



A Telescoping Algorithm for Double Summations

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Identities

A large number of identities are of the form

$$\sum_k F(n, k) = f(n),$$

or, in general,

$$\sum_{k_1, \dots, k_r} F(n, k_1, \dots, k_r) = f(n).$$

They are called **definite summation** .

Examples

1. Li Shanlan's identity

$$\sum_{k=0}^n \binom{n}{k}^2 \binom{m+2n-k}{2n} = \binom{m+n}{n}^2.$$





2. Dixon's identity

$$\sum_k (-1)^k \binom{2n}{k}^3 = (-1)^n \frac{(3n)!}{(n!)^3}.$$

3. Andrews-Paule identity

$$\sum_{i=0}^n \sum_{j=0}^n \binom{i+j}{i}^2 \binom{4n-2i-2j}{2n-2i} = (2n+1) \binom{2n}{n}^2.$$

4. Carlitz's identity

$$\sum_i \sum_j \binom{i+j}{i} \binom{n-i}{j} \binom{n-j}{n-i-j} = \sum_{l=0}^n \binom{2l}{l}.$$





Sister Celine's Method

Basic idea: Try to find the recurrence relations for both sides of the identity.

In 1945, Sister Mary Celine Fasenmyer developed a method for finding recurrence relations for hypergeometric polynomials. H.S. Wilf and D. Zeilberger generalized her result to multisum case:¹

Theorem 1 Suppose $F(n, k_1, \dots, k_r)$ has the following form

$$P(n, k_1, \dots, k_r) \frac{\prod_{s=1}^p (a_s n + b_{s1} k_1 + \dots + b_{sr} k_r + c_s)!}{\prod_{s=1}^q (u_s n + v_{s1} k_1 + \dots + v_{sr} k_r + w_s)!} z_1^{k_1} \dots z_r^{k_r}$$

where $P(n, k_1, \dots, k_r)$ is a polynomial, a_s, b_{si}, u_s, v_{si} are integers. (We call F a proper hypergeometric term.)

¹An algorithmic proof theory for hypergeometric (ordinary and “q”) multisum/integral identities, Invent. Math. 108 (1992) 575-633.



Then it satisfies a nontrivial k -free recurrence relation:

$$\sum_{0 \leq j \leq J} \sum_{0 \leq i_1, \dots, i_r \leq I} a_{j, i_1, \dots, i_r}(n) F(n - j, k_1 - i_1, \dots, k_r - i_r) = 0,$$

where $a_{j, i_1, \dots, i_r}(n)$ are polynomials in n .

Once we get a k -free recurrence relation for F , summing over k , we will get a recurrence for

$$\sum_{k_1, \dots, k_r} F(n, k_1, \dots, k_r).$$

Thus the identity can be proved by checking whether $f(n)$ satisfies the same relation and checking the initial values.





Example

Let

$$F(n, i, j) = \binom{n}{j} \binom{j}{i} x^i y^{j-i} z^{n-j}.$$

Then

$$xF(n-1, i-1, j-1) + yF(n-1, i, j-1) + zF(n-1, i, j) - F(n, i, j) = 0.$$

Therefore, $f(n) = \sum_{i,j} F(n, i, j)$ satisfies

$$(x + y + z)f(n-1) - f(n) = 0.$$

Hence,

$$f(n) = (x + y + z)^n.$$



It is remarkable that the original Sister Celine's Method is not efficient enough. **Kurt Wegschaider** (student of Peter Paule, RISC, Austria) provided an significant improvements of Sister Celine's technique in his Ph.D. thesis. With his package **MultiSum**, we are able to attack multiple binomial sums more efficiently.





