Affine Weyl groups, grids and coloured tableaux

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To Jim Louck on or about his 75th birthday

Motivation

- The evaluation of characters of irreducible representations of classical affine Lie algebras
- Numerator expansions analagous to the Macdonald denominator identity expansions
- Coset representatives of affine Weyl groups with respect to finite Weyl groups and their action on arbitrary weights

Topics

- Embeddings of simple Lie algebras in affine Lie algebras, character formulae and statement of problem
- State of play at Renaissance of Combinatorics'99
- Periodic coloured grids, coloured tableaux and the action of affine Weyl groups
- Coset decomposition of affine Weyl groups, denominator and numerator expansions
- Bruhat graphs, canonical words

Complex simple and affine Lie algebras

Classification

- Complex simple Lie algebras $\overline{\mathbf{g}}$ [Cartan and Killing]

$$A_\ell,\ B_\ell,\ C_\ell, D_\ell,\ E_6,\ E_7,\ E_8,\ F_4,\ G_2.$$

- Affine Lie algebras $\mathbf{g}$ [Kac and Moody]

$$A_\ell^{(1)},\ B_\ell^{(1)},\ C_\ell^{(1)},\ D_\ell^{(1)},\ A_{2\ell}^{(2)},\ A_{2\ell-1}^{(2)},\ D_{\ell+1}^{(2)},$$

$$E_6^{(1)},\ E_7^{(1)},\ E_8^{(1)},\ F_4^{(1)},\ G_2^{(1)},\ E_6^{(2)},\ D_4^{(3)}.$$

- Subscripts determine the rank: rank ℓ in classical cases
- Superscripts (k): untwisted $k = 1$, twisted $k = 2, 3$.

Natural embeddings $\overline{\mathbf{g}} \subset \mathbf{g}$

$A_\ell \subset A_\ell^{(1)}$

$B_\ell \subset B_\ell^{(1)}, A_{2\ell}^{(2)}, D_{\ell+1}^{(2)}$

$C_\ell \subset C_\ell^{(1)}, A_{2\ell-1}^{(2)}$

$D_\ell \subset D_\ell^{(1)}$

$E_6 \subset E_6^{(1)}$

$E_7 \subset E_7^{(1)}$

$E_8 \subset E_8^{(1)}$

$F_4 \subset F_4^{(1)}, E_6^{(2)}$

$G_2 \subset G_2^{(1)}, D_4^{(3)}$

Some terminology

- Let $I = \{0, 1, 2, \ldots, \ell\}$ and $\overline{I} = I \backslash \{0\} = \{1, 2, \ldots, \ell\}$.
- The duals $\mathbf{h}^*$ and $\overline{\mathbf{h}}^*$ of the Coxeter subalgebras of $\mathbf{g}$ and $\overline{\mathbf{g}}$ have bases $\{\Lambda_0, \alpha_i : i \in I\}$ and $\{\alpha_i : i \in \overline{I}\}$, respectively.
- Simple roots α_i and co-roots $\alpha_i^\vee = 2\alpha_i/(\alpha_i|\alpha_i)$ for all $i \in I$.
- Inner product $(\cdot|\cdot)$ on $\mathbf{h}^*$ such that:
$$(\Lambda_0|\Lambda_0) = 0,$$
$$(\Lambda_0|\alpha_i^\vee) = \delta_{0i} \text{ for } i \in I,$$
$$(\alpha_i|\alpha_j^\vee) = A_{ij} \text{ for } i, j \in I.$$
- Marks c_i for $i \in I$ are the smallest positive integers such that $\sum_{i \in I} c_i A_{ij} = 0$ for all $j \in I$.
- Co-marks $c_j^\vee$ for $j \in I$ are the smallest positive integers such that $\sum_{j \in I} c_j^\vee A_{ij} = 0$ for all $i \in I$.
- Coxeter number $h = \sum_{i \in I} c_i$.
Dual Coxeter number $h^\vee = \sum_{i \in I} c_i^\vee$.
Modified dual Coxeter number $\tilde{h}^\vee = \begin{cases} 2h^\vee & \text{if } \mathbf{g} = C_l^{(1)}; \\ h^\vee & \text{otherwise.} \end{cases}$
- Imaginary root $\delta = \sum_{i \in I} c_i \alpha_i$.
$$(\delta|\alpha_j^\vee) = 0 \text{ for } j \in I,$$
$$(\delta|\Lambda_0) = (\alpha_0|\alpha_0)/2,$$
$$(\delta|\delta) = 0.$$

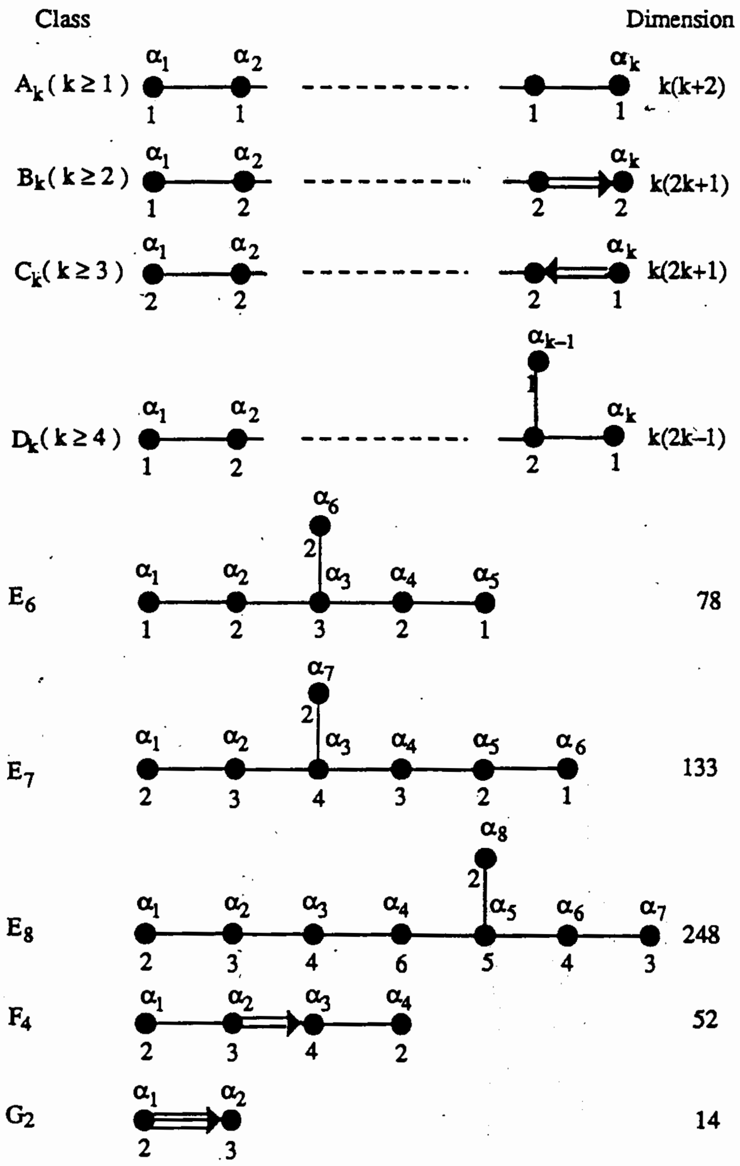


Fig. 9.8.13. Dynkin diagrams of simple Lie algebras.

