A Computerized Proof of Stembridge's TSPP Theorem

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PRELUDE

$$y^p + x\,y^q - 1 = 0 \qquad (p > q \geq 1)$$

Mellin series for principal[*)] solution:

$$\boxed{y(x) = \sum_{k=0}^{\infty} \binom{(qk+1)/p}{k} \frac{(-x)^k}{qk+1}}$$

E.g.: $y^5 + \frac{1}{2} y^2 - 1 = 0$,

$$\sum_{k=0}^{8} \binom{(2k+1)/5}{k} \frac{(-1/2)^k}{2k+1} = \underbrace{0.90107666}_{\text{correct digits of the real root}}34$$

*) I.e., $y(0) = 1$

Multivariate Mellin series $y(x_1,\dots,x_n)$ for

$$y^p + x_1 y^{m_1} + \cdots + x_n y^{m_n} - 1 = 0$$

where the $x_i \in \mathbb{C}$.

Mellin (1915): integral transformations
and residue calculus

Louck (1988): elementary transformation to

$$\frac{\bar{u}+v}{\hat{u}+v+\underline{n}\cdot\underline{z}}\binom{\bar{u}+v+\underline{n}\cdot\underline{z}}{\underline{n}} =$$

$$\sum_{\underline{k}\in\mathbb{N}^p}\frac{\bar{u}}{\hat{u}+\underline{k}\cdot\underline{z}}\binom{\bar{u}+\underline{k}\cdot\underline{z}}{\underline{k}}\frac{v}{v+(\underline{n}-\underline{k})\cdot\underline{z}}\binom{v+(\underline{n}-\underline{k})\cdot\underline{z}}{\underline{n}-\underline{k}}$$

$\bar{u}, v \in \mathbb{C}$, $\underline{z} = (z_1, \dots, z_p) \in \mathbb{C}^p$,

$\underline{n} = (n_1, \dots, n_p)$ and $\underline{k} = (k_1, \dots, k_p) \in \mathbb{N}^p$,

$\underline{n} \cdot \underline{z} := n_1 z_1 + \dots + n_p z_p$, $|\underline{n}| := n_1 + \dots + n_p$,

$$\binom{w}{\underline{n}} := \frac{w(w-1)\cdots(w-|\underline{n}|+1)}{n_1!\, n_2! \cdots n_p!}$$

Raney (1960), Mohanty (1979), Chu (1987);
survey: V. Strehl [Discrete Math. 99 (1992)].

Case $p=1$: Rothe (1792)

Case $p=1$ and $z=0$: Chu–Vandermonde convolution

For general p and z :

NO COMPUTER ALGEBRA PROOF

For fixed parameters $(p = 1, 2, \ldots)$:

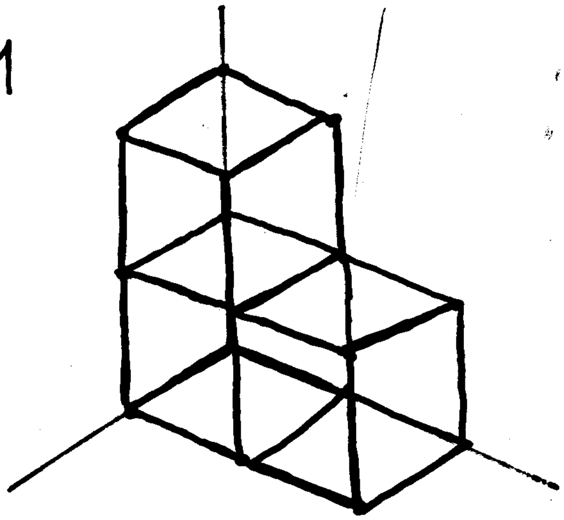
YES!

PLANE PARTITIONS

Ex.: $3 = 2+1 = 1+1+1$

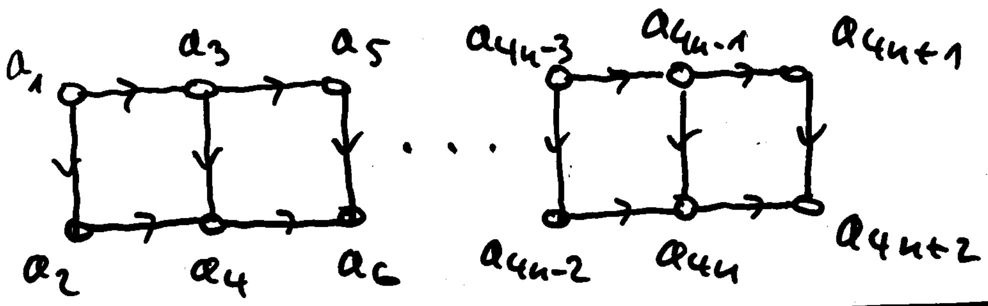
$$= \begin{matrix}2\\+\\1\end{matrix} = \begin{matrix}1+1\\+\\1\end{matrix}$$

$$= \begin{matrix}1\\+\\1\\+\\1\end{matrix}$$



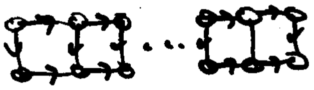
with 2 rows: $N = a_1 + \cdots + a_{4n+2}$ where