Telescoping Algorithm

In 1990, **D. Zeilberger** provided a fast algorithm for proving terminating hypergeometric identities². **H.S. Wilf and D. Zeilberger** also generalized to multisum case.³

The idea is to find a linear difference operator L with coefficients in the ring of polynomials in n

$$L = a_0(n)N^0 + a_1(n)N^1 + \dots + a_l(n)N^l$$

and rational functions R_1, \dots, R_r on n, k_1, \dots, k_r such that

$$LF = \sum_{i=1}^r \Delta_i(R_i F), \tag{1}$$

²D. Zeilberger, A fast algorithm for proving terminating hypergeometric identities. *Discr. Math.* 80 (1990) 207–211

³An algorithmic proof theory for hypergeometric (ordinary and “q”) multisum/integral identities, *Invent. Math.* 108 (1992) 575-633.



where N is the shift operator with respect to n , defined by

$$NF(n, k_1, \dots, k_r) = F(n + 1, k_1, \dots, k_r)$$

and Δ_i is the difference operator with respect to k_i , defined by

$$\Delta_i F(n, k_1, \dots, k_r) = F(n, k_1, \dots, k_{i-1}, k_i + 1, k_{i+1}, \dots, k_r) - F(n, k_1, \dots, k_r).$$

Once we have (1), we derive that

$$L \sum_{k_1, \dots, k_r} F(n, k_1, \dots, k_r) = 0$$

under the condition that there is large enough zone of zeros around the support of $F(n, k_1, \dots, k_r)$.

Thus, we get a recurrence relation for

$$\sum_{k_1, \dots, k_r} F(n, k_1, \dots, k_r).$$





Double Summations

For double summation, it can be rewritten as follows. Given $F(n, i, j)$, we try to find difference operator

$$L = a_0(n)N^0 + a_1(n)N^1 + \cdots a_m(n)N^m$$

and rational functions $R_1(n, i, j), R_2(n, i, j)$ such that

$$LF = \Delta_i(R_1F) + \Delta_j(R_2F).$$

It is solved by the following steps:

1. For fixed m , estimate the denominators of

$$R_1 = \frac{f_1}{g_1} \quad \text{and} \quad R_2 = \frac{f_2}{g_2}.$$



2. For given denominators g_1 and g_2 , estimate the degrees d_1, d_2 of f_1, f_2 respectively. Suppose

$$f_1 = \sum_{r+s \leq d_1} c_{r,s}^{(1)}(n) i^r j^s, \quad \text{and} \quad f_2 = \sum_{r+s \leq d_2} c_{r,s}^{(2)}(n) i^r j^s$$

3. Solve $a_0(n), \dots, a_m(n)$ and $c_{r,s}^{(1)}, c_{r,s}^{(2)}$ from

$$LF = \Delta_i(R_1 F) + \Delta_j(R_2 F)$$

by comparing coefficients. If there exists a non-trivial solution, we are done. Otherwise, increasing m and repeat items 1 – 3.





Example

Let

$$F(n, i, j) = \binom{n}{j} \binom{j}{i} x^i y^{j-i} z^{n-j}.$$

1. Take $m = 0$ and $g_1 = g_2 = 1$.

2. Take

$$f_1 = c^{(1)} y i, \quad f_2 = c^{(2)} z (i - j).$$

3.

$$LF = \Delta_i(R_1 F) + \Delta_j(R_2 F)$$

becomes

$$\begin{aligned} a_0 F(n, i, j) &= c^{(1)} y (i+1) F(n, i+1, j) - c^{(1)} y i F(n, i, j) \\ &+ c^{(2)} z (i - j - 1) F(n, i, j + 1) - c^{(2)} z (i - j) F(n, i, j) \end{aligned}$$





Dividing by $F(n, i, j)$ on both sides, we get

$$a_0 = c^{(1)}x(j - i) - c^{(1)}yi + y(j - n)c^{(2)} - c^{(2)}z(i - j).$$

That is,

$$(c^{(1)}(x + y) + c^{(2)}z)i - (c^{(1)}x + c^{(2)}(y + z))j + a_0 + c^{(2)}yn = 0.$$

Solving the linear equations system

$$\begin{cases} c^{(1)}(x + y) + c^{(2)}z = 0, \\ c^{(1)}x + c^{(2)}(y + z) = 0, \\ a_0 + c^{(2)}yn = 0. \end{cases}$$

we get

$$a_0 = c^{(1)} = c^{(2)} = 0.$$

Since there is only zero solution, we increase m by 1.

