Combinatorics around Painlevé VI

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<u>Presentation</u> ⓪

You know there are many very famous Mathematical Schools in Kyoto/Kansai area of Japan.

① School of SATO on integrable systems, which includes: Date, Jimbo, Kashiwara, Miwa, ...

② School of Algebraic geometry, which includes: Mori, Mukai, K. Saito, ...

③ "Painlevé School", which includes: K. Okamoto, Umemura, Noumi, ...

Being in Japan for a long time, I definitely should participate in activities of these Schools.

④ School of Algebraic Combinatorics in Kyushu: Ito, Bannai, ...

⑤ School of Oshima on representation of Lie algebras: Oshima, Terasoma, ...

<u>However</u>,

"Low of big distances in Japan"

If a distance between the Schools is $N \gg 0$

$\Rightarrow$ communication between the Schools is $1/N$.

My talk will be very Elementary, but follow to request of Professor Alain Lascoux, I will include (some) proofs.

Combinatorics

Let $S \subset \{1,2,3,\dots\}$ be a finite subset which contains 1.

For any subset $T \subset S$ define

① $$d_{T,S} = \prod_{\substack{a \in T \\ b \in S \setminus T}} \frac{a+b}{|a-b|},$$

② $$d_{T,S}(q) = \prod_{\substack{a \in T \\ b \in S \setminus T}} \frac{1-q^{a+b}}{1-q^{|a-b|}}.$$

Problems:

① Describe subsets $S \subseteq \mathbb{N}$, $1 \in S$, such that

$$d_{T,S} \in \mathbb{N} \quad \text{for all subsets } T \subset S.$$

② Describe subsets $S \subset \mathbb{N}$, $1 \in S$, such that

$$d_{T,S}(q) \in \mathbb{N}[q] \quad \text{for all subsets } T \subset S.$$

Examples

① • $\{1,2\}$, $\{1,3\}$;

• $\{1,2,3\}$, $\{1,2,4\}$, $\{1,3,5\}$, $\{1,5,7\}$; (?).

② $S = \{1, 2, \ldots, n\}$;

③ $S = \{1, 2, 3, \ldots, n, n+2, \ldots, n+2m\}$;

④ $S = \{1, n, n+2, \ldots, pn-p-1\}$, $p \equiv 1 (2)$, $n \geq 2$.

$\cdots\cdots\cdots$ **Solution to ①.**

Question:

What is a combinatorial meaning of the <u>integer</u> numbers $d_{T,S}$?

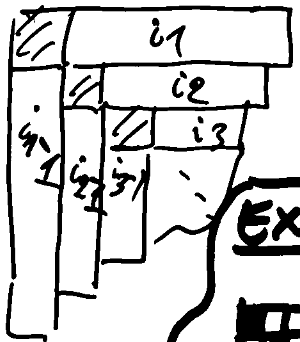
Ⓐ $S = \{1, 2, \ldots, n\}$.

• (J.F. van Diejen, A.N. Kirillov).

① Let $I \subset [1;n]$, $I = \{i_1 > \cdots > i_p\}$. Then

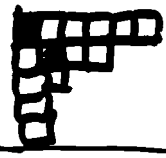
$$d_n(I) := \prod_{\substack{a \in I \\ b \in [1;n] \setminus I}} \frac{a+b}{|a-b|} = \dim^{(\mathfrak{ogl}(n))}_{\lambda(I)}, \text{ where}$$

$$\lambda(I) = \begin{pmatrix} i_1, \ldots, i_p \\ i_1 - 1, \ldots, i_p - 1 \end{pmatrix}$$



$= \lambda(I)$.

Ex $n=5$, $I=(2,5)$



$: \lambda(I)$

In particular, $d_n(I) \in \mathbb{N}$.

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② $d_I(q) = s_{\lambda(I)}(1, q, \ldots, q^{n-1}) \in \mathbb{N}[q]$.

$\boxed{d_I(q) := d_{I,[1;n]}(q)}$

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③ $$\sum_{I \subset [1;n]} d_n(I)\, x^{|I|} = (1+x)^{\binom{n+1}{2}},$$

where $|I| = \sum_{a \in I} a$.

④ $$\sum_{I\subset[1;n]} d_I(q)\, x^{|I|} = \prod_{0\le i\le j\le n-1} (1+q^{i+j}x).$$

Identities ③ & ④ follow from Littlewood's identity

$$\sum_{\lambda=\binom{\alpha_1+1,\dots,\alpha_p+1}{\alpha_1,\dots,\alpha_p}} s_\lambda(\underbrace{X_n}_{=(x_1,\dots,x_n)}) = \prod_{j=1}^{n}(1+x_j^2)\prod_{1\le i<j\le n}(1+x_ix_j),$$

and formulas ① & ②.