$a_i \geq 0$ and



$\boxed{> := \geq}$

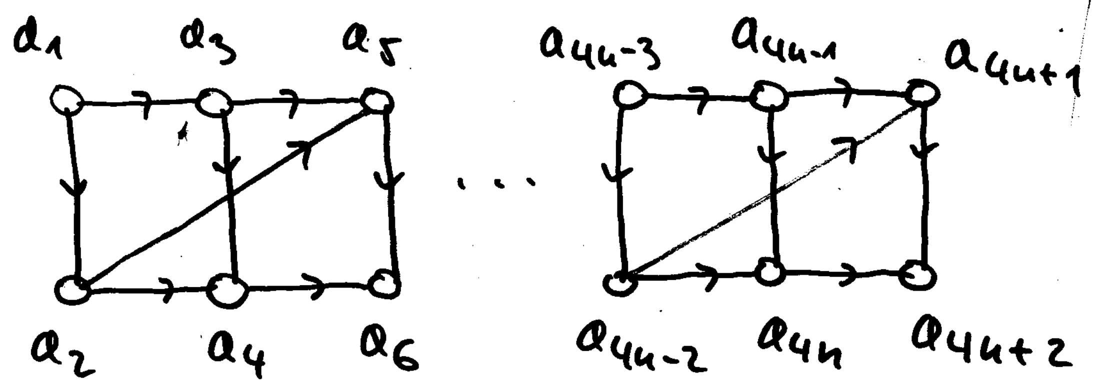
$$\sum q^{a_1 + \cdots a_{4n+2}}$$



$$= \frac{1}{(1-q)\cdot(1-q^2)^2(1-q^3)^2\cdots(1-q^{2n+1})^2\cdot(1-q^{2n+2})}$$

D1
P2

PLANE PARTITIONS WITH DIAGONALS



Generating function:

$$\boxed{\frac{(1+q^2)(1+q^{4n})}{(q;q)_{4n+2}} \cdot Q_1 Q_2 \cdots Q_{n-1}} \;{}^{*)}$$

where

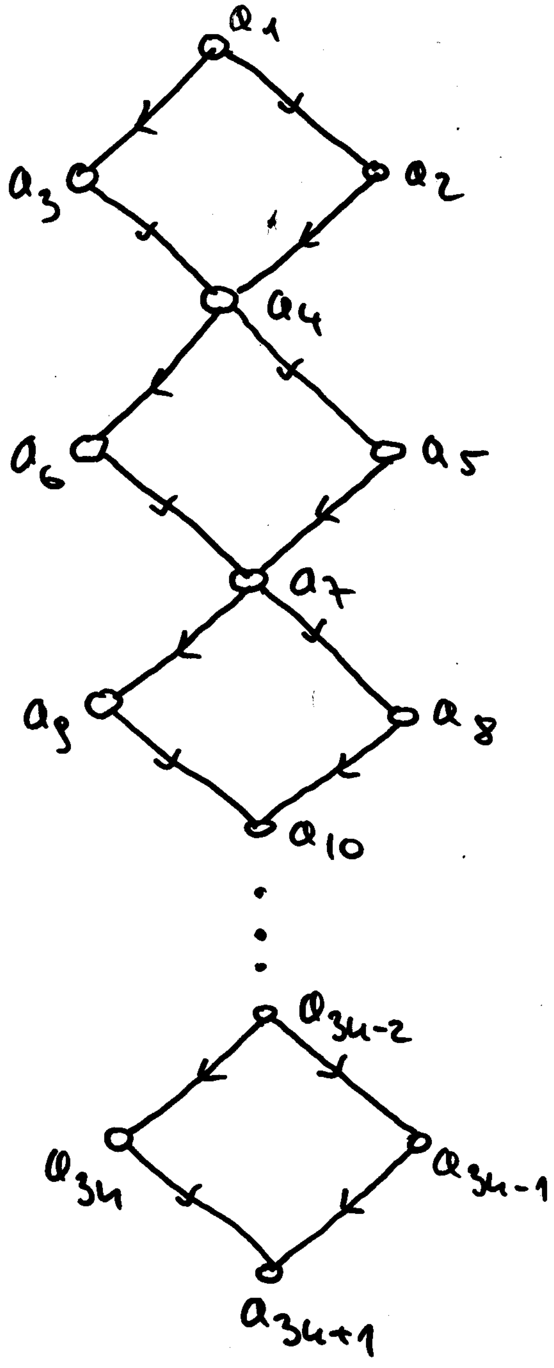
$$Q_k = 1 + q^{4k} + q^{4k+1} + q^{4k+2} + q^{8k+2}$$

Note: $(q;q)_N = (1-q)(1-q^2)\cdots(1-q^N)$

*) G.E. Andrews, PP, and A. Riese

PPD'