4. Now take $m = 1$ and

$$g_1 = g_2 = n + 1 - j.$$



5. Take

$$f_1 = c^{(1)}yi, \quad f_2 = c^{(2)}z(i - j).$$

6. Now

$$LF = \Delta_i(R_1F) + \Delta_j(R_2F)$$

becomes

$$a_0F(n, i, j) + a_1F(n+1, i, j) = \Delta_i \left(\frac{f_1}{g_1} F \right) + \Delta_j \left(\frac{f_2}{g_2} F \right).$$

Dividing F on both sides we get

$$(c^{(1)}x + c^{(1)}y + c^{(2)}z)i - (a_0 + c^{(1)}x + c^{(2)}y + c^{(2)}z)j + (n+1)(a_0 + a_1z + c^{(2)}y) = 0.$$

Solving

$$\begin{cases} c^{(1)}x + c^{(1)}y + c^{(2)}z = 0, \\ a_0 + c^{(1)}x + c^{(2)}y + c^{(2)}z = 0, \\ a_0 + a_1z + c^{(2)}y = 0, \end{cases}$$





we get a non-trivial solution

$$a_0 = y(x + y + z), \quad a_1 = -y, \quad c_1 = z, \quad c_2 = -(x + y).$$

Thus,

$$f(n) = \sum_{i,j} F(n, i, j)$$

satisfies

$$y(x + y + z)f(n) - yf(n + 1) = 0.$$

Therefore,

$$f(n) = (x + y + z)f(n - 1) = \dots = (x + y + z)^n.$$





Gosper's Algorithm: Bivariable Cases (only a trial algorithm)

To estimate the denominators of R_1 and R_2 , we first look at Gosper's algorithm in bivariable cases.

Given a hypergeometric term $F(i, j)$, our aim is to find rational functions $R_1(i, j), R_2(i, j)$ such that

$$F(i, j) = \Delta_i(R_1(i, j)F(i, j)) + \Delta_j(R_2(i, j)F(i, j)).$$

Let

$$R_1(i, j) = \frac{f_1(i, j)}{g_1(i, j)}, \quad R_2(i, j) = \frac{f_2(i, j)}{g_2(i, j)},$$
$$\frac{F(i+1, j)}{F(i, j)} = \frac{r_1(i, j)}{s_1(i, j)}, \quad \frac{F(i, j+1)}{F(i, j)} = \frac{r_2(i, j)}{s_2(i, j)},$$



and

$$u(i, j) = \gcd(s_1(i, j), s_2(i, j)), \quad v(i, j) = \gcd(g_1(i, j), g_2(i, j)),$$

$$s'_1(i, j) = s_1(i, j)/u(i, j) \quad s'_2(i, j) = s_2(i, j)/u(i, j)$$

$$g'_1(i, j) = g_1(i, j)/v(i, j) \quad g'_2(i, j) = g_2(i, j)/v(i, j)$$

We are concerned with those rational functions $R_1(i, j), R_2(i, j)$ satisfying the following three hypotheses.

(H1) For any irreducible factor $p(i, j)$ of $g_1(i, j)$ and $h_1, h_2 \in \mathbb{Z}$,

$$p(i + h_1, j + h_2) \mid g_1(i, j) \Rightarrow p(i + h_1, j + h_2) = p(i, j).$$

(H2) $\gcd(g'_1(i, j), v(i, j)) = \gcd(g'_2(i, j), v(i, j)) = 1$.

(H3) For any $h_1, h_2 \in \mathbb{Z}$, $\gcd(g'_1(i + h_1, j + h_2), g'_2(i, j)) = 1$.