Class Aff$^{(1)}$

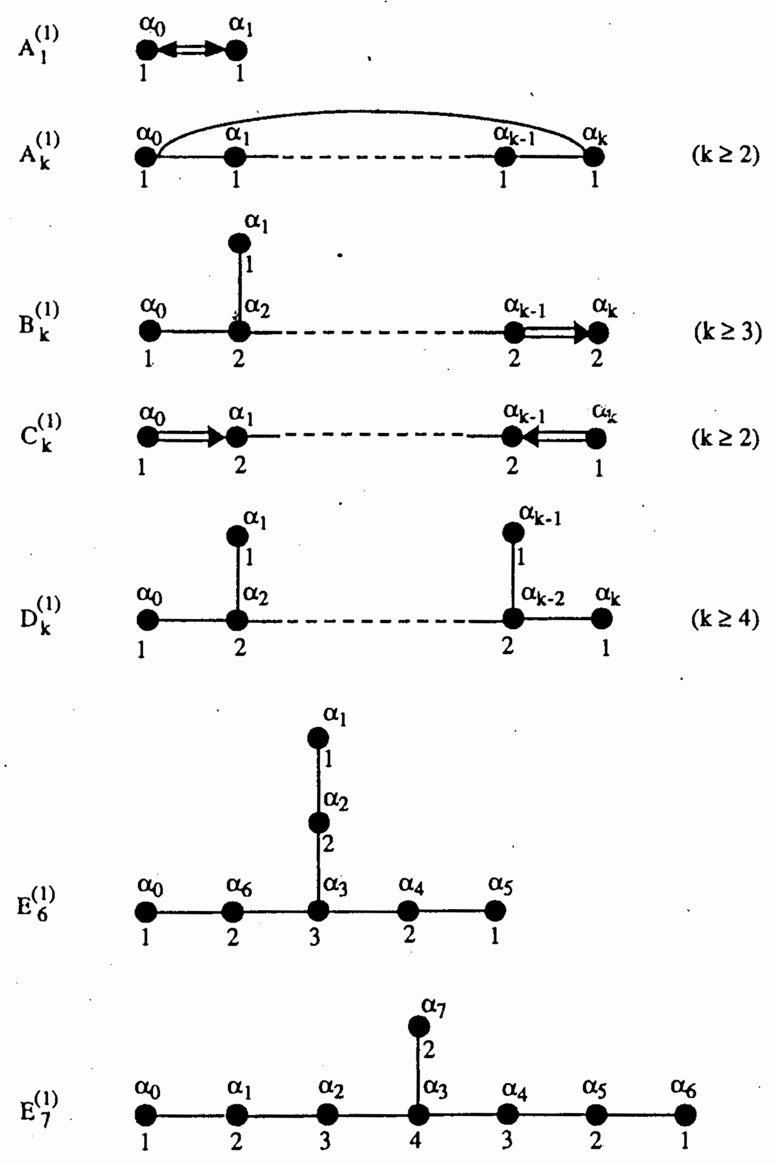


Fig. 14.5.13. Table of affine Dynkin diagrams.

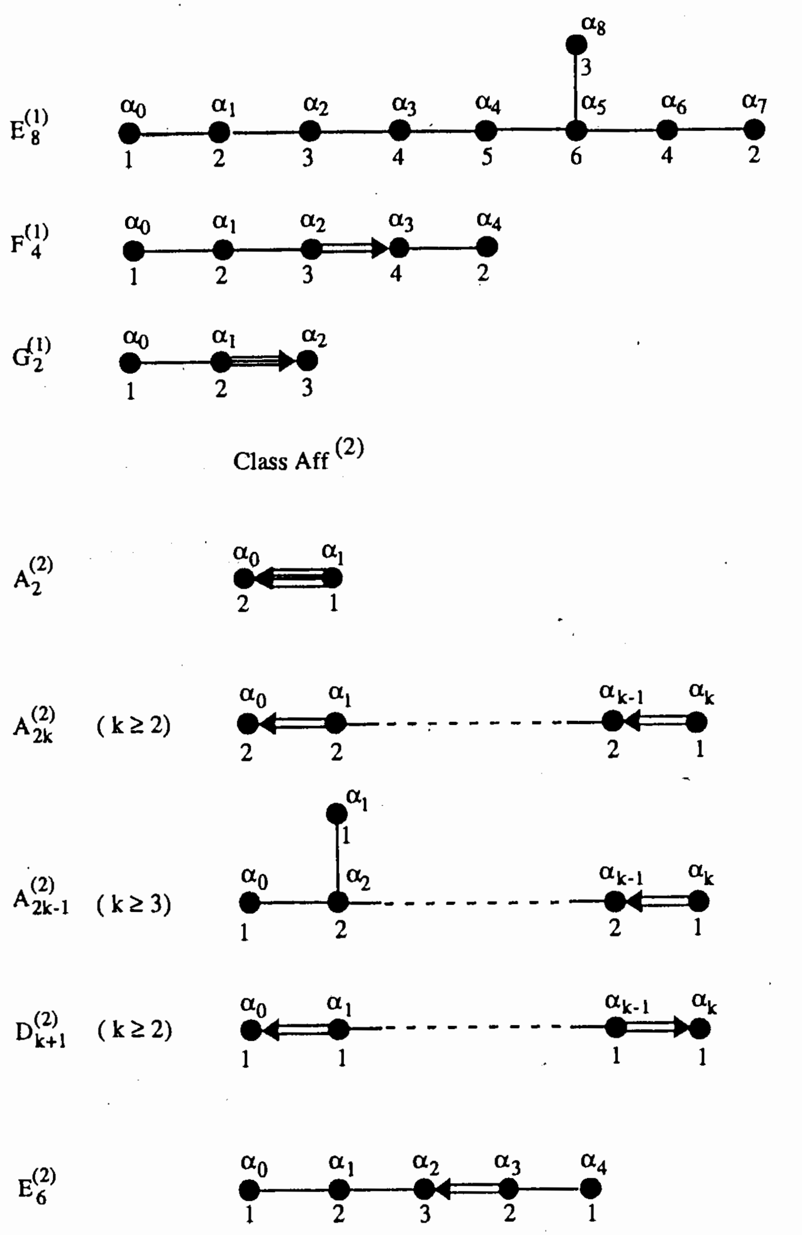


Fig. 14.5.13. Continued.

$D_4^{(3)}$

The weights of an affine Lie algebra

- The fundamental weights of **g** are denoted by Λ_i where

$$(\Lambda_i|\alpha_j^\vee) = \delta_{ij} \text{ for } i, j \in \bar{I},$$
$$(\Lambda_i|\Lambda_0) = 0 \text{ for } i \in \bar{I},$$
$$(\Lambda_i|\delta) = 0 \text{ for } i \in \bar{I}.$$

- For each $\lambda \in \mathbf{h}^*$ we have $\lambda = \sum_{i\in I} n_i(\lambda)\Lambda_i - D(\lambda)\delta$.

Dynkin indices: $n_i(\lambda) = (\lambda|\alpha_i^\vee)$ for $i \in I$.

Depth: $D(\lambda) = -2(\lambda|\Lambda_0)/(\alpha_0|\alpha_0)$.

Level: $L(\lambda) = \sum_{i\in I} c_i^\vee\, n_i(\lambda)$.

Modified level: $\tilde{L}(\lambda) = \begin{cases} \frac{1}{2}L(\lambda) & \text{if } \mathbf{g} = A_{2\ell}^{(2)}; \\ L(\lambda) & \text{otherwise.} \end{cases}$

- The Weyl vector is defined by $\rho = \sum_{i\in I} \Lambda_i$.

Dynkin indices: $n_i(\rho) = (\rho|\alpha_i^\vee) = 1$,

Depth: $D(\rho) = 0$,

Level: $L(\rho) = h^\vee$.

- Simple roots α_k for $k \in I$.

Dynkin indices: $n_i(\alpha_k) = (\alpha_k|\alpha_i^\vee) = A_{ki}$,

Depth: $D(\alpha_k) = -\delta_{k0}$,

Level: $L(\alpha_k) = 0$.

Weyl groups

Affine Weyl group W of $\mathbf{g}$

- W is generated by $\{s_i \,|\, i \in I\}$
- For any $s_i \in W$ and $\lambda \in \mathbf{h}^*$ we have $s_i(\lambda) = \lambda - (\lambda|\alpha_i^\vee)\alpha_i$.
- Each $w \in W$ may be written in the form $w = s_{i_1} s_{i_2} \cdots s_{i_t}$.
- The length $\ell(w)$ of w is the minimum possible value of t.
- The parity or sign of w is given by $\varepsilon(w) = (-1)^{\ell(w)}$.
- The roots of $\mathbf{g}$ are given by $\Delta = \{w(\alpha_i) \,|\, w \in W, i \in I\}$
- $\Delta = \Delta^+ \cup \Delta^-$ where $\Delta^- = \{-\alpha \,|\, \alpha \in \Delta^+\}$

Finite Weyl group $\overline{W}$ of $\overline{\mathbf{g}}$

- $\overline{W}$ is the finite subgroup of W generated by $\{s_i \,|\, i \in \overline{I}\,\}$
- The roots of $\overline{\mathbf{g}}$ are given by $\overline{\Delta} = \{w(\alpha_i) \,|\, w \in \overline{W}, i \in \overline{I}\,\}$
- $\overline{\Delta} \subset \Delta$ and $\overline{\Delta}^+ = \overline{\Delta} \cap \Delta^+$

Coset representatives

- Let $W_s = \{w \in W \,|\, \ell(\overline{w}\,w) \geq \ell(w) \quad \text{for all } \overline{w} \in \overline{W}\}$.
- W_s contains one element from each coset of W with respect to $\overline{W}$
- Each $w \in W_s$ is the unique element of minimal length in the right coset $\overline{W}w$

Irreducible representations and character formulae

Some notation

- Let $\mathbf{h}^* = \overline{\mathbf{h}}^* \oplus \mathbb{C}\Lambda_0 \oplus \mathbb{C}\delta$, then for any $\lambda \in \mathbf{h}^*$ we have:
  $\lambda = \overline{\lambda} + \tilde{L}(\lambda)\Lambda_0 - D(\lambda)\delta$ with $\overline{\lambda} \in \overline{\mathbf{h}}^*$.
- The set of dominant integral weights of $\mathbf{g}$:
  $P^+ = \{\lambda \in \mathbf{h}^* \mid (\lambda|\alpha_i) \in \mathbb{Z}_{\geq 0} \text{ for all } i \in I\}$.
- The set of dominant integral weights of $\overline{\mathbf{g}}$:
  $\overline{P}^+ = \{\lambda \in \overline{\mathbf{h}}^* \mid (\lambda|\alpha_i) \in \mathbb{Z}_{\geq 0} \text{ for all } i \in \overline{I}\}$.