PLANE PARTITION DIAMONDS

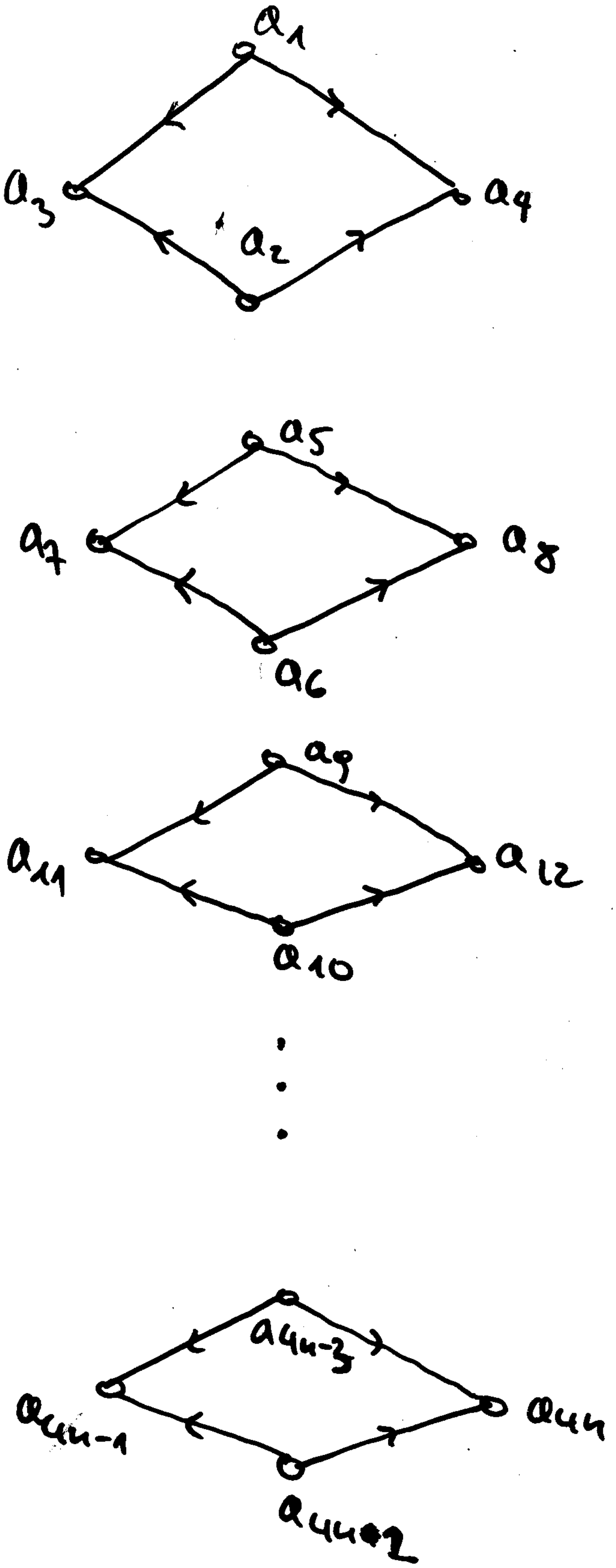


PPD

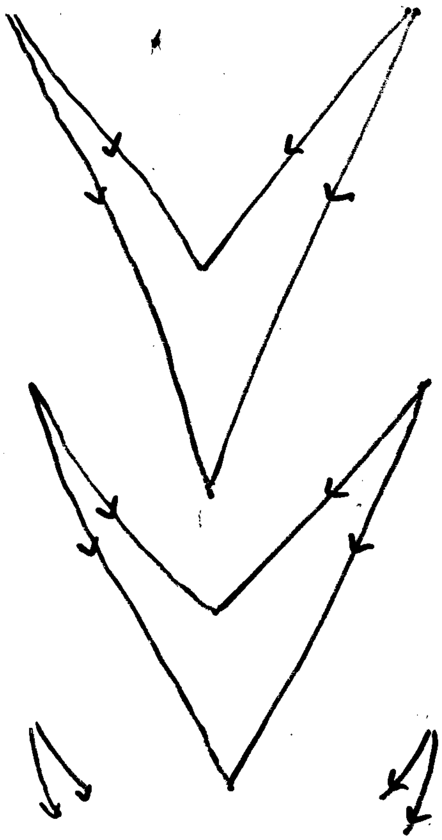
Generating function:

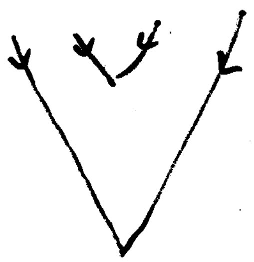
$$\boxed{\frac{(1+q^2)(1+q^5)\cdots(1+q^{3k-1})}{(1-q)(1-q^2)\cdots(1-q^{3k+1})}}$$

CHINESE KITES



C42





C43

Generating function:

$$\boxed{\frac{(1+q)(1+q^3)\cdots(1+q^{2n+1})}{(1-q)(1-q^2)\cdots(1-q^{2n+2})}}$$