Under these hypotheses, we have

Theorem 2 *The denominators $g_1(i, j), g_2(i, j)$ can be factored as follows:*

$$g_1(i, j) = v_1(i)v_2(j)v_3(i + j)v_4(i, j)h_{11}(j)h_{12}(i, j)$$

$$g_2(i, j) = v_1(i)v_2(j)v_3(i + j)v_4(i, j)h_{21}(i)h_{22}(i, j)$$

such that

$$v_1(i) \mid r_1(i - 1, j)s'_2(i - 1, j)$$

$$v_2(j) \mid r_2(i, j - 1)s'_1(i, j - 1)$$

$$v_4(i, j) \mid \gcd(r_1(i - 1, j)s'_2(i - 1, j), r_2(i, j - 1)s'_1(i, j - 1))$$

$$h_{12}(i, j) \mid \gcd(s_1(i, j)s'_2(i, j), r_1(i - 1, j)s'_2(i - 1, j))$$

$$h_{22}(i, j) \mid \gcd(s_2(i, j)s'_1(i, j), r_2(i, j - 1)s'_1(i, j - 1))$$





Proof.

We only consider the case that $p(i, j)$ is an irreducible factor of $v(i, j)$ which depending only on i . The other cases can be obtained analogously.

Substituting the previous notations in

$$F(i, j) = \Delta_i(R_1(i, j)F(i, j)) + \Delta_j(R_2(i, j)F(i, j))$$

and dividing both sides by $F(i, j)$, we get

$$\begin{aligned} & s_1(i, j)s'_2(i, j)g_1(i, j)g'_2(i, j)g_1(i+1, j)g_2(i, j+1) \\ &= f_1(i+1, j)r_1(i, j)s'_2(i, j)g_1(i, j)g'_2(i, j)g_2(i, j+1) \\ &\quad - f_1(i, j)s_1(i, j)s'_2(i, j)g'_2(i, j)g_1(i+1, j)g_2(i, j+1) \\ &\quad + f_2(i, j+1)r_2(i, j)s'_1(i, j)g_1(i, j)g'_2(i, j)g_1(i+1, j) \\ &\quad - f_2(i, j)s_1(i, j)s'_2(i, j)g'_1(i, j)g_1(i+1, j)g_2(i, j+1) \end{aligned}$$



Since $p(i, j)$ is an irreducible factor of $v(i, j)$, we have

$$p(i + 1, j) \mid r_1(i, j)s'_2(i, j)g_1(i, j)g'_2(i, j)g_2(i, j + 1)$$

and

$$p(i, j + 1) \mid r_2(i, j)s'_1(i, j)g_1(i, j)g'_2(i, j)g_1(i + 1, j).$$

Since $p(i, j)$ is a polynomial depending only on i , it follows that $\gcd(p(i + 1, j), g_1(i, j)) = 1$. Otherwise, by hypothesis (H1) we have that $p(i + 1, j) = p(i, j)$ is i -free, which is a contradiction. Similarly, $\gcd(p(i + 1, j), g_2(i, j)) = 1$. Since $p(i, j)$ is a polynomial depending only on i ,

$$\gcd(p(i + 1, j), g_2(i, j + 1)) = \gcd(p(i + 1, j + 1), g_2(i, j + 1)) = 1.$$

Therefore,

$$p(i + 1, j) \mid r_1(i, j)s'_2(i, j).$$

■





Restriction on the Denominators

Now we go back to the telescoping algorithm for double summations.

We focus on those R_1 and R_2 of the form

$$R_1(i, j) = \frac{1}{d(i, j)} \frac{f_1(i, j)}{g_1(i, j)} \quad \text{and} \quad R_2(i, j) = \frac{1}{d(i, j)} \frac{f_2(i, j)}{g_2(i, j)},$$

where $d(n, i, j)$ is defined by

$$\frac{LF(n, i, j)}{F(n, i, j)} = \sum_{l=0}^m a_l(n) \frac{F(n+l, i, j)}{F(n, i, j)} = \frac{c(n, i, j)}{d(n, i, j)}$$

which is independent of a_i .