Complex simple Lie algebras

- [Weyl] For each $\overline{\lambda} \in \overline{P}^+$ we have a finite-dimensional irreducible representation $\overline{V}^{\overline{\lambda}}$ with character

$$\operatorname{ch} \overline{V}^{\overline{\lambda}} = \sum_{w \in \overline{W}} \varepsilon(w) e^{w(\overline{\lambda}+\overline{\rho})-\overline{\rho}} \Big/ \sum_{w \in \overline{W}} \varepsilon(w) e^{w(\overline{\rho})-\overline{\rho}}.$$

Affine Lie algebras

- [Kac] For each $\lambda \in P^+$ we have an infinite-dimensional irreducible representation V^λ with character

$$\operatorname{ch} V^\lambda = \sum_{w \in W} \varepsilon(w) e^{w(\lambda+\rho)-\rho} \Big/ \sum_{w \in W} \varepsilon(w) e^{w(\rho)-\rho}.$$

Proposition [Hussin and King]

For each $\lambda \in P^+$ we have $\operatorname{ch} V^\lambda = M^\lambda / M$ where

$$M^\lambda = e^{\tilde{L}(\lambda)\Lambda_0 - D(\lambda)\delta} \sum_{w \in W_s} \varepsilon(w)\; e^{-D(w\cdot\lambda)\delta} \operatorname{ch} \overline{V}^{\overline{w\cdot\lambda}}$$

and $M = M^0$ while $w \cdot \lambda = w(\lambda + \rho) - \rho$.

The $\epsilon\delta$-basis

- For each classical affine Lie algebra $\mathbf{g}$ set
  $n = \ell + 1$ for $\mathbf{g} = A_l^{(1)}$ and $n = \ell$ otherwise.
- Embed $\mathbf{h}^* = \overline{\mathbf{h}}^* \oplus \boldsymbol{C}\Lambda_0 \oplus \boldsymbol{C}\delta \in E^n \oplus \boldsymbol{C}\Lambda_0 \oplus \boldsymbol{C}\delta$,
  where E^n is an n-dimensional Euclidean vector space.
- Let E_n have basis ϵ_k for $k \in N = \{1, 2, \ldots, n\}$ such that
  $(\epsilon_i|\epsilon_j) = \delta_{ij}$, $(\epsilon_i|\delta) = 0$, $(\epsilon_i|\Lambda_0) = 0$ for all $i, j \in N$.
- Then for all $\lambda \in \mathbf{h}^*$

$$\lambda = \sum_{i \in I} n_i(\lambda)\Lambda_i - D(\lambda)\delta$$

$$= \overline{\lambda} + \tilde{L}(\lambda)\,\Lambda_0 - D(\lambda)\delta \quad \text{with} \quad \overline{\lambda} = \sum_{i \in N} \lambda_i\,\epsilon_i.$$

Statement of problem

Given that $\mathrm{ch}\, V^\lambda = M^\lambda / M$ for all $\lambda \in P^+$:

- Identify all $w \in W_s$.
- Evaluate $w(\rho) - \rho$ for all $w \in W_s$.
- Calculate all terms in the denominator expansion [Macdonald]

$$M = \prod_{\alpha \in \Delta^+ / \overline{\Delta}^+} (1 - e^{-\alpha})^{\mathrm{mult}\,(\alpha)}$$
$$= \sum_{w \in W_s} \varepsilon(w)\, e^{-D(w(\rho)-\rho))\delta}\, \mathrm{ch}\, \overline{V}^{\overline{w(\rho)-\rho}}$$

- Use known products of characters of $\overline{\mathbf{g}}$ to calculate the inverse M^{-1} of M (down to some prescribed finite depth).
- Evaluate $w(\lambda + \rho) - \rho$ for all $w \in W_s$ and all $\lambda \in P^+$.
- Identify all terms in the numerator expansion

$$M^\lambda = e^{\tilde{L}(\lambda)\Lambda_0 - D(\lambda)\delta} \sum_{w \in W_s} \varepsilon(w)\, e^{-D(w(\lambda+\rho)-\rho)\delta}\, \mathrm{ch}\, \overline{V}^{\overline{w(\lambda)+\rho)-\rho}}$$

- Again use known products of characters of $\overline{\mathbf{g}}$ to evaluate $\mathrm{ch}\, V^\lambda = M^\lambda\, M^{-1}$ (down to some prescribed finite depth).

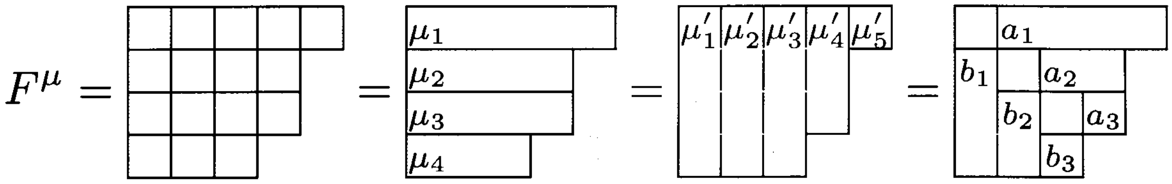
State of play at Renaissance of Combinatorics'99

Identification of $w \in W_s$ Typically we found:

$$
\begin{array}{ll}
A_\ell^{(1)} & \supset A_\ell \ (\ell = 9) \ w_\zeta = (s_0s_1s_2s_3s_9s_8s_7s_6)(s_0s_1s_2s_9s_8)(s_0s_1s_9) \\
B_\ell^{(1)} & \supset B_\ell \ (\ell \geq 5) \ w_\alpha = (s_0s_2s_3s_4)(s_1s_2)(s_0) \\
C_\ell^{(1)} & \supset C_\ell \ (\ell \geq 4) \ w_\gamma = (s_0s_1s_2s_3)(s_0s_1) \\
D_\ell^{(1)} & \supset D_\ell \ (\ell \geq 6) \ w_\alpha = (s_0s_2s_3s_4)(s_1s_2)(s_0) \\
A_{2\ell}^{(2)} & \supset B_\ell \ (\ell \geq 4) \ w_\gamma = (s_0s_1s_2s_3)(s_0s_1) \\
A_{2\ell-1}^{(2)} & \supset C_\ell \ (\ell \geq 5) \ w_\alpha = (s_0s_2s_3s_4)(s_1s_2)(s_0) \\
D_{\ell+1}^{(2)} & \supset B_\ell \ (\ell \geq 5) \ w_\varepsilon = (s_0s_1s_2s_3s_4)(s_0s_1s_2)
\end{array}
$$

Evaluation of $w(\rho) - \rho$

- Each $w = w_\mu \in W_s$ is labelled by a partition μ that specifies a Young diagram F^μ



- Then for $\mu = \zeta$, α, γ or ε we have

$$w_\zeta(\rho) - \rho = \sum_{(i,j) \in F^\zeta} (\epsilon_i - \epsilon_{\ell-j+2} - \delta);$$

$$w_\alpha(\rho) - \rho = \sum_{(i,j) \in F^\alpha} (\epsilon_i - \tfrac{1}{2}\delta);$$

$$w_\gamma(\rho) - \rho = \sum_{(i,j) \in F^\gamma} (\epsilon_i - \tfrac{1}{2}\delta);$$

$$w_\varepsilon(\rho) - \rho = \sum_{(i,j) \in F^\varepsilon} (\epsilon_i - \delta),$$

