu}$$

$$P(f') = t \in \mathrm{WT}(\lambda, \mu, \tilde{n}),$$
$$Q(f') = y \in \mathrm{SYT}(\lambda)$$

$$\lambda(f') = \mathrm{sh}P(f') = \mathrm{sh}Q(f')$$

The element $\begin{bmatrix} \mu & \lambda & P(f') \\ f & Q(f') & \end{bmatrix}$ of the Kostka matrix at the level of bases is determined by a ladder construction, given by the combinatorial growth of the Weyl tableau $t = P(f')$ in the process of consecutive attachments of nodes $j = 1, 2, ..., N$ to already constructed state of j-1 nodes, that is

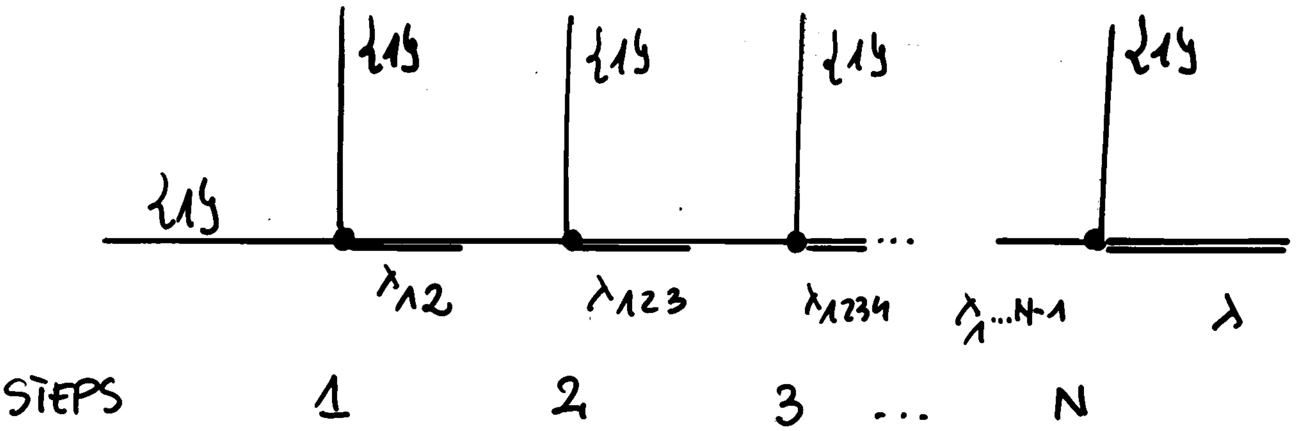
$$\begin{bmatrix} \mu & \lambda & t \\ f & y & \end{bmatrix} =$$

$$\begin{bmatrix} \{1\} & \{1\} & \lambda_{12} \\ f(1) & f(2) & t_{12} \end{bmatrix} \begin{bmatrix} \lambda_{12} & \{1\} & \lambda_{123} \\ t_{12} & f(3) & t_{123} \end{bmatrix} \cdots \begin{bmatrix} \lambda_{1...N-1} & \{1\} & \lambda \\ t_{1...N-1} & f(N) & t \end{bmatrix}.$$

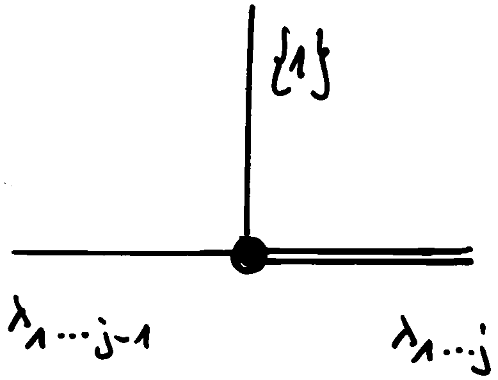
with

$\lambda_{1..j} = \mathrm{sh}y^{j}$, y^{j} - the Young tableau y at the j-th step,

The ladder construction



The j-th step



$$\Rightarrow \begin{bmatrix} \lambda_{1\ldots j-1} & \{1\} & \lambda_{1\ldots j} \\ t_{1\ldots j-1} & f(j) & t_{1\ldots j} \end{bmatrix}$$

$\lambda_{1\ldots j} = \mathrm{sh}\, y^j$, y^j - the Young tableau. y at the j-th step

$t_{1\ldots j}$ - the weyl tableau at the j^{th} step of the ladder Wigwele product.

$f(j) = i_j$; the basis of the fund. irrep $D^{\{1\}}$ on the "$\hat{j}$" node

λ_{12} - irreps of the intermediate rep $D^{\lambda_{12}}$ at the second step.