We also need the previous hypotheses (H1)-(H3).

(H1) For any irreducible factor $p(i, j)$ of $g_1(i, j)$ and integers h_1, h_2 , $p(i + h_1, j + h_2)$ divides $g_1(i, j)$ implies $p(i + h_1, j + h_2) = p(i, j)$. The same does $g_2(i, j)$.

(H2) $\gcd(g'_1(i, j), v(i, j)) = \gcd(g'_2(i, j), v(i, j)) = 1$.

(H3) $\gcd(g'_1(i + h_1, j + h_2), g'_2(i, j)) = 1, \quad \forall h_1, h_2 \in \mathbb{Z}$.

Furthermore, suppose that

$$\begin{aligned} &\gcd(g_1(i, j), d(i + h_1, j + h_2)) \\ &= \gcd(g_2(i, j), d(i + h_1, j + h_2)) = 1, \quad \forall h_1, h_2 \in \mathbb{Z}. \end{aligned}$$





An estimation on the Denominators

Suppose

$$\frac{F(n, i + 1, j)}{F(n, i, j)} = \frac{r_1}{s_1} \quad \text{and} \quad \frac{F(n, i, j + 1)}{F(n, i, j)} = \frac{r_2}{s_2},$$

and

$$s'_1 = s_1 / \gcd(s_1, s_2) \quad s'_2 = s_2 / \gcd(s_1, s_2).$$

From the previous theorem, we derive that

Theorem 3 *Suppose*

$$LF = \Delta_i(R_1F) + \Delta_j(R_2F)$$

and R_1, R_2 satisfy the previous restrictions.



Then denominators $g_1(i, j), g_2(i, j)$ can be factored into polynomials:

$$g_1(i, j) = v_1(i)v_2(j)v_3(i + j)v_4(i, j)h_{11}(j)h_{12}(i, j)$$

$$g_2(i, j) = v_1(i)v_2(j)v_3(i + j)v_4(i, j)h_{21}(i)h_{22}(i, j)$$

such that

$$v_1(i) \mid r_1(i - 1, j)s'_2(i - 1, j)$$

$$v_2(j) \mid r_2(i, j - 1)s'_1(i, j - 1)$$

$$v_4(i, j) \mid \gcd(r_1(i - 1, j)s'_2(i - 1, j), r_2(i, j - 1)s'_1(i, j - 1))$$

$$h_{12}(i, j) \mid \gcd(s_1(i, j)s'_2(i, j), r_1(i - 1, j)s'_2(i - 1, j))$$

$$h_{22}(i, j) \mid \gcd(s_2(i, j)s'_1(i, j), r_2(i, j - 1)s'_1(i, j - 1)).$$





Application

In practice, the denominators g_1, g_2 can be reduced further.

1. Calculate $r_1, r_2, s_1, s_2, s'_1, s'_2$ defined by

$$\frac{F(n, i + 1, j)}{F(n, i, j)} = \frac{r_1}{s_1} \quad \text{and} \quad \frac{F(n, i, j + 1)}{F(n, i, j)} = \frac{r_2}{s_2},$$

and

$$s'_1 = s_1 / \gcd(s_1, s_2) \quad s'_2 = s_2 / \gcd(s_1, s_2).$$

2. Let $v_1(i)$ be the maximal factor of $r_1(i - 1, j)s'_2(i - 1, j)$ depending only on i and $v_2(j)$ be the maximal factor of $r_2(i, j - 1)s'_1(i, j - 1)$ depending only on j . Then take $v(i)$ to be greatest common divisor of $v_1(i)$ and $v_2(i)$.