Denominator expansions

Let $q = e^{-\delta}$ and let $\mathcal{P}$ be the set of all partitions, then

$$A_\ell^{(1)} \supset A_\ell : \qquad M = \textstyle\sum_{\zeta\in\mathcal{F}} (-1)^{|\zeta|}\, q^{|\zeta|}\, \mathrm{ch}\,\overline{V}^{\{\zeta;\overline{\zeta'}\}};$$

$$B_\ell^{(1)} \supset B_\ell : \qquad M = \textstyle\sum_{\alpha\in\mathcal{A}} (-1)^{|\alpha|/2}\, q^{|\alpha|/2}\, \mathrm{ch}\,\overline{V}^{[\alpha]};$$

$$C_\ell^{(1)} \supset C_\ell : \qquad M = \textstyle\sum_{\gamma\in\mathcal{C}} (-1)^{|\gamma|/2}\, q^{|\gamma|/2}\, \mathrm{ch}\,\overline{V}^{\langle\gamma\rangle};$$

$$D_\ell^{(1)} \supset D_\ell : \qquad M = \textstyle\sum_{\alpha\in\mathcal{A}} (-1)^{|\alpha|/2}\, q^{|\alpha|/2}\, \mathrm{ch}\,\overline{V}^{[\alpha]};$$

$$A_{2\ell}^{(2)} \supset B_\ell : \qquad M = \textstyle\sum_{\gamma\in\mathcal{C}} (-1)^{|\gamma|/2}\, q^{|\gamma|/2}\, \mathrm{ch}\,\overline{V}^{[\gamma]};$$

$$A_{2\ell-1}^{(2)} \supset C_\ell : \qquad M = \textstyle\sum_{\alpha\in\mathcal{A}} (-1)^{|\alpha|/2}\, q^{|\alpha|/2}\, \mathrm{ch}\,\overline{V}^{\langle\alpha\rangle};$$

$$D_{\ell+1}^{(2)} \supset B_\ell : \qquad M = \textstyle\sum_{\varepsilon\in\mathcal{E}} (-1)^{(|\varepsilon|+p)/2}\, q^{|\varepsilon|}\, \mathrm{ch}\,\overline{V}^{[\varepsilon]},$$

where in Frobenius notation

$$\mathcal{A} = \left\{\alpha \in \mathcal{P} \mid \alpha = \begin{pmatrix} a_1 & a_2 & \cdots & a_p \\ a_1{+}1 & a_2{+}1 & \cdots & a_p{+}1 \end{pmatrix}\right\};$$

$$\mathcal{C} = \left\{\gamma \in \mathcal{P} \mid \gamma = \begin{pmatrix} b_1{+}1 & b_2{+}1 & \cdots & b_p{+}1 \\ b_1 & b_2 & \cdots & b_p \end{pmatrix}\right\};$$

$$\mathcal{E} = \left\{\varepsilon \in \mathcal{P} \mid \varepsilon = \begin{pmatrix} a_1 & a_2 & \cdots & a_p \\ a_1 & a_2 & \cdots & a_p \end{pmatrix}\right\};$$

$$\mathcal{F} = \left\{\zeta \in \mathcal{P} \mid \zeta = \begin{pmatrix} a_1 & a_2 & \cdots & a_p \\ b_1 & b_2 & \cdots & b_p \end{pmatrix}\right\} = \mathcal{P}.$$

These expansions are valid for all values of the rank ℓ provided that the characters are interpreted in terms of standard characters by means of modification rules.

Numberings of Young diagrams

$$F^{\mu} = \begin{ytableau} 0 & 9 & 8 & 7 & 6 \\ 1 & 0 & 9 & 8 \\ 2 & 1 & 0 & 9 \\ 3 & 2 & 1 \end{ytableau} \qquad F^{\alpha} = \begin{ytableau} 0 & 2 & 3 & 4 \\ 1 & 0 & 2 \\ 2 & 1 & 0 \\ 3 & 2 & \text{[illegible]} \\ 4 \end{ytableau}$$

$$F^{\gamma} = \begin{ytableau} 0 & 0 & 1 & 2 & 3 \\ 1 & 0 & 0 & 1 \\ 2 & 1 \\ 3 \end{ytableau} \qquad F^{\varepsilon} = \begin{ytableau} 0 & 1 & 2 & 3 & 4 \\ 1 & 0 & 1 & 2 \\ 2 & 1 \\ 3 & 2 \\ 4 \end{ytableau}$$

Evaluation of $w(\lambda+\rho)-\rho$

- Let $\lambda = \sum_{k\in I} n_k(\lambda)\Lambda_i - D(\lambda)\delta$ be integral dominant.
- Note $w(\lambda+\rho)-\rho = \lambda + w(\lambda) - \lambda + w(\rho) - \rho$.
- With the above numberings η_{ij} in the position (i,j)

$$w_\zeta(\lambda) - \lambda = \sum_{(i,j)\in F^\zeta} n_{\eta_{ij}}(\lambda)\ (\epsilon_i - \epsilon_{\ell-j+2} - \delta);$$

$$w_\alpha(\lambda) - \lambda = \sum_{(i,j)\in F^\alpha} n_{\eta_{ij}}(\lambda)\ (\epsilon_i - \tfrac{1}{2}\delta);$$

$$w_\gamma(\lambda) - \lambda = \sum_{(i,j)\in F^\gamma} n_{\eta_{ij}}(\lambda)\ (\epsilon_i - \tfrac{1}{2}\delta);$$

$$w_\varepsilon(\lambda) - \lambda = \sum_{(i,j)\in F^\varepsilon} n_{\eta_{ij}}(\lambda)\ (\epsilon_i - \delta)$$

- It has only been necessary to scale contributions to $w(\rho)-\rho$ by the components $n_k(\lambda)$ of λ with $k=\eta_{ij}$.

The evaluation of $w(\lambda) - \lambda$ for all $w \in W$

For $w = s_i s_j \cdots s_k$ set $w = w' s_k$ and proceed recursively.

Lemma Let $w = w' s_k$ for some $k \in I$, then for all $\lambda \in \mathbf{h}^*$

$$w(\lambda) = w'(\lambda) - n_k(\lambda) w'(\alpha_k),$$

Proof The result follows from

$$w(\lambda) - w'(\lambda) = w'(s_k(\lambda) - \lambda),$$

with $\quad s_k(\lambda) = \lambda - (\lambda|\alpha_k^\vee)\alpha_k,$

where $\quad (\lambda|\alpha_k^\vee) = n_k(\lambda).$

- Since $n_k(\rho) = 1$, we have as a special case
  $w(\rho) = w'(\rho) - w'(\alpha_k).$
- To obtain $w(\lambda)$ we then scale by $n_k(\lambda)$
  $w(\lambda) = w'(\lambda) - n_k(\lambda) w'(\alpha_k).$
- To prepare for the next iteration we use $n_k(\alpha_i) = A_{ik}$
  $w(\alpha_i) = w'(\alpha_i) - A_{ik} w'(\alpha_k).$
- Note that $w'(\alpha_k) \in \Delta$.
- In fact $w'(\alpha_k) \in \Delta^\pm$ according as $\ell(w) = \ell(w') \pm 1$.

Simple roots in the $\epsilon\delta$-basis

$$\text{For all } \mathbf{g}: \quad \alpha_i = \epsilon_i - \epsilon_{i+1} \qquad i = 1, 2, \ldots, \ell - 1$$

$$\begin{array}{rll}
A_\ell^{(1)}: & \alpha_0 = \delta - \epsilon_1 + \epsilon_{\ell+1} & \alpha_\ell = \epsilon_\ell - \epsilon_{\ell+1} \\
B_\ell^{(1)}: & \alpha_0 = \delta - \epsilon_1 - \epsilon_2 & \alpha_\ell = \epsilon_\ell \\
C_\ell^{(1)}: & \alpha_0 = \delta - 2\epsilon_1 & \alpha_\ell = 2\epsilon_\ell \\
D_\ell^{(1)}: & \alpha_0 = \delta - \epsilon_1 - \epsilon_2 & \alpha_\ell = \epsilon_{\ell-1} + \epsilon_\ell \\
A_{2\ell}^{(2)}: & \alpha_0 = \delta - 2\epsilon_1 & \alpha_\ell = \epsilon_\ell \\
A_{2\ell-1}^{(2)}: & \alpha_0 = \delta - \epsilon_1 - \epsilon_2 & \alpha_\ell = 2\epsilon_\ell \\
D_{\ell+1}^{(2)}: & \alpha_0 = \delta - \epsilon_1 & \alpha_\ell = \epsilon_\ell
\end{array}$$

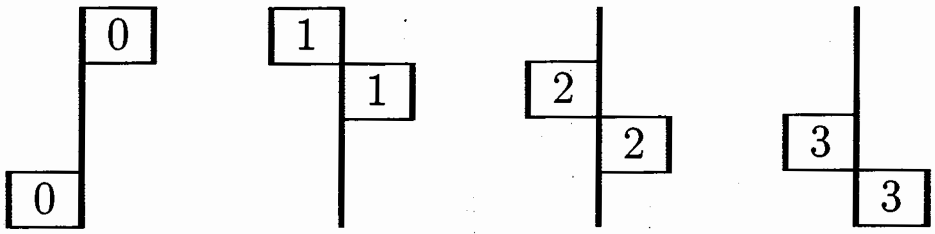
Definitions of $\lambda(w)$, $d(w)$ and $F(w)$

- Let $w(\rho) - \rho = \sum_{k \in N} \lambda_i(w)\, \epsilon_i - d(w)\, \delta.$
- Then $F(w)$ is the diagram constructed by placing $|\lambda_i(w)|$ boxes in the ith row to the left or right of a reference axis according as $\lambda_i(w) > 0$ or $\lambda_i(w) < 0$, respectively, for $i = 1, 2, \ldots, n$.