$t_{1..j}$ - the Weyl lableau at the j-th step of the ladder The intermediate Wigner - Clebsh - Gordan coefficient

$$\begin{bmatrix} \lambda_{1..j-1} & \{1\} & \lambda_{1..j} \\ t_{1..j-1} & f(j) & t_{1..j} \end{bmatrix}$$

describes the coupling at the j-th step of growth, according to the Littlewood - Richardson decomposition

$$D^{\lambda_{1...j-1}} \otimes D^{\{1\}} = \sum_{\lambda_{1...j}} \oplus D^{\lambda_{1...j}}$$

in the group U(n)

Such states are also eigenstates of the Jucys - Murphy operators

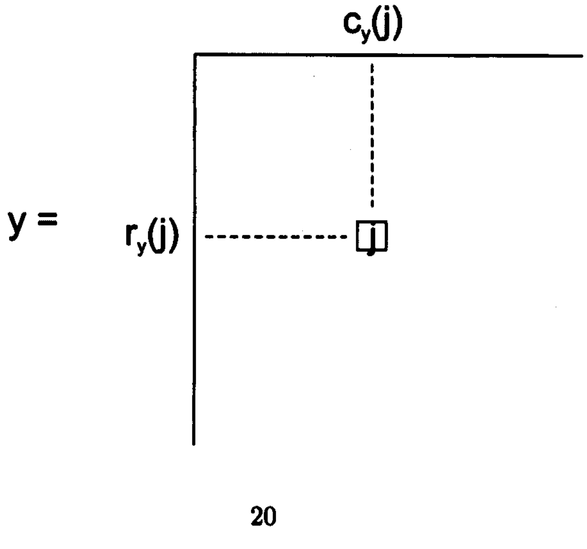
$$\hat{M}_j = \sum_{j'=1}^{j-1} (j, j'), \quad j = 2, 3, \ldots, N,$$

which form a maximal set of commuting operators within a carrier space of Δ^λ, that is

$$\hat{M}_j \, | \mu \, \lambda \, t \, y \rangle = m_j(y) \, | \mu \, \lambda \, t \, y \rangle,$$

with

$$m_j(y) = c_j(y) - r_j(y),$$



In this way, Young tableaux $\boldsymbol{y} \in \mathrm{SYT}(\boldsymbol{\lambda})$ classify the spectrum of Jucys - Murphy operators, and $|\boldsymbol{\mu}\,\boldsymbol{\lambda}\,\boldsymbol{t}\,\boldsymbol{y}\rangle$ is a realisation of the orthogonal Young basis for irreps $\boldsymbol{\Delta}^{\boldsymbol{\lambda}}$ of $\boldsymbol{\Sigma}_{\boldsymbol{N}}$

$$\tilde{\boldsymbol{n}}^{\tilde{\boldsymbol{N}}} = \bigcup_{\boldsymbol{\mu}\models\boldsymbol{N},||\boldsymbol{\mu}||\leq\boldsymbol{n}} \mathcal{O}_{\boldsymbol{\mu}}$$

$\boldsymbol{\mu}$ is a composition of N, or a weight

$$\mathcal{O}_{\boldsymbol{\mu}} \xrightarrow{RSK} \bigcup_{\boldsymbol{\lambda}\trianglerighteq\boldsymbol{\mu}} \mathrm{WT}(\boldsymbol{\lambda},\boldsymbol{\mu},\tilde{\boldsymbol{n}}) \times \mathrm{SYT}(\boldsymbol{\lambda}) \xrightarrow{KKR}$$

$$\rightarrow \bigcup_{\boldsymbol{\lambda}\trianglerighteq\boldsymbol{\mu}} \mathrm{RC}\,(\boldsymbol{\lambda},\boldsymbol{1}^{\boldsymbol{N}})$$

$$
\begin{array}{ccccc}
\boldsymbol{f} & \xmapsto{\text{RSK}} & (\boldsymbol{P}(\boldsymbol{f}), \boldsymbol{Q}(\boldsymbol{f})) & \xmapsto{\text{KKR}} & \boldsymbol{\nu}\alpha \\
 & \Downarrow & & \Downarrow & \\
 & \begin{bmatrix} \boldsymbol{\mu} & \boldsymbol{\lambda} & \boldsymbol{t} \\ \boldsymbol{f} & \boldsymbol{y} & \end{bmatrix} & & ? &
\end{array}
$$

at L.B. it corresponds

$$(\boldsymbol{\lambda}\, \boldsymbol{t}\, \boldsymbol{y}) = \mathrm{RSK}\,(\boldsymbol{f}')$$

where $\boldsymbol{\nu}$ - rigged string configurations

Kostka matrix at the level of bases can be determined KINEMATICALLY, by a ladder coupling of irreps

The corresponding transformation for KKR requires solution of the spectral eigenproblem of the Hamiltonian or solution of the system of Bethe equations (highly nonlinear) or (hope?) also by kinematics?

Conclusions

Kinematics of the Heisenberg chain can be expressed in terms of some orthonormal bases

(a) the initial basis $\tilde{\boldsymbol{n}}^{\tilde{N}}$ of magnetic configurations

(b) the basis of wavelets which is the Fourier transform of the basis of magnetic configurations on orbits of the translation group $\boldsymbol{C_N} \subset \boldsymbol{\Sigma_N}$ of the chain $\tilde{\boldsymbol{N}}$

(c) the irreducible basis of the Weyl duality between actions of the symmetric group $\boldsymbol{\Sigma_N}$ and the unitary group $\boldsymbol{U(n)}$; the corresponding Kostka matrices at the level of basis are constructed by a consecutive coupling of N copies of the fundamental representation $\boldsymbol{D}^{\{1\}}$ of the group U(n) along the Robinson - Schensted - Knuth algorithm

(d) the basis of eigenstates of the Heisenberg Hamiltonian, labelled by rigged string configurations of Kerov, Kirillov and Reshetikhin; till now we have no simple kinematical prescription for such an explicit construction