3. Take $h_{11}(j)$ to be the maximal factor of $s_1(i, j)s'_2(i, j)$ which depends only on j and take $h_{21}(i)$ to be the maximal factor of $s_1(i, j)s'_2(i, j)$ which depends only on i .

4. Take $h_{12}(i, j)$ to be the maximal factor of

$$\gcd(s_1(i, j)s'_2(i, j), r_1(i-1, j)s'_2(i-1, j))$$

which depends on i . And take $h_{22}(i, j)$ to be the maximal factor of

$$\gcd(s_1(i, j)s'_2(i, j), r_2(i, j-1)s'_1(i, j-1))$$

which depends on j .

Finally,

$$g_1(i, j) = v(i)h_{11}(j)h_{12}(i, j) \quad g_2(i, j) = v(i)h_{21}(i)h_{22}(i, j)$$

In most cases, $g_1(i, j)$ and $g_2(i, j)$ can be further reduced by leaving out factors of degree 1 and 2 from g_1 and g_2 respectively.



Difficulty and Opportunity: We don't know anything at this moment about the factors $v_3(i+j)$, $h_{11}(j)$ and $h_{21}(i)$. However, this does not stop us getting a nice algorithmic proof because we can take the simplest case: setting them to 1.

This is the idea of getting a sufficient proof for insufficient reasons. Or we can say that the proposed algorithm works a special class of identities.

Rota: Don't fight with success!





The Andrews-Paule Sums

$$\sum_{i=0}^n \sum_{j=0}^n \binom{i+j}{i}^2 \binom{4n-2i-2j}{2n-2i} = (2n+1) \binom{2n}{n}^2.$$

Andrews-Paule Proof: Transform it to

$$\sum_{i=0}^{\lfloor \frac{m}{2} \rfloor} \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \binom{i+j}{i}^2 \binom{m+n-2i-2j}{n-2i} = \frac{\lfloor \frac{m+n+1}{2} \rfloor! \lfloor \frac{m+n+2}{2} \rfloor!}{\lfloor \frac{m}{2} \rfloor! \lfloor \frac{m+1}{2} \rfloor! \lfloor \frac{n}{2} \rfloor! \lfloor \frac{n+1}{2} \rfloor!}.$$

and use the operator $NM - N - M$, (how was it obtain?)

Suppose we can use the telescoping method, then we would have

$$g_1(i, j) = (2n - 2i + 1)(n - i + 1)(j + 1)^2,$$

$$g_2(i, j) = (2n - 2i + 1)(n - i + 1)(i + 1)^2.$$



Leaving out the factors $(n-i+1)$ and $(i+1)^2$ from $g_1(i, j)$ and $g_2(i, j)$ respectively, we obtain

$$\begin{aligned}\tilde{g}_1(i, j) &= (2n - 2i + 1)(j + 1)^2 \\ \tilde{g}_2(i, j) &= (2n - 2i + 1)(n - i + 1).\end{aligned}$$

The total degrees of the numerators can be chosen as 1 bigger than the total degrees of the denominators.

Note that if we do not leave out the above factors, the computation is still possible, and it is more likely that we will succeed.

In fact,

$$(2n + 1)F(n, i, j) = \Delta_i R_1 F(n, i, j) + \Delta_j R_2 F(n, i, j)$$

where

$$R_1 = \frac{i^2(6n^2 + 5n + 1 + \dots - 4ijn)}{(2n - 2i + 1)(1 + j)^2}$$



$$R_2 = \frac{-2n^2 + 2jn^2 + 6in^2 + \dots - 4i^2}{(2n - 2i + 1)}$$

Summing $i, j = 0, \dots, n$, we get

$$\begin{aligned} & (2n + 1) \sum_{i,j} F(n, i, j) \\ &= \sum_j (R_1 F(n + 1, j) - R_1 F(0, j)) + \\ & \quad \sum_i (R_2 F(i, n + 1) - R_2 F(i, 0)) \\ &= \sum R_1 F(n + 1, j) + (G(n + 1) - G(0)) \\ &= (n + 1)^2 \binom{2n + 1}{n}, \end{aligned}$$

where

$$G(i) = \frac{(-2n + i - 1)(-4n + 2i - 1)i}{-1 + 2i - 2n} \binom{4n - 2i}{2n - 2i}.$$



One see that there is only one term left in the summation: $\sum_j R_1 F(n+1, j)$. The second summation reduces to zero because $G(n+1) = G(0) = 0$. We are done!