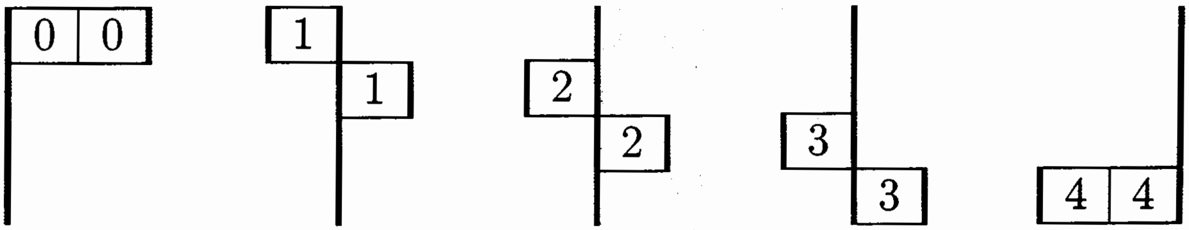
Origin of the numberings

Consider the step by step evaluation of $w(\rho) - \rho$ using $w(\rho) - \rho = w'(\rho) - \rho - w'(\alpha_k)$ for $w = w's_k$.

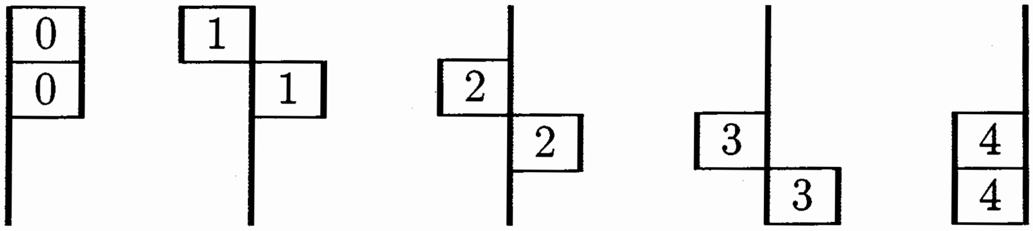
- First for $\ell(w) = 1$ and $w = s_k$ we have $w' = 1$ so that $s_k(\rho) - \rho = -\alpha_k = -\sum_{i \in N} (\alpha_k|\epsilon_i)\, \epsilon_i - \delta_{k0}\delta$.
- To form $F(s_k)$ we enter $m = -(\alpha_k|\epsilon_i)$ boxes of colour k in row i for $i = 1, 2, \ldots, n$.
- These are to the right and left of the vertical axis in row i for $m > 0$ and $m < 0$, respectively.
- For example in the case of $A_3^{(1)}$ and $k = 0, 1, 2, 3$ we obtain:



- In the case of $C_4^{(1)}$ and $k = 0, 1, 2, 3$ we obtain:



- In the case of $D_4^{(1)}$ and $k = 0, 1, 2, 3$ we obtain:



Continuing

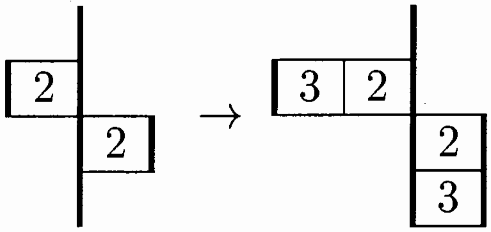
- For $\ell(w) = 2$ and $w = s_j s_k$ we have $w' = s_j$

  so that $s_j s_k(\rho) - \rho = s_j(\rho) - \rho - s_j(\alpha_k)$.

  Hence $s_j s_k(\rho) - \rho = -\alpha_j - \alpha_k + A_{kj}\alpha_j$.

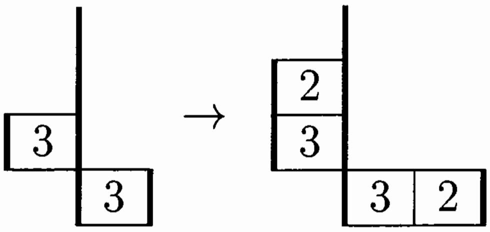
- For $A_3^{(1)}$ and $(j,k) = (3,2)$ and $(2,3)$ we obtain for the second steps:

s_3 :



$$s_3 s_2(\rho) - \rho = -2\epsilon_2 + \epsilon_3 + \epsilon_4$$

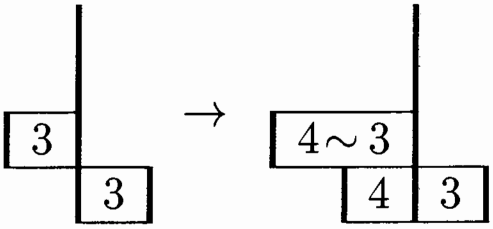
s_2 :



$$s_2 s_3(\rho) - \rho = -\epsilon_2 - \epsilon_3 + 2\epsilon_4$$

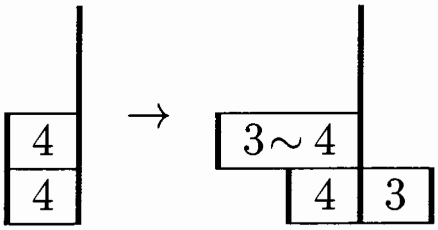
- In the case of $D_4^{(1)}$ and $(j,k) = (4,3)$ and $(3,4)$ we obtain:

s_4 :



$$s_4 s_3(\rho) - \rho = -2\epsilon_3$$

s_3 :



$$s_3 s_4(\rho) - \rho = -2\epsilon_3$$

Periodic grids

For each classical affine Lie algebra **g** of rank ℓ construct a grid as follows:

- Let $I = \{0, 1, \ldots, \ell\}$ and let $n = \ell + 1$ for $\mathbf{g} = A_\ell^{(1)}$ and $n = \ell$ otherwise.
- Each grid is an n-rowed array of nodes, each coloured with an element $k \in I$.
- Each colour k is associated with a simple root α_k and hence with a node of the Dynkin diagram of **g**.
- The nodes in the column immediately to the right of a vertical axis are coloured $0, 1, \ldots, n-1$ from top to bottom.
- The sequence of colours reading across each row is obtained from the labels on the Dynkin diagram by:

  going round a closed loop for $A_\ell^{(1)}$;

  passing to and fro with reflections at each end

  in all other cases;

  duplicating k and joining the pair of nodes

  to give $k-k$ if $(\alpha_k|\alpha_k) = 4$;

  at each branching entering $i \sim j$ as an

  unordered pair if $j = i \pm 1$ and $(\alpha_i|\alpha_j) = 0$.

- The result in each case is a sequence having period $\tilde{h}^\vee$.

Coloured grids - classical untwisted affine algebras

$A_4^{(1)}$: Rank 4, Period $\tilde{h}^\vee = 5$.

2	1	0	4	3	2	1	0	4	3	2	1	0	4	3	2	1
3	2	1	0	4	3	2	1	0	4	3	2	1	0	4	3	2
4	3	2	1	0	4	3	2	1	0	4	3	2	1	0	4	3
0	4	3	2	1	0	4	3	2	1	0	4	3	2	1	0	4
1	0	4	3	2	1	0	4	3	2	1	0	4	3	2	1	0

$B_4^{(1)}$: Rank 4, Period $\tilde{h}^\vee = 7$.

2	1	0	2	3	4	3	2	1∼	0	2	3	4	3	2	1∼	0
3	2	1∼	0	2	3	4	3	2	1∼	0	2	3	4	3	2	1∼
4	3	2	1∼	0	2	3	4	**3**	2	1∼	0	2	3	4	3	2
3	4	3	2	1∼	0	2	3	4	3	2	1∼	0	2	3	4	3

$C_4^{(1)}$: Rank 4, Period $\tilde{h}^\vee = 10$

2	1	0−	0	1	2	3	4−	4	3	2	1	0−	0	1	2	3
3	2	1	0−	0	1	2	3	4−	4	3	2	1	0−	0	1	2
−4	3	2	1	0−	0	1	2	3	4−	4	3	2	1	0−	0	1
4−	4	3	2	1	0−	0	1	2	3	4−	4	3	2	1	0−	0

$D_4^{(1)}$: Rank 4, Period $\tilde{h}^\vee = 6$.