PLANE PARTITIONS

with symmetry condititions

Symmetric PPs fitting into $B(2,2,2)$:

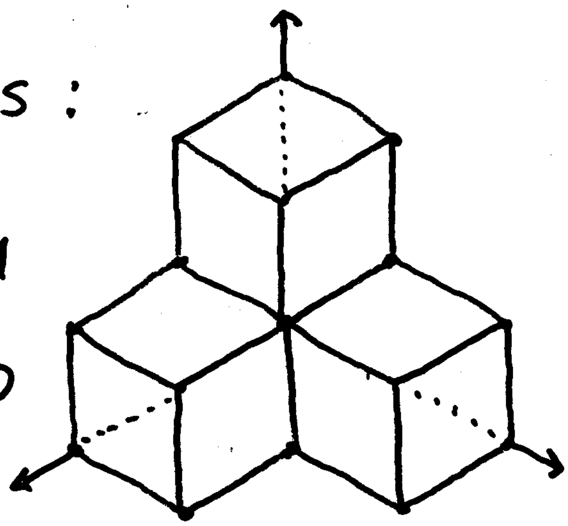
$$\begin{array}{ccc}2&+&2\\+&&+\\2&+&2\end{array}\qquad\begin{array}{ccc}2&+&2\\+&&+\\2&+&1\end{array}\qquad\begin{array}{ccc}2&+&2\\+&&+\\2&+&0\end{array}\qquad\begin{array}{ccc}2&+&1\\+&&+\\1&+&1\end{array}\qquad\begin{array}{ccc}2&+&1\\+&&+\\1&+&0\end{array}$$

$$\begin{array}{ccc}2&+&0\\+&&+\\0&+&0\end{array}\qquad\begin{array}{ccc}1&+&1\\+&&+\\1&+&1\end{array}\qquad\begin{array}{ccc}1&+&1\\+&&+\\1&+&0\end{array}\qquad\begin{array}{ccc}1&+&0\\+&&+\\0&+&0\end{array}\qquad\begin{array}{ccc}0&+&0\\+&&+\\0&+&0\end{array}$$

Totally symmetric PPs :

$$\begin{array}{ccc} 2 & + & 2 \\ + & & + \\ 2 & + & 2 \end{array} \qquad \begin{array}{ccc} 2 & + & 2 \\ + & & + \\ 2 & + & 1 \end{array} \qquad \begin{array}{ccc} 2 & + & 1 \\ + & & + \\ 1 & + & 0 \end{array}$$

$$\begin{array}{ccc} 1 & + & 0 \\ + & & + \\ 0 & + & 0 \end{array} \qquad \begin{array}{ccc} 0 & + & 0 \\ + & & + \\ 0 & + & 0 \end{array}$$



Generating function for SPPs : MacMahon ; G.E. Andrews (1978), I. Macdonald (1979)

PLANE PARTITIONS

with symmetry conditions: TSPP

TSPP(n) := no. of totally symmetric plane ptns. fitting into $B(n,n,n)$

$n=0,\dots,6$: 1, 2, 5, 16, 66, 352, 2431

Theorem (Stembridge, 1995) [conjectured by: Andrews, Macdonald, Stanley]

$$\boxed{\mathrm{TSPP}(n) = \prod_{1\le i\le j\le k\le n} \frac{i+j+k-1}{i+j+k-2}}$$

$$= \prod_{k=1}^{n} t_1(k)$$

where $t_1(k) := \prod_{s=1}^{k} \frac{k+2s-1}{k+s-1}$

Theorem (S. Okada, 1989)

$$\boxed{TSPP(n)^2 = \begin{cases} \det(M_{n+2}(x))\, x^{-1}, & \text{if } n \text{ odd} \\ \det(M_{n+2}(x)), & \text{if } n \text{ even} \end{cases}}$$ P6

where P7

$$M_n(x) := \left(\mu_1(i,j)\right)_{0 \le i,j \le n-1}$$

with

$$\mu_1(i,j) = \begin{cases} x, & \text{if } i=j=0 \\ (-1)^{j-1}, & \text{if } i=0,\ j>0 \\ (-1)^{i}, & \text{if } j=0,\ i>j \\ 0, & \text{if } i=j>0 \\ \mu(i-1,j-1), & \text{if } j>i\geq 1 \\ -\mu(j-1,i-1), & \text{if } 1\leq j<i \end{cases}$$

and

$$\mu(i,j) = \begin{cases} 0, & \text{if } j\leq i \\ 2^{j-1}+(-1)^{j-1}, & \text{if } i=0,\ i<j \\ (-1)^{j-i-1}+\sum_{s=i}^{j-1}\binom{i+j-2}{s}, & \text{if } 0<i<j \end{cases}$$

How to show that

$$\det(M_{n+2}(1)) = \prod_{k=1}^{n} t_1(k)^2 \ ?$$

PROBLEM: evaluate $\det(M)$

GEA-STRATEGY: *)

find $W = \begin{pmatrix} \ddots & * \\ 0 & \ddots \end{pmatrix}$ with $\det(W) = 1$

such that

(+) $$MW \overset{(+)}{=} \begin{pmatrix} \ddots & 0 \\ * & \ddots \end{pmatrix};$$

then

$$\det(M) = \det(MW)$$

$$= \text{product of diagonal elts}$$

IN PRACTICE: $\boxed{\text{GUESS } W \text{ and VERIFY (+)}}$!

*) Also known as LU decomposition: $M = L \cdot U = \begin{pmatrix} \ddots & 0 \\ * & \ddots \end{pmatrix} \cdot W^{-1}$

TSM
3A
P9

Ex.:

$$\begin{pmatrix} 1 & x_1 & x_1^2 & x_1^3 \\ 1 & x_2 & x_2^2 & x_2^3 \\ 1 & x_3 & x_3^2 & x_3^3 \\ 1 & x_4 & x_4^2 & x_4^3 \end{pmatrix} \begin{pmatrix} 1 & -x_1 & x_1x_2 & -x_1x_2x_3 \\ 0 & 1 & -(x_1+x_2) & x_1x_2+x_1x_3+x_2x_3 \\ 0 & 0 & 1 & -(x_1+x_2+x_3) \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ * & x_2-x_1 & 0 & 0 \\ * & * & (x_3-x_1)(x_3-x_2) & 0 \\ * & * & * & (x_4-x_1)(x_4-x_2)(x_4-x_3) \end{pmatrix}$$

TSPP
3B

P10

$$\Updownarrow$$

$$\boxed{\begin{aligned}&-x_1x_2x_3 + x_i(x_1x_2 + x_1x_3 + x_2x_3)\\&\quad - x_i^2(x_1+x_2+x_3) + x_i^3\\&= (x_i - x_1)(x_i - x_2)(x_i - x_3)\end{aligned}}$$

(P13)

The matrix $W_n(x)$: [Conjectured by Andrews ~ 1990]

$$\boxed{W_n(x) := \left(e(i,j)\right)_{0 \le i,j \le n-1}}$$

where

$$e(i,j) := \begin{cases} 0, & \text{if } i>j \\ 1, & \text{if } i=j \\ e_1(i,j) - \dfrac{t_1(j-1)}{t_1(j-2)} x^{(-1)^j} e_1(i,j-1), \end{cases}$$

$$e_1(i,j) := \begin{cases} 0 & , \text{if } i>j \\ 1 & , \text{if } i=j \\ r_1(j) & , \text{if } i=0,\ i<j \\ r_2(j) & , \text{if } i=1,\ i<j \\ f_1(j-i, \frac{j}{2}), & \text{if } 2 \le i < j,\ j \text{ even} \\ f_2(j-i, \frac{j-1}{2}), & \text{if } 2 \le i<j,\ j \text{ odd}, \end{cases}$$

$$r_1(j) := \begin{cases} t_1(j-1), & \text{if } j \text{ even} \\ 0, & \text{if } j \text{ odd}, \end{cases}$$

$$r_2(j) = \begin{cases} \dfrac{t_1(j-1)}{2}, & \text{if } j \text{ even} \\ \dfrac{t_1(j-1)}{2} + \dfrac{f_2(j-2, \frac{j-1}{2})}{2}, & \text{if } j \text{ odd}, \end{cases}$$

$$f_1(c,j) := (-1)^c \sum_{s=0}^{\lfloor c/2 \rfloor} \frac{(-1)^s \binom{j-1-s}{c-2s} (j)_s (-3j+1)_s (3j-3s-1)}{4^s s! (-2j+\frac{3}{2})_s (3j-1)},$$

$$f_2(c,j) := (-1)^c \sum_{s=0}^{\lfloor c/2 \rfloor} (-1)^s \left\{ {j-s \atop c-2s} \right\} r_3(s,j),$$

$$r_3(s,j) := 4^{-s} \sum_{k=0}^{s} \frac{(j-k)(j)_k (-3j-1)_k}{j \cdot k! (-2j+\frac{1}{2})_k},$$

and

$$\left\{ {x \atop u} \right\} = \frac{1}{2}\left(\binom{x}{u} + \binom{x-1}{u} \right).$$

THE PROOF

$$M_n(x)\,W_n(x) = \big(\mu_1(i,j)\big)\big(e(i,j)\big)$$

$$= \Bigg(\underbrace{\sum_{k=0}^{j} \mu_1(i,k)\,e(k,j)}_{=:\,G(i,j)}\Bigg)_{0 \le i,j \le n-1}$$

to show

$$= \begin{pmatrix} x & 0 & 0 & 0 & \cdots & 0 \\ * & \frac{t_1(0)^2}{x} & 0 & 0 & \cdots & 0 \\ * & * & t_1(1)^2 x & 0 & \cdots & 0 \\ * & * & * & \frac{t_1(2)^2}{x} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ * & * & * & * & \cdots & t_1(n-2)^2 x^{(-1)^{n-1}} \end{pmatrix}$$

$$\begin{cases} \text{ODD CASE: } j = 2m+1; \\ \text{EVEN CASE: } j = 2m. \end{cases}$$

$\boxed{m \geq 1}$ $\quad (\ldots) - \frac{1}{x}(\ldots) = G(0, 2m+1) = 0$

$$\Updownarrow$$

$$\sum_{k=0}^{2m-1} (-1)^k f_1(k,m) = \frac{1}{2} t_1(2m-1)$$

and

$$\sum_{k=0}^{2m} (-1)^k f_2(k,m) = \frac{1}{2} t_1(2m)$$

$\boxed{m \geq 1}$ $\quad (\ldots) - \frac{1}{x}(\ldots) = G(1, 2m+1) = 0$

$$\Updownarrow$$

$$\sum_{k=0}^{2m-1} 2^{2m-1-k} f_1(k,m) = t_1(2m-1)$$

and

$$\sum_{k=0}^{2m} 2^{2m-k} f_2(k,m) = t_1(2m) - f_2(2m,m)$$

(p17)

$$\boxed{2 \le i \le 2m+1}$$

$$(\ldots) - \frac{1}{x}(\ldots) = G(i, 2m+1) = \begin{cases} 0 & , \text{ if } 2 \le i \le 2m \\ \frac{t_1(2m)^2}{x} & , \text{ if } i = 2m+1 \end{cases}$$

$$\Updownarrow$$

For $2 \le i \le 2m+1$:

$$\left(\Sigma_4 - \Sigma_4'\right)(i,m) = \frac{1}{2}\left(2^{-2+i} + (-1)^i\right) t_1(2m) + f_2(2m-1, m)$$

and

For $2 \le i \le 2m$:

$$\left(\Sigma_3 - \Sigma_3'\right)(i,m) = \frac{1}{2}\left(2^{-2+i} - (-1)^i\right) t_1(2m-1)$$

and $(i = 2m+1)$

$$\left(\Sigma_3 - \Sigma_3'\right)(2m+1, m) = \left(-t_1(2m) + 2^{2m-2} + \frac{1}{2}\right) t_1(2m-1)$$

(I) Compute $\boxed{r\text{-free}}$ polys $p_i(u)$ and $g(u,r)$ s.t.

$$\boxed{p_0(u)\, f(u,r) + p_1(u)\, f(u+1,r) + p_2(u)\, f(u+2,r) \overset{(*)}{=} \Delta_r\, g(u,r).}$$

(II) Sum (*) over all r to obtain

$$\boxed{p_0(u)\, S(u) + p_1(u)\, S(u+1) + p_2(u)\, S(u+2) = 0.}$$

Zeilberger's Algorithm

Given: $S(n) := \sum_r \underbrace{f(n,r)}_{\text{hg in } n \text{ and } r}$,

find: a recurrence for $S(n)$

A2
(+A2!

Multiple Sums

Given: $S(n) := \sum_{r} \sum_{s} \underbrace{F(n,r,s)}_{\text{hg in } n,r,s}$,

find: a recurrence for $S(n)$.

: WZ-method

A_2'

(I) Compute (r,s)-free polys $p_i(n)$ and $G_j(n,r,s)$ s.t.

$$\boxed{\begin{aligned} &p_0(n)\,F(n,r,s) + p_1(n)\,F(n+1,r,s) + p_2(n)\,F(n+2,r,s) \\ &\overset{(*)}{=} \Delta_r\,G_1(n,r,s) + \Delta_s\,G_2(n,r,s). \end{aligned}}$$

(II) Sum $(*)$ over all r and s to obtain

$$\boxed{p_0(n)\,S(n) + p_1(n)\,S(n+1) + p_2(n)\,S(n+2) = 0.}$$

NOTE: It's very hard to design a good algorithm to execute the WZ-method efficiently!

→ Wegschaider's Algorithm (RISC-Linz)

Example Prove that

A.L. Schmidt et al., V. Strehl

$$\boxed{\sum_k \sum_j \binom{n}{k}\binom{n+k}{k}\binom{k}{j}^3 = \sum_k \binom{n}{k}^2\binom{n+k}{k}^2}$$

The summand of the double sum is annihilated by

$$\begin{cases} (1+n)^3 - (3+2\,n)\,(39+51\,n+17\,n^2)\,N + (2+n)^3\,N^2 \\ -\Delta_j(-(1+n)^3\,K - (2+n)^3\,K\,N^2 \\ \qquad + (3+2\,n)\,(21+6\,j-12\,j^2-2\,k+30\,j\,k-12\,k^2+27\,n+9\,n^2)\,N) \\ -\Delta_k(-(1+n)^3 - (2+n)^3\,N^2) \\ \qquad + (3+2\,n)\,(-21-48\,j-24\,j^2+26\,k+24\,j\,k-6\,k^2+3\,n+n^2)\,J\,N) \end{cases}$$

The summand of the single sum is annihilated by

$$\boxed{(1+n)^3 - (3+2n)(39+51n+17n^2)N + (2+n)^3N^2}$$
$$-\Delta_k(-(1+n)^3 + (3+2n)(3+4k+8k^2+3n+n^2)N - (2+n)^3N^2)$$

[Proof obtained automatically by Wegschaider's MULTISUM.]

A.2. Verification of (1) by the Wilf–Zeilberger method

The certifying polynomial $p(j,k,n)$ of Section 3.5 is given by

$$
\begin{aligned}
&108j^5k^3n-69k^2jn^4-12j^6kn-2944j^2k^2n^2+130j^3k^4n^2-960j^3kn^3\\
&+792j^4n^3+132j^4n^4-103670kj-13350k^2j+34043kj^2+5816j^3k^3\\
&-5266j^4k^2+10122j^3k^2+628j^5k-5040j^4k+8478j^3k-5691j^2k^3\\
&-11598j^2k^2+52jk^3+102j^3k^5-880j^4k^4+1772j^3k^4+1270j^5k^3\\
&-3876j^4k^3-568j^6k^2+2322j^5k^2-26j^6k-18j^2k^5-1532j^2k^4\\
&+30jk^5-48j^3k^6-30j^4k^5+150j^5k^4-166j^6k^3+60j^7k^2\\
&+72j^2k^6-48jk^6-3579kn-14604jn+13302j^2n+7749k^2n\\
&+2679j^3n-1641k^3n-588j^4k^3n+694j^4kn^2-12645n-414k^2jn^3\\
&+54j^3k^5n-196j^4k^3n^2+3006j^3k^2n-618j^4k^2n-12j^6k^2n-8274n^2\\
&-116052j-2309j^4+65507k-20093j^3-114j^6+1401j^5\\
&+29838k^2+87861j^2+4151k^3+1530k^5+12k^6-2706n^3\\
&+41338-451n^4+8370j^3kn+390j^3k^4n+54j^2k^5n+18j^3k^5n^2\\
&-834kj^2n+696jk^3n+354j^5kn+210j^5k^2n-2799k^2jn-360j^5n^3\\
&-1515j^2k^3n-216j^4k^4n-6780j^2k^2n+12j^2k^4n+756j^3k^3n-2724j^4kn\\
&-12837kjn+210j^3k^2n^2+280j^4k^2n^2-4j^6k^2n^2+70j^5k^2n^2-1554k^2jn^2\\
&+1350j^3kn^2+18j^2k^5n^2-71kJ^2n^2+232jk^3n^2+36j^5k^3n^2-206j^5kn^2\\
&-16j^2k^3n^4-649j^2k^3n^2-72j^4k^4n^2+4j^2k^4n^2+108j^3k^3n^2-4594kjn^2\\
&-4j^6kn^2+54j^4k^2n^4+324j^4k^2n^3-16j^3k^3n^4-96j^3k^3n^3-84j^6n\\
&+969j^5n-3573j^4n+324k^5n+12j^4k^6+6j^5k^5-6j^6k^4\\
&-4307kn^2-4562jn^2+5037j^2n^2+3042k^2n^2+317j^3n^2-1033k^3n^2\\
&+44j^6n^2-217j^5n^2-3j^4n^2+108k^5n^2-324k^3n^3-54k^3n^4\\
&+51k^2n^4+306k^2n^3-60j^5n^4+8j^6n^4+48j^6n^3+402j^2n^3\\
&+67j^2n^4-64j^3n^4-384j^3n^3+34jn^4+204jn^3-346kn^4\\
&-2076kn^3-88j^3k^2n^4-528j^3k^2n^3-160j^3kn^4+1068j^4kn^3+23j^2n^4\\
&+138kj^2n^3+178j^4kn^4-76j^2k^2n^4-456j^2k^2n^3-36j^5kn^4-216j^5kn^3\\
&-35kjn^4-210kjn^3-96j^2k^3n^3
\end{aligned}
$$

and similarly the certifying polynomial $q(j,k,n)$ is given by

$$
\begin{aligned}
&18j^5k^3n-66^{j6}kn-117k^4jn^2+38j^2k^2n^2-30228kj-16819k^2j\\
&+40207kj^2-342j^4k^2+1452j^3k^2+740j^5k-2767j^4k-5848j^3k\\
&+688j^2k^2-50j^4k^4+200j^5k^3-190j^4k^3-134j^6k^2+102j^5k^2\\
&-84j^6k+72j^5k^4-152j^6k^3+120j^7k^2+2268kn-5928jn\\
&\quad+216j^2n+2430k^2n-3642j^3n-204j^4k^3n-363j^4kn^2-351k^4jn\\
&\quad-68j^4k^3n^2+156j^3k^2n-702j^4k^2n+30j^6k^2n-29180j+8348j^4\\
&\quad+10752k-15682j^3-300j^6+1092j^5+11520k^2+3466j^2\\
&\quad+768k^5-1518j^3kn+8217kj^2n+888j^5kn+330j^5k^2n-3537k^2jn\\
&\quad-18j^4k^4n+114j^2k^2n-1089j^4kn-6336kjn+52j^3k^2n^2-234j^4k^2n^2\\
&\quad+10j^6k^2n^2+110j^5k^2n^2-1179k^2jn^2-506j^3kn^2+2739kj^2n^2+6j^5k^3n^2\\
&\quad+296j^5kn^2-6j^4k^4n^2-2112kjn^2-22j^6kn^2-372j^6n-678j^5n\\
&\quad+2244j^4n+162k^5n+6j^4k^6-6j^5k^5-12j^6k^4+756kn^2\\
&\quad-1976jn^2+72j^2n^2+810k^2n^2-1214j^3n^2-124j^6n^2+226j^5n^2\\
&\quad+748j^4n^2+54k^5n^2-1643k^4j.
\end{aligned}
$$

A4

(+ A1)

Multiple Sums : C. Schneider's Algo.

Given: $$S(n) := \sum_r \underbrace{\sum_s \underbrace{F(n,r,s)}_{\text{hg in } n,r,s}}_{=:\, f(n,r)},$$

find: a recurrence for $S(n)$

How to find: $\Delta_r\, g(u,r) =$

$p_0(u)\, f(u,r) + p_1(u)\, f(u+1,r) + p_2(u)\, f(u+2,r)$?

AS

+

AS'

+

AS''

+

$AS^{(3)}$

+

$AS^{(4)}$

Δ_1

(1) By Z's algo.:

$$\boxed{f(n,r+2) \overset{(1)}{=} \mu_0(n,r)\, f(n,r) + \mu_1(n,r)\, f(n,r+1)}$$

(2) By an extension of Z's algo.: [PP]

$$\boxed{f(n+1,r) = \nu_0(n,r)\, f(n,r) + \nu_1(n,r)\, f(n,r+1)}$$

NOTE: The $\mu_i(n,r)$ and $\nu_i(n,r)$ are RATIONAL FUNCTIONS in n and r.

[redacted]

AS'

NOTE: (1) & (2) imply rational functions $\gamma_i(n,r)$ s.t.

$$\boxed{f(n+2,r) \overset{(3)}{=} \gamma_0(n,r)\, f(n,r) + \gamma_1(n,r)\, f(n,r+1)}$$

Consequently, (2) & (3) imply:

$$p_0(n)\, f(n,r) + p_1(n)\, f(n+1,r) + p_2(n)\, f(n+2,r)$$

$$= \big(p_0(n) + p_1(n)\, \nu_0(n,r) + p_2(n)\, \gamma_0(n,r)\big)\, \underline{f(n,r)}$$

$$+ \big(p_1(n)\, \nu_1(n,r) + p_2(n)\, \gamma_1(n,r)\big)\, \underline{f(n,r+1)}.$$

Δ_4

AS'''
$'' AS^{(3)}$

ANSATZ with unknown rational functions $\alpha_i(u,r)$:

$$\boxed{g(u,r) = \alpha_0(u,r)\, f(u,r) + \alpha_1(u,r)\, f(u,r+1)}$$

Then: $\Delta_r\, g(u,r) =$

$$\left(-\alpha_0(u,r) + \alpha_1(u,r+1)\,\mu_0(u,r)\right) \underline{f(u,r)}$$

$$+\left(\alpha_0(u,r+1) - \alpha_1(u,r) + \alpha_1(u,r+1)\,\mu_1(u,r)\right) \underline{f(u,r+1)}$$

NOTE: (1) had been used to express $f(u,r+2)$ in terms of $f(u,r)$ and $f(u,r+1)$.

Δ_3

$$\Updownarrow$$

$$\left(-\alpha_0(n,r)+\alpha_1(n,r+1)\mu_0(n,r)\right)\ f(n,r)$$
$$+\left(\alpha_0(n,r+1)-\alpha_1(n,r)+\alpha_1(n,r+1)\mu_1(n,r)\right)f(n,r+1)=$$

$$\left(p_0(n)+p_1(n)\,\nu_0(n,r)+p_2(n)\,\gamma_0(n,r)\right)f(n,r)$$
$$+\left(p_1(n)\,\nu_1(n,r)+p_2(n)\,\gamma_1(n,r)\right)f(n,r+1)$$

The coefficient comparison leads to solving a parameterized difference equation for a rational function $\alpha_1(u,r) \in \mathbb{C}(u)(r)$:

$$\boxed{\begin{aligned}&\mu_0(u,r+1)\,\alpha_1(u,r+2) + \mu_1(u,r)\,\alpha_1(u,r+1) - \alpha_1(u,r)\\ &= p_0(u) + \big(\gamma_0(u,r+1)+\gamma_1(u,r)\big)\,p_1(u) + \big(\gamma_0(u,r+1)+\gamma_1(u,r)\big)\,p_2(u)\end{aligned}}$$

furthermore,

$$\alpha_0(u,r) = \text{fu}\big(\alpha_1(u,r+1),\, p_0(u),\, p_1(u),\, p_2(u)\big)$$

$$\in \mathbb{C}(u)(r)$$

$\square$