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⑤ Generalization

Let k be an integer, $0\le k\le n$, $n\geq 2$. Then

Th AK
$$\sum_{I\subset[k+1,\dots,n]} z^{|I|}\cdot d_n(I)\prod_{\substack{a\in I\\ j\in[1;k]}}\frac{a-j}{a+j} = d_n([1;k])\, C_n^{(k)}(-z)\cdot (1+z)^{\binom{n-k+1}{2}},$$
where

$$C_n^{(k)}(z) = \sum_{(\wp_1,\dots,\wp_k)} z^{\sum_i \mathrm{picks}(\wp_i)},$$ if $k>0$, $\boxed{C_n^{(0)}\equiv 1}$

summed over k-tuples $(\wp_1,\dots,\wp_k)$ of non-crossing Dyck's paths $\wp_1,\dots,\wp_n$ s.t. the path $\wp_j$ starts from $(0,j)$ and ends at $(0,2n-j)$.

In particular, $$C_n^{(k)}(1) = \prod_{1\le i\le j\le n-1}\frac{2k+i+j}{i+j} =$$

$$= \sum_{\lambda,\ \lambda_1 \le 2k,\ \lambda'_i \equiv 0\,(2)} |STY(\lambda, \le n-1)|.$$

J. Stembridge, G. Viennot, I. Gessel

(5′) $$V_n^{(1)}(z) = \sum_{k=1}^{n} \underbrace{\frac{1}{n}\binom{n}{k}\binom{n}{k-1}}_{\text{Narayana's numbers.}} z^{k-1}$$

Problems:

(1) Find a formula similar to that in (5) for the following sums:
Let $K \subset [1;n]$ be a subset, define the sum

$$\sum_{I \subset [1;n] \setminus K} d_n(I) \prod_{\substack{a \in I \\ b \in K}} \frac{a-b}{a+b}\, z^{|I|} := g_K(z).$$

(2) Prove that $C_n^{(k)}(z)$ is an unimodal polynomial.

Remark It follows from Cauchy's identity that

$$\det\left| z^a \delta_{a,b} + \frac{a}{a+b}\binom{n+a}{a}\binom{n}{a}\prod_{j=1}^{k}\frac{a-j}{a+j}\right|_{k<a,b\le n} =$$

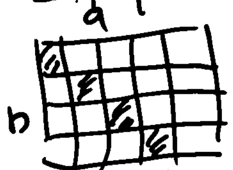
$$:= g_{[1;k]}(z) \overset{(5)}{=} (1+z)^{\binom{n-k+1}{2}} d_{[1;k],[1;n]}\, C_n(z).$$

$\boxed{\text{Exercise (A.N.K)}}$ Let $p \geq -1$ be an integer, and $I = \{j_1 > j_2 > \ldots > j_p\} \subset [1;n]$ be a subset. Consider partition $\lambda_{p,J}$ with Frobenius' symbol

$(j_1+p, \dots, j_r+p \mid j_r-1, \dots, j_r-1)$. Then

$$\dim V_{\lambda_{p,J}}^{(\mathfrak{gl}(n))} = \prod_{\substack{a \in J \\ b \in [1;n] \setminus J}} \frac{a+b+p}{|a-b|} = d_n(\lambda_{p,J}). \quad (\star)$$

For example, take $a \geq b, c$ three positive integers. Let $n = b+c$, $p = a-b-1$, $J = [1;b] \subset [1;n]$

$\lambda_{p,J} =$ 

Then $$\dim_{\lambda_{p,J}}^{(\mathfrak{gl}(n))} = \prod_{i=0}^{a-1} \prod_{j=0}^{b-1} \prod_{k=0}^{c-1} \frac{i+j+k+2}{i+j+k+1} = \mathrm{RHS}(\star)$$

This number is equal to that of plane partitions with all parts $\leq c$ and shape $\lambda \subset$  $(= \lambda_{p,J})$

Similarly, $$s_{\lambda_{p,J}}(1, \dots, q^{n-1}) \doteq \prod_{\substack{a \in J \\ b \in [1;n] \setminus J}} \frac{1-q^{a+b+p}}{1-q^{|a-b|}}.$$

Ⓑ $S_{n,m} := \{1, 2, \dots, n, n+2, \dots, n+2m\}$.

Theorem (J.F. van Diejen, A.N. Kirillov)

Let $I \subset S_{n,m}$, then

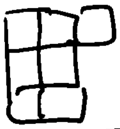
$$d_{n,m}(I) = \prod_{\substack{a \in I \\ b \in S_{n,m} \setminus I}} \frac{a+b}{|a-b|} \in \mathbb{N};$$

$$d_{n,m}(I \| q) := \prod_{\substack{a \in I \\ b \in S_{n,m} \setminus I}} \frac{1-q^{a+b}}{1-q^{|a-b|}} \in \mathbb{N}[q].$$

Example. Take $J \subset [1,2,\dots,n]$, then

$$d_{n,m}(J) = \frac{1}{h(\lambda_J)} \prod_{j \in J} \left\{ \prod_{a=1}^{\min(m,j)} (n-j+(2a-1))(n-j+2\max(m,j)+2a) \prod_{b=1}^{2(j-m)_+} (n-j+2m+a) \right\},$$

where $\lambda_J = \begin{pmatrix} j_1, \dots, j_p \\ j_1-1, \dots, j_p-1 \end{pmatrix}$ is the partition with Frobenius' symbol displayed, and $h(\lambda)$, for any diagram λ, is the product of hooks in λ.

E.g.  $h(\lambda) = 5 \cdot 4 \cdot 1 \cdot 3 \cdot 2 \cdot 2 \cdot 1$

PROBLEM:

Find combinatorial interpretations of the numbers $d_{n,m}(J)$, and polynomials $d_{n,m}(J \| q)$.

We expect that $d_{n,m}(J \| q)$ is a unimodal (log-concave ?) polynomial.

Theorem (F. Diejen, A.N. Kirillov)

$$\sum_{I \subset S_{n,m}} c(I) \cdot d_{n,m}(I)\, z^{|I|} = \left(\frac{z+1}{1-z}\right)^{\binom{m+1}{2}} (z+1)^{\binom{n+m+1}{2}},$$

where $c(I) = (-1)^{\sum_{a \in J} c(a)}$, and for any $a \in S_{n,m}$,

$$c(a) = \begin{cases} 0, & \text{if } a \le n, \\ \frac{a-n}{2}, & \text{if } a > n. \end{cases}$$

Remark It's not difficult to see that the LHS is equal to the following determinant

$$LHS = \det \left| z^a \delta_{a,b} + \frac{a}{a+b} \prod_{j=1}^{a} \frac{(n-1-a+2j)}{j} \frac{(n-a+2m+2j)}{j} \right|_{a,b \in S_{n,m}}$$

So the problem is to evaluate this determinant!

Generalization

Theorem (A.N.K.) Let $0 \le p \le n$ be an integer, consider matrix $(M^{(p)}_{ab})$, where

$$M^{(p)}_{a,b} = z^a \delta_{a,b} + \frac{a}{a+b} \cdot \prod_{j=1}^{a} \frac{n-1-a+2j}{j} \frac{n-a+2m+2j}{j} \prod_{j=1}^{p} \frac{a-j}{a+j},$$

where $a, b \in S_{n,m} \setminus [1;p]$. Then

$$\det(M^{(p)}_{a,b}) = d_{n,m}([1;p]) \, (1+z)^{\binom{n+m-p+1}{2}} \cdot C^{(p)}_{n,m}(z),$$

where

ⓐ $C^{(p)}_{n,m}(z)$ is a polynomial in $(-z)$ with non-negative (rational coefficients);

ⓑ $C^{(p)}_{n,m}(z)$ is divisible by $(1-z)^{\binom{(m-p+1)_+}{2}}$,

where $(a)_+ := \max(a,0)$;

ⓒ $$C^{(p)}_{n,m}(-1) = 2^{\binom{m+1}{2}} \underbrace{\prod_{1 \le i \le j \le n+m-p} \frac{2p+i+j}{i+j}}_{\text{integer}} \underbrace{\prod \frac{(n+2a)_p}{(n+2m-2a+3)_p}}_{\text{rational number in general}},$$

where $(a)_p = a(a+1)\dots(a+p-1)$;

Remark $d_{n,m}([1;p]) \cdot C^{(p)}_{n,m} \in \mathbb{N}$.

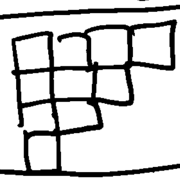
Example: $C^{(2)}_{42}(z) = \cancel{300}\ \frac{1}{5}(5, -35, 155, -294, 408, -294, 155, -35, 5)\ \cancel{(1+z)^{10}}\ (1-z)^3\ d_{42}([1;2])$

$$\boxed{C^{(2)}_{42}(-1) = \frac{1}{5} \cdot 2^4 \cdot 3^2 \cdot 7 \cdot 11}$$

Theorem (F. Degen & A.N.K)

$$\sum_{I\subset S_{n,m}}' c(I) \prod_{\substack{a\in I\\ b\in S_{n,m}\setminus I}} \frac{1-q^{a+b}}{|q^a-q^b|}\, z^{|I|} =$$

$$= \prod_{x\in\delta_m} (1-q^{c(x)}z) \prod_{x\in\delta_{n+m}} (1+q^{c(x)}z),$$ where

$\delta_N=(N,N-1,\dots,1)$ denotes the staircase Young diagram, (Ex.  $=\delta_4$) and for any $x=(i,j)\in\delta_N$, $c(x):=j-i$, denotes its content.

PROBLEM

Does there exist a Littlewood type identity behind the identities of (B)

GENERALIZATION:
(4-parameter identity)

Given four non-negative integers n_+, m_+, n_-, m_-, consider the sets

$$\sigma_\pm = \{1,2,\dots,n_\pm, n_\pm+2,\dots,n_\pm+2m_\pm\},$$ and

$\sigma=\sigma_+\cup\sigma_-$.

If $I\subset\sigma$, we define $I_\pm = I\cap\sigma_\pm$.

Let $\varepsilon=\pm1$, define function $c^\varepsilon:\sigma\to\{\pm1\}$,

$c^\varepsilon(a)=(-1)^{(a-\varepsilon a)/2}$, if $a\in\sigma_\pm$, $a\le n_\pm$, and

$$c^{\varepsilon}(a) = (-1)^{(a=\varepsilon n_\pm)/2}, \quad \text{if } a \in \sigma_\pm, \quad a \geq n_\pm .$$

Finally, let $\varepsilon_1 = \pm 1$, $\varepsilon_2 = \pm 1$, and $I \subset \sigma$, define

$$c^{\varepsilon_1 \varepsilon_2}(I) = \prod_{a \in I_+} c^{\varepsilon_1}(a) \prod_{a \in I_-} c^{\varepsilon_2}(a)$$

DEFINITION $\quad T^{\varepsilon_1\varepsilon_2}_{n_\pm, m_\pm}(z) =$

$$\sum_{I \subset \sigma} c^{\varepsilon_1\varepsilon_2}(I) \prod_{\substack{a \in I_+,\, b \notin I_+ \\ \text{or} \\ a \in I_-,\, b \notin I_-}} \frac{1-q^{a+b}}{|q^a - q^b|} \prod_{\substack{a \in I_+,\, b \notin I_- \\ \text{or} \\ a \in I_-,\, b \notin I_+}} \frac{1+q^{a+b}}{q^a + q^b} \cdot z^{|I|}.$$

THEOREM (F. Diejen, A.N. Kirillov)

$$T^{\varepsilon_1\varepsilon_2}_{n_\pm, m_\pm}(z) = \tau^{\varepsilon_1}\left(n_+ + m_+, \left|\frac{\varepsilon_1+\varepsilon_2}{2}\right| n_- + m_-\right) \cdot \tau^{-\varepsilon_1}\left(m_+, \left|\frac{\varepsilon_1-\varepsilon_2}{2}\right| n_- + m_-\right),$$

where

$$\tau^\pm(a,b) = \tau^\pm(z|a,b) = \begin{cases} \mu^\pm\left(z, \frac{a+b}{2}\right) \nu^\pm\left(z, \frac{|a-b|}{2}\right) & \text{if } a+b \equiv 0\ (2), \\ \mu^\pm\left(z, \frac{|a-b|-1}{2}\right) \nu^\pm\left(z, \frac{a+b+1}{2}\right) & \text{if } a+b \equiv 1\ (2), \end{cases}$$

and

$$\mu^\pm(z,m) = \prod_{k=1}^{m} \left[(1 \pm q^{2k-1} z)(1 \pm q^{-2k+1} z)\right]^{m+1-k},$$

$$\nu^\pm(z,m) = \prod_{k=1}^{m} (1 \pm q^{2k} z)^{m-k} (1 \pm q^{-2k+2} z)^{m+1-k}.$$

Corollary (The case $m_+ = m_- = 0$)

$$T^{\varepsilon_1\varepsilon_2}_{n_+,n_-}(z) = \prod_{x\in\delta_{2(n_++n_-)}} (1+\varepsilon_1 q^{c(x)} z), \quad \underline{\text{if } \varepsilon_1\varepsilon_2 = +1},$$

$$T^{\varepsilon_1\varepsilon_2}_{n_+,n_-}(z) = \prod_{x\in\delta_{2n_+}} (1+\varepsilon_1 q^{c(x)} z) \prod_{x\in\delta_{2n_-}} (1+\varepsilon_2 q^{c(x)} z),$$

if $\varepsilon_1\varepsilon_2 = -1$

$$\boxed{= T^{1,\varepsilon_1}_{n_+,0}(z)\, T^{1,\varepsilon_2}_{n_-,0}(z)}$$

This factorization is unclear from the begining.

Example. $\mathfrak{O}_+ = \{1,3\}$, $\mathfrak{O}_- = \{1\}$, i.e. $n_+ = m_+ = 1$, $n_- = 1$, $m_- = 0$.

$T^{++}_{1,1;1,0}(q,z) =$

$$\det \begin{vmatrix} z + \frac{(1+q^2)^2}{2q^2}, & \frac{1+q^2}{2q^2}, & \frac{1-q^4}{2q^2} \\ -\frac{(1-q^6)(1+q^4)}{(1-q^4)\,q^2}, & z^3 - \frac{1+q^4}{q^2}, & -\frac{1-q^6}{q^2} \\ \frac{(1-q^2)(1+q^4)}{2q^2(1+q^2)}, & \frac{1-q^2}{2q^2}, & z + \frac{1+q^4}{2q^2} \end{vmatrix} \qquad \begin{matrix} z^2 - \frac{2q^2}{1+q^2}z + q^2 \\ \\ 1 \\ \\ z^2 - q^2 \end{matrix}$$

$$= (1+q^2 z)(1+q^{-2}z)(1+z)^2(z-1) = \qquad (z^2-1)(z+q^{-2})$$

$$= \tau^+(z|2,1)\,\tau^-(z|1,0).$$

Main steps of proof

First proof: The case $\mathfrak{A}_- = \emptyset$.

Theorem Let $\mathfrak{A} := \{1, 2, \ldots, n, n+2, \ldots, n+2m\}$.
Consider determinant $D_{n,m}(z;q) =$

$$\det \left| z^a \delta_{a,b} + \frac{1-q^a}{1-q^{a+b}} \, d_a(n,m) \right|_{a,b \in \mathfrak{A}}, \text{ where}$$

$$d_a(n,m) = \frac{\prod_{k=1}^{a} (1-q^{n-a+2k-1}) \prod_{k=1}^{a} (1-q^{n+2m-a+2k})}{(q;q)_a \, (q;q)_a \, q^{a(n+m)-a(a+1)/2}}.$$

THEN

$$D_{n,m}(z,q) = \prod_{x \in \delta_m} (z - q^{c(x)}) \prod_{x \in \delta_{n+m}} (z + q^{c(x)}),$$

where $\delta_n = (n, n-1, \ldots, 2, 1)$ denotes the staircase Young diagram of height n, and $c(x) = j - i$, if $x \in \delta_n$, $x = (i,j)$ denotes the content of the cell x.

$$\begin{array}{|c|c|c|c|} \hline 0 & 1 & 2 & 3 \\ \hline -1 & 0 & 1 & \\ \cline{1-3} -2 & -1 & & \\ \cline{1-2} -3 & & & \\ \cline{1-1} \end{array} \; := \delta_4$$

$\boxed{\text{Lemma 1}}$ $D_{n,m}(z,q) = T^{++}_{n,m;0,0}(q,z) =$

$$= \sum_{I \subset \mathfrak{A}_{n,m}} c(I) \prod_{\substack{a \in I \\ b \in I^c}} \left| \frac{1-q^{a+b}}{q^a - q^b} \right| z^{|I|} .$$

Main Lemma For each $a \in \mathfrak{A}_{n,m}$, there exists a polynomial $f_a(z;q) \in \mathbb{Q}(q)[z]$ in z of degree $n+2m-a$, with coefficients in the field of rational functions $\mathbb{Q}(q)$ such that

$$\sum_{a \in \mathfrak{A}_{n,m}} \left(z^a \delta_{a,b} + \frac{1-q^a}{1-q^{a+b}} d_a(n,m) \right) f_a(z;q) = \qquad (!)$$

$$= \prod_{j=0}^{m-1} (z - q^{-j}) \prod_{j=0}^{n+m-1} (z + q^{-j}), \quad \text{for all } b \in \mathfrak{A}_{n,m}$$

Lemma Let $a \in \mathfrak{A}_{m,n}$, and $f_a(z;q) :=$

$$= \sum_{k=0}^{n+2m} (-1)^k d_k^{(a)}(n,m \mid q)\, z^{n+2m-a-k} .$$

The rational functions

$$d_k^{(b)}(n,m \mid q) := q^{k(-n-m+b+k+1)} \frac{[b]!\,[n+2m-b]!}{[b+k]!\,[n+2m-b-k]!} \cdot$$

$$\left\{ \sum_{r \geq 0} q^{r(n-b-2k+2r)} \begin{bmatrix} m \\ r \end{bmatrix}_{q^2} \begin{bmatrix} n \\ k-2r \end{bmatrix}_q \prod_{j=1}^{r} \frac{1-q^{n+b+2j-1}}{1-q^{n+2m-b-2j+1}} \right\}$$

are a UNIQUE SOLUTION of relations (!).

Key Lemma

$$\alpha_k^{(b)}(n,m|q) = q^{-(n+2m-b-2k)}\,\alpha^{(b)}_{n+2m-b-k},$$

for all $0 \le k \le n+2m-b$.

In other words, $f_a(z,q)$ is a symmetric polynomial $\forall a \in \Omega$

- Main Lemma $\Longrightarrow$ Theorem (not so difficult)
- Key Lemma $\Longrightarrow$ Lemma $\Longrightarrow$ Main Lemma (not so easy)
- Key Lemma $\Longleftarrow$ Transformation formula for some $_8\varphi_7$. Namely, $\alpha_k^{(b)}(n,m|q) =$

$$q^{k(-n-m+b+k+1)}\,\frac{\begin{bmatrix} n+2m-b \\ k \end{bmatrix}_q \begin{bmatrix} n \\ k \end{bmatrix}_q}{\begin{bmatrix} b+k \\ k \end{bmatrix}_q}.$$

$$_8\varphi_7\left(\begin{matrix} q^{-m}, q^{-m}, q^{-k/2}, -q^{-k/2}, q^{-\frac{k-1}{2}}, -q^{-\frac{k-1}{2}}, q^{\frac{n+b+1}{2}}, -q^{\frac{n+b+1}{2}} \\ -q, q^{\frac{n-k+1}{2}}, -q^{\frac{n-k+1}{2}}, q^{\frac{n-k+2}{2}}, -q^{\frac{n-k+2}{2}}, q^{-\frac{n+2m-b-1}{2}}, -q^{\frac{n+2m-b-1}{2}} \end{matrix}; q, q\right)$$

$$\boxed{\text{This is } \underline{\text{not}} \text{ very-well-poised } {}_8\varphi_7!}$$

$$\alpha_k^{(b)}(n,m|q) = q^{-(n+2m-b-2k)}\,\alpha^{(b)}_{n+2m-b-k}(n,m|q)$$

follows from a certain transformation law

Second proof (main ideas),
case $\sigma_- = \phi$, $\varepsilon_1 = \varepsilon_2 = +1$.

Recall: $\sigma = \{1, 2, \dots, n, n+2, \dots, n+2m\}$,

$$T_{n,m} := T_{n,m}(q,z) = \sum_{I \subset \sigma} d_{n,m}(I)\, c(I)\, z^{|I|},$$

where $C(I) = \prod_{j \in I} c(j)$, $c(j) = \begin{cases} 0, & j \le n, \\ \frac{j-n}{2}, & j \geq n. \end{cases}$

THEOREM: (Recurrence relation)

$$T_{n,m+1}\, \widetilde{T}_{n,m-1} = (1 - z^2)\, T_{n,m}(qz)\, T_{n,m}(q^{-1} z),$$

where $\widetilde{T}_{n,m} = T_{n,m}$ if $m > 0$, and

$\widetilde{T}_{n,0} = T_{n-1,0}$, if $n \geq 1$.

$\Rightarrow$ Reduce general case to the case $m = 0$.

Recall: $d_{n,m}(I) = \prod_{\substack{a \in I \\ b \in I^c}} \frac{1 - q^{a+b}}{|q^a - q^b|}$.

Definition let $I, J \subset \sigma$, define

$$f(z) := f_{I,J}(z) = \prod_{a \in I} \frac{1 - q^{a+2} z}{q^a - q^2 z} \prod_{a \in J} \frac{q^a - z}{1 - q^a z} + \prod_{a \in I} \frac{q^a - z}{1 - q^a z} \prod_{a \in J} \frac{1 - q^{a+2}}{q^a - q^2 z}.$$

Lemma 1 $f_{I,J}(z) = q^{|I|-|J|} + q^{|J|-|I|} +$

$$+ \sum_{a\in I} \frac{c_a^{I,J}\, z}{(q^a - q^2 z)(1-q^a z)} + \sum_{b\in J} \frac{c_b^{J,I}\, z}{(q^b-q^2 z)(1-q^b z)}$$

$$+ \frac{D_1}{(1-qz)^2}.$$

Proof Residue calculus. Moreover,

$$c_a^{I,J} = q^{-a}(q^2-q^{2a})(1-q^{2a}) \prod_{\substack{i\in I\\ i\neq a}} \frac{1-q^{i+a}}{q^i-q^a} \prod_{j\in J} \frac{q^j-q^{j+a}}{1-q^{j+a-2}}$$

$$D_1 = \begin{cases} 0, & \text{if } 1 \notin I\cap J, \\ \text{explicit expression}, & \text{if } i \in I\cap J. \end{cases}$$

KEY LEMMA: Let $a \in I$ ($a>1$), put

$$I_1 = (I \setminus \{a\})^c, \quad J_1 = (J \cup \{a-2\})^c$$

Then

$$c_a^{IJ}\, d_{nm}(I)\, d_{nm}(J) = (-1)^A \cdot$$

$$c_a^{I_1 J_1} \cdot d_{n,m}(I_1)\, d_{n,m}(J_1), \text{ where}$$

$$A = \begin{cases} 0, & \text{if } a \le n, \\ 1, & \text{if } a > n. \end{cases}$$ $\boxed{\text{similar result if } a = 1}$

Proof Using explicit formula for $c_a^{I,J}$ (and D_1).

Lemma 2. (a) $d_{n,m+1}(I) = d_{n,m}(I) \prod\limits_{a\in I} \dfrac{q^{n+2m+2+a} - 1}{q^{n+2m+2} - q^a}$ (16a

(b) If $n+2m \notin I$, then

$$d_{n,m-1}(I) = d_{n,m}(I) \prod_{a\in I} \frac{q^a - q^{n+2m}}{1 - q^{n+2m+a}}$$

Proof of recurrence relation ($m \geq 1$).

$$T_{n,m+1} T_{n,m-1} = \sum_{\substack{I\subset \mathfrak{A}_{n,m+1} \\ J\subset \mathfrak{A}_{n,m-1}}} d_{n,m+1}(I)\, d_{n,m-1}(J)\, c(I)\, c(J)\, z^{|I|+|J|} =$$

$$= \sum_{\substack{n+2m+2 \notin I \\ J\subset \mathfrak{A}_{n,m-1}}} (\dots) + \sum_{\substack{n+2m+2 \in I \\ J\subset \mathfrak{A}_{n,m-1}}} (\dots).$$ In the second sum change $I \to \underbrace{\mathfrak{A}_{n,m+1} \setminus I}_{= I^c}$, $J \to \underbrace{\mathfrak{A}_{n,m-1} \setminus J}_{= J^c}$.

Important remarks:

$$\boxed{\begin{aligned} &d_{n,m}(I) = \\ &= d_{n,m}(I^c) \end{aligned}}$$

$$c(I)\,c(J) = -\,c(I^c)\,c(J^c)$$
$$|I^c| + |J^c| = |I| + |J| + 2 \text{, if } n+2m+2 \in I.$$

$$= \sum_{\substack{n+2m+2 \notin I \\ J\subset \mathfrak{A}_{n,m-1}}} d_{n,m+1}(I)\, d_{n,m-1}(J)\, c(I)\, c(J)\, z^{|I|+|J|}$$

$$- \sum_{\substack{n+2m+2 \notin I \\ J\subset \mathfrak{A}_{n,m-1}}} d_{n,m+1}(I)\, d_{n,m-1}(J)\, c(I)\, c(J)\, z^{|I|+|J|+\dots} =$$

Using Lemma 2, Symmetry of $d_{n,m}$'s and Definition of the function $f_{I,J}(z)$,

$$= \frac{1}{2} \sum_{I,J \subset \Omega_{n,m}} f_{I,J}(q^{n+2m})\, d_{n,m}(I)\, d_{n,m}(J)\, c(I)\, c(J) \cdot z^{|I|+|J|}$$

$$- \frac{1}{2} \sum_{I,J \subset \Omega_{n,m}} f_{I,J}(q^{n+2m})\, d_{n,m}(I)\, d_{n,m}(J)\, c(I)\, c(J)\, z^{|I|+|J|+2} =$$

KEY LEMMA

$$= \frac{1}{2} \Bigg\{ \sum_{I,J \subset \Omega_{n,m}} \underbrace{q^{|I|-|J|} + q^{|J|-|I|}} + \sum_{\substack{a \in I \\ a-2 \notin J \\ a \neq 1}} \frac{C_a^{I,J}\, q^{n+2m}}{(q^a - q^{n+2m+2})(1 - q^{a+n+2m})}$$

$$+ \sum_{\substack{b \in J \\ b \neq 1}} \frac{C_I^{I,J}\, q^{n+2m}}{(q^a - q^{n+2m+2})(1 - q^{a+n+2m})} + \frac{D_1^{I,J}}{(1-qz)^2} \Bigg\} \cdot$$

$$d_{n,m}(I)\, d_{n,m}(J)\, c(I)\, c(J) \cdot z^{|I|+|J|}$$

$$- \frac{1}{2} \sum_{I,J \subset \Omega_{n,m}} \Big\{ \underbrace{q^{|I|-|J|} + q^{|J|-|I|}} + \sum_{\substack{a \in I \\ a \neq 1}} (\cdots)$$

$$+ \sum_{\substack{b \in J \\ b \neq 1}} (\cdots) + \frac{D_1^{I,J}}{(1-qz)^2} \Big\} d_{n,m}(I)\, d_{n,m}(J) \cdot c(I) \cdot c(J)\, z^{|I|+|J|} \cdot z^2$$

Marked terms give

$(1-z^2)\, T_{n,m}(qz)\, T_{n,m}(q^{-1}z)$

Main observation: for any $a \in I$, $a-2 \notin I$, $a \neq 1$,

Involution $\begin{cases} I \longrightarrow I_1 = (I \setminus \{a\})^c \\ J \longrightarrow J_1 = (J \cup \{a-2\})^c \end{cases}$

transforms the first summand to the second one. Indeed,

$|I_1| + |J_1| = |I^c| + |J^c| + 2;$

$c(I_1) + c(J_1) \equiv c(I) + c(J) + c(a) - c(a-2) \pmod 2,$

so that $\quad c(I_1)\, c(J_1) = (-1)^{A(a)} c(I)\, c(J).$

By KEY Lemma

put $d_J = d_{n,m}(J)$

$c_a^{I,J} d_I d_J = c_a^{I_1,J_1} d_{I_1} d_{J_1} (-1)^{A(a)}.$

Therefore the sums are <u>canceled</u>!

If $1 \in I \cap J$, using the same involution, one can kill terms with $D_1^{I,J}$.

QED.

This proof can be generalized to prove identities for

$T^{\epsilon_1,\epsilon_2}_{n_+,m_+;\,n_-,m_-}(q,z)$.

Recurrence relation the <u>same</u>!

Universal characters

(D. Littlewood, R. King, K. Koike, I. Terada)

Let λ, μ be partitions, $\ell(\lambda)=n$, $\ell(\mu)=m$, $t^{(1)}=(t_1^{(1)}, t_2^{(1)}, \ldots)$, $t^{(2)}=(t_1^{(2)}, t_2^{(2)}, \ldots)$ two sets of variables.

<u>Definition</u> $S_{\lambda,\mu}(t^{(1)}, t^{(2)}) =$

$$\det \left| \begin{array}{c|l} q_{\mu_{m-i+1}+i-j}(t^{(2)}) & 1 \le i \le m, \ j=1,2,\ldots,n+m \\ \hdashline p_{\lambda_i - i + j}(t^{(1)}) & m < i \le n+m, \ j=1,\ldots,n+m \end{array} \right|$$

where $\sum_{k \ge 0} p_k(t^{(1)}) z^k = \exp\left(\sum_{j \ge 1} t_j^{(1)} z^j\right)$, $p_k = 0$, if $k<0$

$$\sum_{k \ge 0} q_k(t^{(2)}) z^k = \exp\left(\sum_{j \ge 1} t_j^{(2)} z^j\right), \ q_k = 0, \text{ if } k<0.$$

It is ~~easy~~ to see that after substitution

$$t_j^{(1)} := \frac{x_1^j + \ldots + x_n^j}{j}$$

$$t_j^{(2)} := \frac{y_1^j + \ldots + y_m^j}{j}$$

so-called Muira transformation

$\Rightarrow p_k(t^{(1)}) = h_k(x)$, $q_k(t^{(2)}) = h_k(y)$, we obtain function $s_{\lambda,\mu}(x,y)$ | r ... rel$_{Gl_n}$ (bi)-characters

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Now take $\lambda_n = (n, n-1, \dots, 1)$, $\mu_m = (m, m-1, \dots, 1)$, and put $S_{n,m} := s_{\lambda_n,\mu_m}(t^{(1)}, t^{(2)})$. Finally, consider specialization

$$s_{n,m}(t,s) := S_{n,m}(t^{(1)}, t^{(2)}) \Big| \begin{array}{l} t^{(1)}_j = -\dfrac{t}{2} + \dfrac{2s+n-m}{j} \\ t^{(2)}_j = \dfrac{t}{2} + \dfrac{2s+n-m}{j} \end{array}$$

<u>Theorem</u> $\quad s_{n,m}(t,s) =$

$$\frac{1}{c_{n,m}} \sum_{I \subset \sigma_{|n-m|,m}} c(I)\, d_{|n-m|,m}(I)\, t^{|I|} \prod_{j \in I^c} e_j^{(n-m)}(s),$$

where $\quad c_{n,m} = \prod_{j=1}^{(n+m-2)_+} (2j+1) \qquad (a)_+ = \max(a,0)$

$$\begin{cases} e^{(n)}_{2k}(s) = \prod_{\ell=-k}^{k-1} (s-k-2\ell-1), \\ e^{(n)}_{2k+1}(s) = \prod_{\ell=-k}^{k} (s-n-2\ell). \end{cases}$$

Work in progress:

elliptic generalization, elliptic Painlevé IV, and elliptic hypergeometric ${}_{10}E_9$.
8-parameter generalization

Generalizations: T. Masuda

1) q-P_{VI} (M. Jimbo, S. Sakai,)
2) discrete Painlevé VI (Grammatikos, Ohta, ...)
3) elliptic Painlevé VI (Kajiwara, Masuda, Noumi, Yamada)

We can construct a 2-parameter family of solutions to P_{VI}, q-P_{VI}, and elliptic P_{VI} ?!.

Combinatorics comes when we consider some special limit/degeneration of rational solutions we constructed. Namely, rational solutions to P_{VI} we constructed, have a form $q_i = \frac{Q_1 Q_2}{Q_3 Q_4}$, where Q_i's are polynomials (in t), which after some special limit, have integer coefficients.

$\boxed{\text{The main goal of my talk is to describe some properties of the latter}}$

Apriori, there are no reasons to expect that Q_i's should have integer coefficients.

~~A~~ It was a big surprise for me that the same integers appear in the study of spherical functions on some (quantum) homogenious spaces, and the study of eigenfunctions of the BC_1 Calogero-Moser (known also as Pöschl-Teller) model, and a "combinatorial" formula for Askey-Wilson's functions.

Problem To explain Why?