2	1	0	2	4∼	3	2	1∼	0	2	4∼	3	2	1∼	0	2	4∼
∼3	2	1∼	0	2	4∼	3	2	1∼	0	2	4∼	3	2	1∼	0	2
4∼	3	2	1∼	0	2	4∼	3	2	1∼	0	2	4∼	3	2	1∼	0
2	4	3	2	1∼	0	2	4∼	3	2	1∼	0	2	4∼	3	2	1∼

Coloured grids - classical twisted affine algebras

$A_8^{(2)}$: Rank 4, Period $\tilde{h}^\vee = 9$.

$$\begin{array}{cc|ccccccccccccccc}
2 & 1 & 0 \,- & 0 & 1 & 2 & 3 & 4 & 3 & 2 & 1 & 0 \,- & 0 & 1 & 2 & 3 & 4 \\
3 & 2 & 1 & 0 \,- & 0 & 1 & 2 & 3 & 4 & 3 & 2 & 1 & 0 \,- & 0 & 1 & 2 & 3 \\
4 & 3 & 2 & 1 & 0 \,- & 0 & 1 & 2 & 3 & 4 & 3 & 2 & 1 & 0 \,- & 0 & 1 & 2 \\
3 & 4 & 3 & 2 & 1 & 0 \,- & 0 & 1 & 2 & 3 & 4 & 3 & 2 & 1 & 0 \,- & 0 & 1
\end{array}$$

$A_7^{(2)}$: Rank 4, Period $\tilde{h}^\vee = 8$.

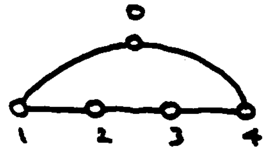
$$\begin{array}{cc|ccccccccccccccc}
2 & 1 & 0 & 2 & 3 & 4 \,- & 4 & 3 & 2 & 1 \sim & 0 & 2 & 3 & 4 \,- & 4 & 3 & 2 \\
3 & 2 & 1 \sim & 0 & 2 & 3 & 4 \,- & 4 & 3 & 2 & 1 \sim & 0 & 2 & 3 & 4 \,- & 4 & 3 \\
-4 & 3 & 2 & 1 \sim & 0 & 2 & 3 & 4 \,- & 4 & 3 & 2 & 1 \sim & 0 & 2 & 3 & 4 \,- & 4 \\
4 \,- & 4 & 3 & 2 & 1 \sim & 0 & 2 & 3 & 4 \,- & 4 & 3 & 2 & 1 \sim & 0 & 2 & 3 & 4 \,-
\end{array}$$

$D_5^{(2)}$: Rank 4, Period $\tilde{h}^\vee = 8$.

$$\begin{array}{cc|ccccccccccccccc}
2 & 1 & 0 & 1 & 2 & 3 & 4 & 3 & 2 & 1 & 0 & 1 & 2 & 3 & 4 & 3 & 2 \\
3 & 2 & 1 & 0 & 1 & 2 & 3 & 4 & 3 & 2 & 1 & 0 & 1 & 2 & 3 & 4 & 3 \\
4 & 3 & 2 & 1 & 0 & 1 & 2 & 3 & 4 & 3 & 2 & 1 & 0 & 1 & 2 & 3 & 4 \\
3 & 4 & 3 & 2 & 1 & 0 & 1 & 2 & 3 & 4 & 3 & 2 & 1 & 0 & 1 & 2 & 3
\end{array}$$

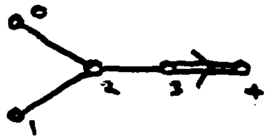
$A_4^{(1)}$:

2	1	0	4	3	2	1	0	4	3	2	1	0	4	3	2	1
3	2	1	0	4	3	2	1	0	4	3	2	1	0	4	3	2
4	3	2	1	0	4	3	2	1	0	4	3	2	1	0	4	3
0	4	3	2	1	0	4	3	2	1	0	4	3	2	1	0	4
1	0	4	3	2	1	0	4	3	2	1	0	4	3	2	1	0



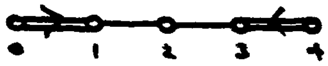
$B_4^{(1)}$:

2	1∼0		2	3	4	3	2	1∼0		2	3	4	3	2	1∼0	
3	2	1∼0		2	3	4	3	2	1∼0		2	3	4	3	2	1∼
4	3	2	1∼0		2	3	4	3	2	1∼0		2	3	4	3	2
3	4	3	2	1∼0		2	3	4	3	2	1∼0		2	3	4	3



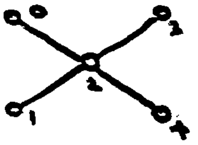
$C_4^{(1)}$:

2	1	0—	0	1	2	3	4—	4	3	2	1	0—	0	1	2	3
3	2	1	0—	0	1	2	3	4—	4	3	2	1	0—	0	1	2
4	3	2	1	0—	0	1	2	3	4—	4	3	2	1	0—	0	1
4	4	3	2	1	0—	0	1	2	3	4—	4	3	2	1	0—	0



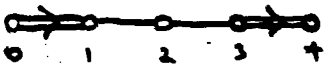
$D_4^{(1)}$:

2	1∼0		2	4∼3		2	1∼0		2	4∼3		2	1∼0		2	4∼
∼3	2	1∼0		2	4∼3		2	1∼0		2	4∼3		2	1∼0		2
4∼3		2	1∼0		2	4∼3		2	1∼0		2	4∼3		2	1∼0	
2	4∼3		2	1∼0		2	4∼3		2	1∼0		2	4∼3		2	1∼



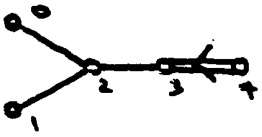
$A_8^{(2)}$:

2	1	0—	0	1	2	3	4	3	2	1	0—	0	1	2	3	4
3	2	1	0—	0	1	2	3	4	3	2	1	0—	0	1	2	3
4	3	2	1	0—	0	1	2	3	4	3	2	1	0—	0	1	2
3	4	3	2	1	0—	0	1	2	3	4	3	2	1	0—	0	1



$A_7^{(2)}$:

2	1∼0		2	3	4—	4	3	2	1∼0		2	3	4—	4	3	2
3	2	1∼0		2	3	4—	4	3	2	1∼0		2	3	4—	4	3
4	3	2	1∼0		2	3	4—	4	3	2	1∼0		2	3	4—	4
4	4	3	2	1∼0		2	3	4—	4	3	2	1∼0		2	3	4



$D_5^{(2)}$:

2	1	0	1	2	3	4	3	2	1	0	1	2	3	4	3	2
3	2	1	0	1	2	3	4	3	2	1	0	1	2	3	4	3
4	3	2	1	0	1	2	3	4	3	2	1	0	1	2	3	4
3	4	3	2	1	0	1	2	3	4	3	2	1	0	1	2	3

Coloured tableaux

- For each $\mathbf{g}$ and each $w \in W$ there exists a unique coloured tableaux $T(w)$ of shape $F(w)$ defined by a vector $\lambda(w) \in E^n$.

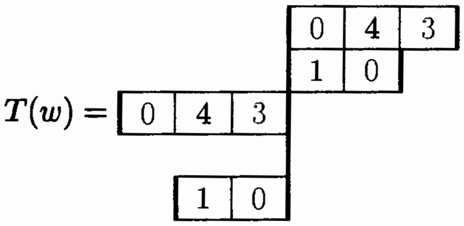
- The profile $P(w)$ of $T(w)$ is the set of all vertical edges that are:
  - (i) at the end of each row of $F(w)$, whether positive or negative;
  - (ii) on the vertical axis if the row length of $F(w)$ is zero,
  - (iii) or if the row length of $F(w)$ is one and the vertical axis splits an unordered pair.

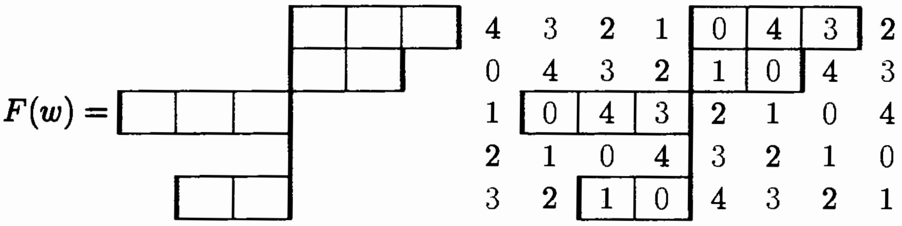
- A node is adjacent to the profile $P(w)$ of $T(w)$ if it is:
  - (i) adjacent to an edge in $P(w)$;
  - (ii) tied to a node that is adjacent to an edge in $P(w)$;
  - (iii) one of an unordered pair of nodes adjacent to an edge of $P(w)$.