Conceptually, the above proof is really simple. Note that we get the same denominators of R_1 and R_2 as the proof of Kurt Wegschaider for the variation of Andrews-Paule identity.

Conjecture: There exists a Racah style proof of the Andrews-Paule identity, and the Racah trick can be computerized.





Carlitz's identity I

$$\sum_i \sum_j \binom{i+j}{i} \binom{n-i}{j} \binom{n-j}{n-i-j} = \sum_{l=0}^n \binom{2l}{l}.$$

We have

$$g_1(i, j) = (j+1)^2(-n+j), g_2(i, j) = (i+1)^2(-n+i).$$

Cancelling the factors $(-n+j)$ and $(i+1)(-n+i)$, we obtain

$$\tilde{g}_1(i, j) = (j+1)^2 \quad \text{and} \quad \tilde{g}_2(i, j) = i+1.$$

Noting that for recurrence of degree 2 we have

$$d(i, j) = (-n+i-1+j)^2(-n+i-2+j)^2$$



We finally get

$$P(N, n)F(n, i, j) = \Delta_i R_1 F(n, i, j) + \Delta_j R_2 F(n, i, j)$$

where

$$P(N, n) = (4n + 6) - (8 + 5n)N + (n + 2)N^2$$

and the denominators of R_1, R_2 are $d(i, j)\tilde{g}_1(i, j)$ and $d(i, j)\tilde{g}_2(i, j)$ respectively.





Carlitz's identity II

$$\begin{aligned} \sum_i \sum_j \binom{i+j}{i} \binom{m-i+j}{j} \binom{n-j+i}{i} \binom{m+n-i-j}{m-i} \\ = \frac{(m+n+1)!}{m!n!} \sum_k \frac{1}{2k+1} \binom{m}{k} \binom{n}{k}. \end{aligned}$$

We have

$$g_1(i, j) = (n - j + i)(1 + j)^2, g_2(i, j) = (m - i + j)(i + 1)^2.$$

Cancelling the factors $(1 + j)$ and $(i + 1)^2$, we obtain

$$\tilde{g}_1(i, j) = (n - j + i)(1 + j) \quad \text{and} \quad \tilde{g}_2(i, j) = m - i + j.$$

Noting that for recurrence of degree 2 we have

$$d(i, j) = (-n + j - 1)^2(-n + j - 2)^2$$



We finally get

$$P(N, n)F(n, i, j) = \Delta_i R_1 F(n, i, j) + \Delta_j R_2 F(n, i, j)$$

where

$$P(N, n) = 2(m + 3 + n)(2 + m + n)^2 - (3m + 2nm + 4n^2 + 14 + 15n)(n + m + 3)N + (2n + 5)(n + 2)^2 N^2$$

and the denominators of R_1, R_2 are $d(i, j)\tilde{g}_1(i, j)$ and $d(i, j)\tilde{g}_2(i, j)$ respectively. The degrees of denominators and numerators of R_1, R_2 are both smaller than those given by Wilf-Zeilberger [WZ]⁴.

⁴An algorithmic proof theory for hypergeometric (ordinary and “q”) multisum/integral identities, Invent. Math. 108 (1992) 575-633.



A wrong idea: ...

A naive idea: ... Algorithmic approaches to multiple summations (in particular, a computerized Racah trick, or Saalschutz the trick) is the future to get into the territory of Jim Louck.

A Chinese idea of compromise:

