

# Counting occurrences of 231 in an involution

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Joint work with

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## The aim of this talk

- We study the generating function for the number of **involutions** on  $n$  letters **containing exactly  $r \geq 0$  occurrences of 231**.
- It is shown that finding this function for a given  $r$  **amounts to a routine check** of all involutions of length less than or equal to  $2r + 2$ .

## Permutations

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- In such a context,  $\tau$  is usually called a **pattern**. We denote the number of occurrences of  $\tau$  in  $\pi$  by  $\tau(\pi)$ .

**Example.** Let  $\pi = 5624713 \in S_7$ , then the number of occurrences of the pattern 132 in  $\pi$  is  $132(\pi) = 3$ , namely

first occurrence:  $\pi = 5624713$ ,

second occurrence:  $\pi = 5624713$ ,

third occurrence:  $\pi = 5624713$ ,

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For the case  $r > 0$  there exist only a few papers, usually restricting themselves to the patterns of length three.

Using two simple involutions (reverse and complement) on  $S_n$  it is immediate that with respect to being equidistributed, the six patterns of  $S_3$  fall into two classes, namely  $\{123, 321\}$  and  $\{132, 213, 231, 312\}$ .

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A general approach to the counting problem was suggested by Noonan and Zeilberger; they gave another proof of Noonan's result, and conjectured that

$$S_1^{132}(n) = \binom{2n-3}{n-3}$$

and

$$S_2^{123}(n) = \frac{59n^2 + 117n + 100}{2n(2n-1)(n+5)} \binom{2n}{n-4}.$$

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A general conjecture of Noonan and Zeilberger states that the sequence  $\{S_r^\tau(n)\}_{n \geq 0}$  is  $P$ -recursive in  $n$  for any  $r$  and  $\tau$ .

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In 2002, Mansour and Vainshtein suggested a new approach to this problem in the case  $\tau = 132$ , which allows to get **an explicit expression for  $S_r^{132}(n)$  for any given  $r$** . More precisely, they presented an algorithm that computes the generating function  $\sum_{n \geq 0} S_r^{132}(n)x^n$  for any  $r \geq 0$ . To get the result for a given  $r$ , the algorithm performs certain routine checks for each permutation of  $S_{2r}$ .

## Involutions

An **involution**  $\pi$  is a permutation in  $S_n$  such that  $\pi = \pi^{-1}$ ; let  $\mathcal{I}_n$  be the set of all the involutions in  $S_n$ .

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We denote  $I_r^\tau(n)$  the number of involutions  $\pi \in \mathcal{I}_n$  with  $\tau(\pi) = r$ , and  $I_r^\tau(x)$  the corresponding generating function, that is,  $I_r^\tau(x) = \sum_{n \geq 0} I_r^\tau(n)x^n$ .

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Recently, Mansour suggested a new approach to this problem in the case of  $\tau = 3412$ , which allows us to obtain an explicit expression for the generating function  $\sum_{n \geq 0} I_r^{3412}(n)x^n$  for any given  $r \geq 0$ .

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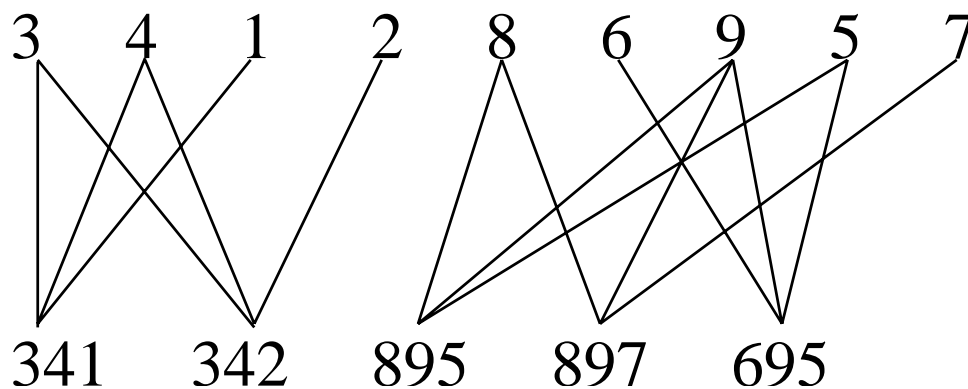
To get the result for a given  $r$ , the algorithm performs certain routine checks for each element in  $\bigcup_{k=1}^{2r+2} \mathcal{I}_k$ .

The algorithm has been implemented in C, and yielded explicit results for  $0 \leq r \leq 7$ .

## Preliminary results

For any involution  $\pi \in \mathcal{I}_n$ , we can assign a bipartite graph  $G_\pi$  in the following way:

The vertices in one part of  $G_\pi$ , denoted  $V_1$  are the entries of  $\pi$ , and the vertices of the second part, denoted  $V_3$ , are the occurrences of 231 in  $\pi$ . Entry  $i \in V_1$  is connected by an edge to occurrence  $j \in V_3$  if  $i$  enters  $j$ .



The graph  $G_{341286957}$ .

- Let  $\tilde{G}$  be an arbitrary connected component of  $G_\pi$ , and let  $\tilde{V}$  be its vertex set. We denote  $\tilde{V}_1 = \tilde{V} \cap V_1$ ,  $\tilde{V}_3 = \tilde{V} \cap V_3$ ,  $t_1 = |\tilde{V}_1|$ ,  $t_3 = |\tilde{V}_3|$ . Denote by  $G_\pi^n$  the connected component of  $G_\pi$  containing entry  $n$ .

- For any  $\pi \in \mathcal{I}_n$  where  $\pi_j = n$  and  $|V_1(G_\pi^n)| > 1$ , suppose that  $i_k$  is the minimal index such that  $\pi_{i_k} > j$  and  $i_k < j$ .

- If there exists a  $i_k < m < j$  such that  $\pi_m < i_k$ , then assume that  $i_1$  is the minimal index such that there exists a subsequence

$$(\pi_{i_1}, \pi_{i_2}, i_1, \pi_{i_3}, i_2, \dots, i_h, \pi_{i_{h+2}}, i_{h+1}, \dots, \pi_{i_k}, i_{k-1}, \pi_j, i_k, j)$$

where  $i_1 < i_2 < i_3 < \dots < i_k < j$ .

- We call this subsequence **connected sequence**.

Otherwise,  $i_1 = i_k$ . For our convenience, we call  $i_1$  the **initial index**.

**Definition 1.** For any  $\pi \in \mathcal{I}_n$  and  $\pi_j = n$ , we define the **231-tail** by

$$\chi_\pi = \begin{cases} (n, \pi_{j+1}, \dots, \pi_{n-1}, j), & \text{if } |V_1(G_\pi^n)| = 1, \\ (\pi_{i_1}, \pi_{i_1+1}, \dots, \pi_n), & \text{if } |V_1(G_\pi^n)| > 1, \end{cases}$$

where  $i_1$  is the initial index of  $\pi$ .

- For example, the 231-tail of the involution 216483957 is 6483957.
- Denote  $l_\pi$  and  $c_\pi$  the length of  $\pi$  and the number of the occurrences of 231 in  $\pi$ .

Now, the question:

What kind of 231-tail we have???

In fact, for any  $\pi \in \mathcal{I}_n$  with  $|V_1(G_\pi^n)| = 1$ , the 231-tail  $\chi_\pi$  of  $\pi$  can be represented as

$$\chi_\pi = (n, n - 1, \dots, n - s + 1, \lambda).$$

**Lemma 1.** Let  $\pi \in \mathcal{I}_n$ , the permutation of 231-tail of  $\pi$ ,  $\chi_\pi$ , is an involution, and there exists an involution  $\pi'$  such that  $\pi = (\pi', \chi_\pi)$ .

**Lemma 2.** Let  $\pi \in \mathcal{I}_n$  with

$$\chi_\pi = (n, n - 1, \dots, n - s + 1, \lambda),$$

where  $\lambda$  is nonempty, such that  $c_{\chi_\pi} = r$ , then  $l_{\chi_\pi} \leq 2r + 2$ .

**Lemma 3.** For any  $\pi \in \mathcal{I}_n$  with  $|V_1(G_\pi^n)| > 1$ , the subsequence of  $\pi$  contained in the connected component  $G_\pi^n$  is just the 231-tail  $\chi_\pi$  of  $\pi$ .

- Studying occurrences of 132 in a permutation which leads to consideration of 231 in a permutation, Mansour and Vainshtein have proved that the relation  $t_1 \leq 2t_3 + 1$ . It is clear that the set of involutions is a subset of permutations. So we have

**Lemma 4.** For any connected component  $\tilde{G}$  of  $G_\pi$ , one has  $t_1 \leq 2t_3 + 1$ .

## Main results

Denote by  $K_t$  the subset of  $\bigcup_{k \leq 2t+2} \mathcal{I}_k$  whose elements can be expressed as

$$(k, k - 1, \dots, k - s + 1, \lambda)$$

where  $\lambda$  is nonempty,

and by  $H_t$  be the subset of  $\bigcup_{k \leq 2t+1} \mathcal{I}_k$  such that the corresponding bipartite graph of each element is connected. It is obvious that  $K_t \cap H_t = \emptyset$ .

Then the main result of this paper can be formulated as follows.

**Theorem** For any  $r \geq 1$ ,

$$I_r^{231}(x) = \frac{x}{1-x} I_r^{231}(x) + \sum_{\mu \in K_r \cup H_r} x^{l_\mu} I_{r-c_\mu}^{231}(x). (*)$$

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**Proof.** Denote by  $F_r^\mu(x)$  the generating function for the number of involutions  $\pi \in \mathcal{I}_n$  such that  $\chi_\pi$  is just order-isomorphic to  $\mu$ .

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We discuss three cases to find  $F_r^\mu(x)$ :

1. If  $\pi \in \mathcal{I}_n$  with  $\chi_\pi = (n, n-1, \dots, n-s+1)$ , then

$l_\mu = s$  and  $\mu = (s, s-1, \dots, 1)$ , so we have

$$F_r^\mu(x) = x^s I_r^{231}(x).$$

2. If  $\pi \in \mathcal{I}_n$  with  $\chi_\pi = (n, n-1, \dots, n-s+1, \lambda)$  where

$\lambda$  is nonempty, then  $\mu \in K_r$  by Lemma (2), thus we

$$\text{have } F_r^\mu(x) = x^{l_\mu} I_{r-c_\mu}^{231}(x).$$

3. If  $\pi \in \mathcal{I}_n$  with  $\chi_\pi = (\pi_{i_1}, \pi_{i_1+1}, \dots, \pi_n)$  where  $i_1$  is

the initial index of  $\pi$ , then Lemma's (3-4) yield  $\mu \in H_r$

$$\text{and } F_r^\mu(x) = x^{l_\mu} I_{r-c_\mu}^{231}(x).$$

Hence, summing over all

$$\mu \in \{(s, s-1, s-2, \dots, 2, 1) \mid s \geq 1\} \cup K_r \cup H_r$$

we get the desired result.  $\square$

Let us start from the case  $r = 0$ . Observe that  $(*)$  remains valid for  $r = 0$ , provided the left hand side is replaced by  $I_0^{231}(x) - 1$ ; subtracting 1 here accounts for the empty permutation. Note that when  $r = 0$ ,  $K_0 \cup H_0$  is empty. Hence we get

$$I_0^{231}(x) - 1 = \frac{x}{1-x} I_0^{231}(x),$$

equivalently

$$I_0^{231}(x) = \frac{1-x}{1-2x}, (**)$$

which is the result of Simion and Schmidt.

Let now  $r = 1$ . Observe that  $K_1 \cup H_1 = \{4231\}$ .

Therefore, (\*) amounts to

$$I_1^{231}(x) = \frac{x}{1-x} I_1(x) + x^4 I_0^{231}(x),$$

and we get the following result from Formula (\*\*).

**Corollary** (see Egge and Mansour) The generating function  $I_1^{231}(x)$  for the number of involutions containing exactly one occurrence of the pattern 231 is given by

$$I_1^{231}(x) = \frac{x^4(1-x)^2}{(1-2x)^2};$$

equivalently, for  $n \geq 5$ ,  $I_1^{231}(n) = (n-1)2^{n-6}$ .

Let  $r = 2$ . Exhaustive search adds four new elements to the previous list; these are 653421, 52431, 53241, and 3412, therefore we get

**Corollary** The generating function  $I_2^{231}(x)$  is given by

$$I_2^{231}(x) = \frac{x^4(1-x)^2}{(1-2x)^3} (1 - 3x^2 - 2x^3 + x^4 - x^5);$$

equivalently, for  $n \geq 9$ ,

$$I_2^{231}(n) = (n^2 + 137n - 234)2^{n-11}.$$

Let  $r = 3, 4, 5, 6, 7$ ; exhaustive search in  $\mathcal{I}_8, \mathcal{I}_{10}, \mathcal{I}_{12}, \mathcal{I}_{14}$ , and  $\mathcal{I}_{16}$  reveals 13, 24, 41, 69, and 103 elements, respectively, and we get

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$$I_r^{231}(x) = \frac{(1-x)^2}{(1-2x)^{r+1}} Q_r(x),$$

for all  $r = 3, 4, 5, 6, 7$ , where

$$Q_3(x) = x^5(4 - 14x + 8x^2 + 11x^3 - 6x^4 - 2x^5 + 2x^6 + 5x^7 - 2x^8 + x^9);$$

$$Q_4(x) = x^6(6 - 32x + 49x^2 + 7x^3 - 73x^4 + 40x^5 + 30x^6 - 37x^7 + 2x^8 + 4x^{10} - 9x^{11} + 3x^{12} - x^{13});$$

$$Q_5(x) = x^6(8 - 58x + 146x^2 - 120x^3 - 40x^4 - 24x^5 + 290x^6 - 184x^7 - 197x^8 + 228x^9 + 30x^{10} - 132x^{11} + 62x^{12} + 13x^{14} - 16x^{15} + 14x^{16} - 4x^{17} + x^{18});$$

$$Q_6(x) = x^6(4 - 31x + 80x^2 - 56x^3 + 4x^4 - 384x^5 + 1097x^6 - 830x^7 - 483x^8 + 660x^9 + 685x^{10} - 1091x^{11} - 59x^{12} + 722x^{13} - 195x^{14} - 338x^{15} + 285x^{16} - 92x^{17} + 20x^{18} - 45x^{19} + 35x^{20} - 20x^{21} + 5x^{22} - x^{23});$$

$$Q_7(x) = x^7(17 - 199x + 969x^2 - 2502x^3 + 3642x^4 - 3274x^5 + 3324x^6 - 4714x^7 + 1874x^8 + 6326x^9 - 8262x^{10} - 231x^{11} + 5474x^{12} - 637x^{13} - 4022x^{14} + 1933x^{15} + 1340x^{16} - 1129x^{17} - 518x^{18} + 982x^{19} - 498x^{20} + 166x^{21} - 92x^{22} + 105x^{23} - 62x^{24} + 27x^{25} - 6x^{26} + x^{27}).$$

Equivalently, for  $n \geq 4r + 1$ ,  $I_r^{231}(n) = 2^n \cdot Q_r(n)$ , where

$$Q_3(n) = \frac{1}{3 \cdot 2^{17}} (n^3 + 414n^2 + 12227n - 30762)$$

$$Q_4(n) = \frac{1}{3 \cdot 2^{24}} (n^4 + 830n^3 + 108275n^2 + 476710n - 3117432)$$

$$Q_5(n) = \frac{1}{3 \cdot 5 \cdot 2^{29}} (n^5 + 1385n^4 + 416765n^3 + 20952295n^2 + 85955874n - 544257360)$$

$$Q_6(n) = \frac{1}{3^2 \cdot 5 \cdot 2^{35}} (n^6 + 2079n^5 + 1124515n^4 + 158232165n^3 + 3797599444n^2 + 1475950836n - 72974470320)$$

$$Q_7(n) = \frac{1}{3^2 \cdot 5 \cdot 7 \cdot 2^{40}} (n^7 + 2912n^6 + 2476306n^5 + 685388480n^4 + 51462119569n^3 + 764352578528n^2 - 3749997108516n - 326124489600)$$

## Further results

As an easy consequence of the Main Theorem we get the following result.

**Corollary** For any  $r \geq 1$  there exist a polynomial  $P_{5r-1}(x)$  of degree  $5r - 1$  with integer coefficients such that

$$I_r^{231}(x) = \frac{(1-x)^2}{(1-2x)^{r+1}} P_{5r-1}(x).$$

In other words, for  $n \rightarrow \infty$

$$I_r^{231}(n) \approx \frac{2^{n-5r-1} n^r}{r!}.$$

- Another direction would be to match the approach of this paper with the previous results on restricted 231-avoiding involutions.
- Let  $\Phi_r(x; k)$  be the generating function for the number of involutions in  $\mathcal{I}_n$  containing  $r$  occurrences of 231 and avoiding the pattern  $12\dots k \in \mathfrak{S}_k$ .
- Our new approach allows to get a recursion for  $\Phi_r(x; k)$  for any given  $r \geq 0$ .

We denote by  $e_\lambda$  the length of the longest increasing subsequence of any involution  $\lambda$ . For example, let  $\lambda = 3412$ , then  $e_\lambda = 2$ . We denote by  $K_t(k) \cup H_t(k)$  the set of all involutions  $\lambda \in K_t \cup H_t$  such that  $e_\lambda \leq k - 1$ .

**Theorem** For any  $r \geq 1$  and  $k \geq 3$ , Then the generating function  $\Phi_r(x; k)$  is given by

$$\frac{x}{1-x} \Phi_r(x; k-1) + \sum_{\mu \in K_r(k) \cup H_r(k)} x^{l_\mu} \Phi_{r-c_\mu}^{231}(x; k - e_\mu).$$

Besides,  $\Phi_r(x; 1) = \Phi_r(x; 2) = 0$ , and  $\Phi_0(x; 1) = 1$  and  $\Phi_0(x; 2) = \frac{1}{1-x}$ .

**Corollary**(Egge and Mansour)

For all  $k \geq 1$ ,

$$\Phi_0(x; k) = \sum_{j=0}^{k-1} \left( \frac{x}{1-x} \right)^j ;$$

$$\Phi_1(x; k) = x^4 \sum_{j=0}^{k-3} (j+1) \left( \frac{x}{1-x} \right)^j .$$

- Another direction would be to match the approach of this note with the previous results on restricted 231-avoiding even or odd involutions.
- We say  $\pi$  an even (resp; odd) involution if the number of inversion in  $\pi$ , namely  $21(\pi)$  is even (resp; odd).
- We define the sign  $h_\lambda$  of any permutation  $\lambda$  as  $(-1)^{21(\lambda)}$ . For example, if  $\lambda = (6, 5, 3, 4, 2, 1)$  then  $h_\lambda = 1$ .
- We denote by  $K_r^+ \cup H_r^+$  the set of all the involutions  $\lambda \in K_r \cup H_r$  such that  $h_\lambda = 1$  and denote by  $K_r^- \cup H_r^-$  the set of all involutions  $\lambda \in K_r \cup H_r$  such that  $h_\lambda = -1$ .

- Let  $I_r^+(x)$  (resp;  $I_r^-(x)$ ) be the generating function for the number of even (resp; odd) involutions in  $\mathcal{I}_n$  containing  $r$  occurrences of 231. Our new approach allows to get:

**Theorem** For all  $r \geq 1$ ,

$$\begin{aligned}
 I_r^+(x) &= \frac{x+x^4}{1-x^4} I_r^+(x) + \frac{x^2+x^3}{1-x^4} I_r^-(x) \\
 &\quad + \sum_{\mu \in K_r^+ \cup H_r^+} x^{l_\mu} I_{r-c_\mu}^+(x) + \sum_{\mu \in K_r^- \cup H_r^-} x^{l_\mu} I_{r-c_\mu}^-(x); \\
 I_r^-(x) &= \frac{x+x^4}{1-x^4} I_r^-(x) + \frac{x^2+x^3}{1-x^4} I_r^+(x) \\
 &\quad + \sum_{\mu \in K_r^+ \cup H_r^+} x^{l_\mu} I_{r-c_\mu}^-(x) + \sum_{\mu \in K_r^- \cup H_r^-} x^{l_\mu} I_{r-c_\mu}^+(x).
 \end{aligned}$$

In particular, we have

$$\begin{aligned}
 I_0^+(x) - 1 &= \frac{x+x^4}{1-x^4} I_0^+(x) + \frac{x^2+x^3}{1-x^4} I_0^-(x) \text{ and} \\
 I_0^-(x) &= \frac{x+x^4}{1-x^4} I_0^-(x) + \frac{x^2+x^3}{1-x^4} I_0^+(x).
 \end{aligned}$$

As an example of the above theorem we get

**Corollary** For  $0 \leq r \leq 2$ ,

$$I_r^+(x) = \frac{E_r(x)}{(1-2x)^{r+1}(1-x+2x^2)^{r+1}},$$

$$I_r^-(x) = \frac{O_r(x)}{(1-2x)^{r+1}(1-x+2x^2)^{r+1}};$$

where

$$E_0(x) = 1 - 2x + 2x^2 - 2x^3;$$

$$E_1(x) = 2x^6(1 - 2x + 2x^2 - 2x^3);$$

$$E_2(x) = x^4(1 - 5x + 11x^2 - 15x^3 + 10x^4 + 5x^5 - 11x^6 - 5x^7 + 47x^8 - 94x^9 + 86x^{10} - 62x^{11} + 16x^{12});$$

$$O_0(x) = x^2;$$

$$O_1(x) = x^4(1 - 4x + 8x^2 - 12x^3 + 13x^4 - 8x^5 + 4x^6);$$

$$O_2(x) =$$

$$x^6(2 - 6x + 6x^2 - 2x^3 - 9x^4 + 4x^5 + 20x^6 - 36x^7 + 53x^8 - 24x^9 + 8x^{10}).$$

**Corollary** Let  $P_t(x)$  be a polynomial of degree  $t$  with integer coefficients; for any  $r \geq 0$ ,

$$I_r^+(x) = \frac{P_{m_r}(x)}{(1-2x)^{r+1}(1-x+2x^2)^{r+1}},$$

$$I_r^-(x) = \frac{P_{n_r}(x)}{(1-2x)^{r+1}(1-x+2x^2)^{r+1}},$$

where  $m_r, n_r \leq \frac{r(r+9)}{2}$ .

THANKS!