- Given $w = w's_k$ then $T(w)$ is formed from $T(w')$ by adding or deleting all nodes of the underlying grid that are coloured k and that are adjacent to $P(w')$.

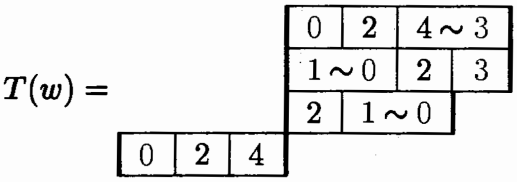
Coloured tableaux generated by Weyl group action

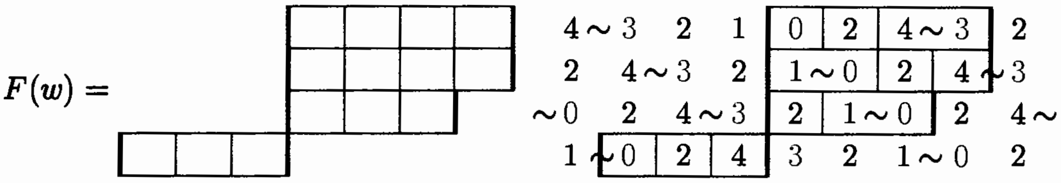
Ex 1 For $A_4^{(1)}$ and $w = s_0s_3s_4s_3s_1s_0$ we find $T(w)$ and $F(w)$ as below, with $\lambda(w) = (3, 2, -3, 0, -2)$.





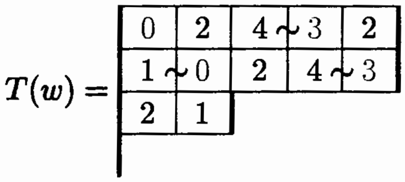
Ex 2 For $D_4^{(1)}$ and $w = s_0s_2s_1s_4s_2s_3s_0$ we find $T(w)$ and $F(w)$ as below, with $\lambda(w) = (4, 4, 3, -3)$.

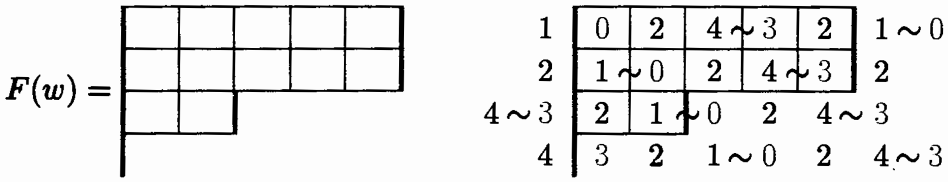




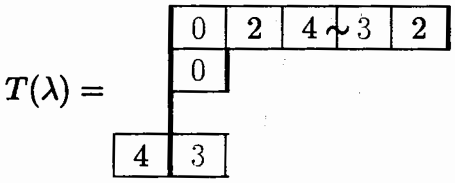
Coloured tableaux generated by Weyl group action

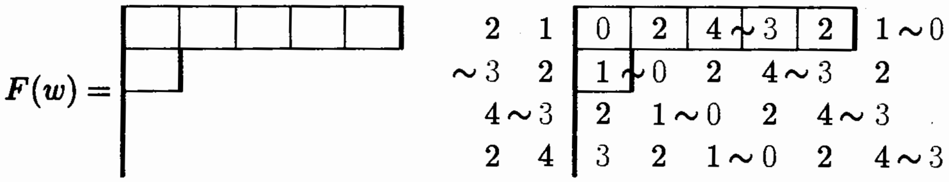
Ex 3 For $D_4^{(1)}$ and $w = s_0s_2s_1s_4s_3s_2s_4s_3$ we find $T(w)$ and $F(w)$ as below, with $\lambda(w) = (5,5,2,0)$.





Ex 4 For $D_4^{(1)}$ and $w = s_0s_2s_4s_3s_2$ we find $T(w)$ and $F(w)$ as below, with $\lambda(w) = (5,1,0,0)$.





Recursive construction of tableaux

Ex 5 For $C_4^{(1)}$ and $w' = s_0 s_1 s_2 s_3 s_4 s_0 s_1 s_0$ we have

$$T(w') = \begin{array}{|c|c|c|c|c|c|c|}
\hline
0 & 0 & 1 & 2 & 3 & 4 & 4 \\
\hline
1 & 0 & 0 & 1 \\
\cline{1-4}
2 & 1 & 0 & 0 \\
\cline{1-4}
3 \\
\cline{1-1}
\end{array} .$$

For $w = w's_k$ and $k = 0$ and $k = 3$ we obtain:

$$T(w's_0) = \begin{array}{|c|c|c|c|c|c|c|}
\hline
0 & 0 & 1 & 2 & 3 & 4 & 4 \\
\hline
1 & 0 & 0 & 1 \\
\cline{1-4}
2 & 1 \\
\cline{1-2}
3 \\
\cline{1-1}
\end{array} \qquad T(w's_3) = \begin{array}{|c|c|c|c|c|c|c|c|}
\hline
0 & 0 & 1 & 2 & 3 & 4 & 4 & 3 \\
\hline
1 & 0 & 0 & 1 \\
\cline{1-4}
2 & 1 & 0 & 0 \\
\cline{1-4}
\end{array}$$

Ex 6 For $\mathbf{g} = B_4^{(1)}$ and $w' = s_0 s_2 s_3 s_4 s_3 s_2$ we have

$$T(w') = \begin{array}{|c|c|c|c|c|c|}
\hline
0 & 2 & 3 & 4 & 3 & 2 \\
\hline
0 \\
\cline{1-1}
\end{array} .$$

Then we find first

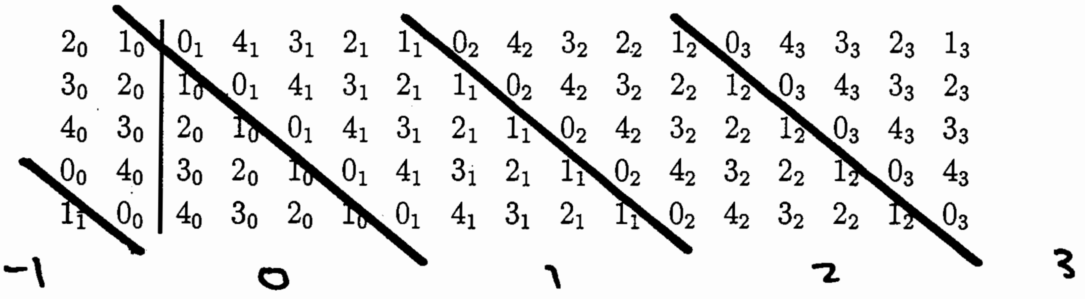
$$T(w's_1) = \begin{array}{|c|c|c|c|c|c|c|}
\hline
0 & 2 & 3 & 4 & 3 & 2 & 1 \\
\hline
\multicolumn{2}{|c|}{1 \sim 0} \\
\cline{1-2}
\end{array}$$

and then

$$T(w's_1s_0) = \begin{array}{|c|c|c|c|c|c|c|}
\hline
0 & 2 & 3 & 4 & 3 & 2 & 1 \sim 0 \\
\hline
1 \\
\cline{1-1}
\end{array} .$$

Grid with depth factors added

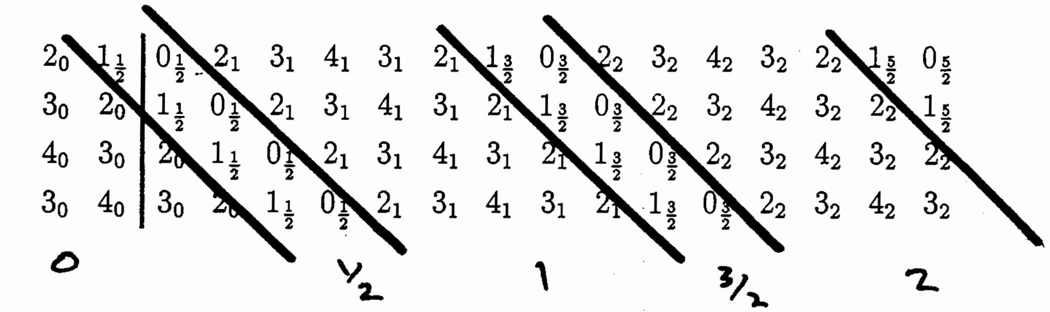
$A_4^{(1)}$



2_0	1_0	0_1	4_1	3_1	2_1	1_1	0_2	4_2	3_2	2_2	1_2	0_3	4_3	3_3	2_3	1_3
3_0	2_0	1_0	0_1	4_1	3_1	2_1	1_1	0_2	4_2	3_2	2_2	1_2	0_3	4_3	3_3	2_3
4_0	3_0	2_0	1_0	0_1	4_1	3_1	2_1	1_1	0_2	4_2	3_2	2_2	1_2	0_3	4_3	3_3
0_0	4_0	3_0	2_0	1_0	0_1	4_1	3_1	2_1	1_1	0_2	4_2	3_2	2_2	1_2	0_3	4_3
1_1	0_0	4_0	3_0	2_0	1_0	0_1	4_1	3_1	2_1	1_1	0_2	4_2	3_2	2_2	1_2	0_3

-1 0 1 2 3

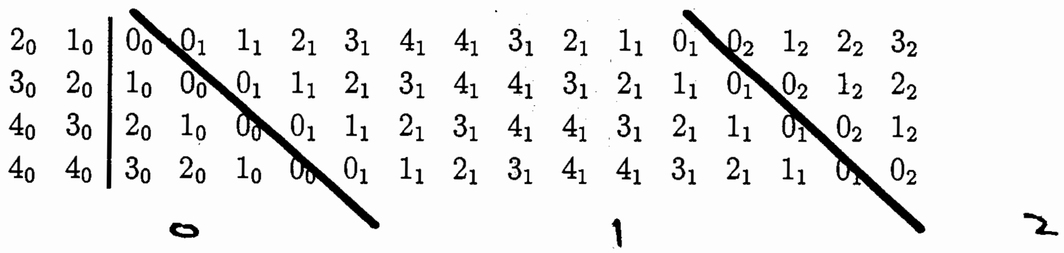
$B_4^{(1)}$:



2_0	$1_{\frac{1}{2}}$	$0_{\frac{1}{2}}$	2_1	3_1	4_1	3_1	2_1	$1_{\frac{3}{2}}$	$0_{\frac{3}{2}}$	2_2	3_2	4_2	3_2	2_2	$1_{\frac{5}{2}}$	$0_{\frac{5}{2}}$
3_0	2_0	$1_{\frac{1}{2}}$	$0_{\frac{1}{2}}$	2_1	3_1	4_1	3_1	2_1	$1_{\frac{3}{2}}$	$0_{\frac{3}{2}}$	2_2	3_2	4_2	3_2	2_2	$1_{\frac{5}{2}}$
4_0	3_0	2_0	$1_{\frac{1}{2}}$	$0_{\frac{1}{2}}$	2_1	3_1	4_1	3_1	2_1	$1_{\frac{3}{2}}$	$0_{\frac{3}{2}}$	2_2	3_2	4_2	3_2	2_2
3_0	4_0	3_0	2_0	$1_{\frac{1}{2}}$	$0_{\frac{1}{2}}$	2_1	3_1	4_1	3_1	2_1	$1_{\frac{3}{2}}$	$0_{\frac{3}{2}}$	2_2	3_2	4_2	3_2

0 1/2 1 3/2 2

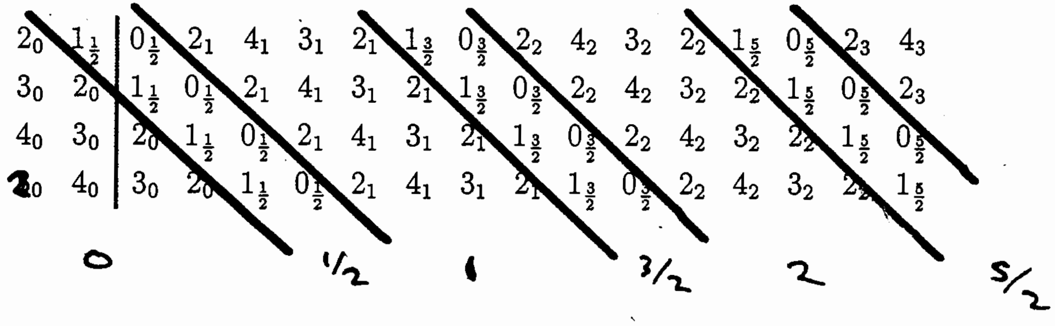
$C_4^{(1)}$:



2_0	1_0	0_0	0_1	1_1	2_1	3_1	4_1	4_1	3_1	2_1	1_1	0_1	0_2	1_2	2_2	3_2
3_0	2_0	1_0	0_0	0_1	1_1	2_1	3_1	4_1	4_1	3_1	2_1	1_1	0_1	0_2	1_2	2_2
4_0	3_0	2_0	1_0	0_0	0_1	1_1	2_1	3_1	4_1	4_1	3_1	2_1	1_1	0_1	0_2	1_2
4_0	4_0	3_0	2_0	1_0	0_0	0_1	1_1	2_1	3_1	4_1	4_1	3_1	2_1	1_1	0_1	0_2

0 1 2

$D_4^{(1)}$:



2_0	$1_{\frac{1}{2}}$	$0_{\frac{1}{2}}$	2_1	4_1	3_1	2_1	$1_{\frac{3}{2}}$	$0_{\frac{3}{2}}$	2_2	4_2	3_2	2_2	$1_{\frac{5}{2}}$	$0_{\frac{5}{2}}$	2_3	4_3
3_0	2_0	$1_{\frac{1}{2}}$	$0_{\frac{1}{2}}$	2_1	4_1	3_1	2_1	$1_{\frac{3}{2}}$	$0_{\frac{3}{2}}$	2_2	4_2	3_2	2_2	$1_{\frac{5}{2}}$	$0_{\frac{5}{2}}$	2_3
4_0	3_0	2_0	$1_{\frac{1}{2}}$	$0_{\frac{1}{2}}$	2_1	4_1	3_1	2_1	$1_{\frac{3}{2}}$	$0_{\frac{3}{2}}$	2_2	4_2	3_2	2_2	$1_{\frac{5}{2}}$	$0_{\frac{5}{2}}$
2_0	4_0	3_0	2_0	$1_{\frac{1}{2}}$	$0_{\frac{1}{2}}$	2_1	4_1	3_1	2_1	$1_{\frac{3}{2}}$	$0_{\frac{3}{2}}$	2_2	4_2	3_2	2_2	$1_{\frac{5}{2}}$

0 1/2 1 3/2 2 5/2

Grids with depth factors added

$A_8^{(2)}$:

$$\begin{array}{cc|ccccccccccccccc}
2_0 & 1_0 & 0_0 & 0_1 & 1_1 & 2_1 & 3_1 & 4_1 & 3_1 & 2_1 & 1_1 & 0_1 & 0_2 & 1_2 & 2_2 & 3_2 & 4_2 \\
3_0 & 2_0 & 1_0 & 0_0 & 0_1 & 1_1 & 2_1 & 3_1 & 4_1 & 3_1 & 2_1 & 1_1 & 0_1 & 0_2 & 1_2 & 2_2 & 3_2 \\
4_0 & 3_0 & 2_0 & 1_0 & 0_0 & 0_1 & 1_1 & 2_1 & 3_1 & 4_1 & 3_1 & 2_1 & 1_1 & 0_1 & 0_2 & 1_2 & 2_2 \\
3_0 & 4_0 & 3_0 & 2_0 & 1_0 & 0_0 & 0_1 & 1_1 & 2_1 & 3_1 & 4_1 & 3_1 & 2_1 & 1_1 & 0_1 & 0_2 & 1_2
\end{array}$$

0 1 2

$A_7^{(2)}$

$$\begin{array}{cc|ccccccccccccccc}
2_0 & 1_{\frac{1}{2}} & 0_{\frac{1}{2}} & 2_1 & 3_1 & 4_1 & 4_1 & 3_1 & 2_1 & 1_{\frac{3}{2}} & 0_{\frac{3}{2}} & 2_2 & 3_2 & 4_2 & 4_2 & 3_2 & 2_2 \\
3_0 & 2_0 & 1_{\frac{1}{2}} & 0_{\frac{1}{2}} & 2_1 & 3_1 & 4_1 & 4_1 & 3_1 & 2_1 & 1_{\frac{3}{2}} & 0_{\frac{3}{2}} & 2_2 & 3_2 & 4_2 & 4_2 & 3_2 \\
4_0 & 3_0 & 2_0 & 1_{\frac{1}{2}} & 0_{\frac{1}{2}} & 2_1 & 3_1 & 4_1 & 4_1 & 3_1 & 2_1 & 1_{\frac{3}{2}} & 0_{\frac{3}{2}} & 2_2 & 3_2 & 4_2 & 4_2 \\
\cancel{3}4_0 & 4_0 & 3_0 & 2_0 & 1_{\frac{1}{2}} & 0_{\frac{1}{2}} & 2_1 & 3_1 & 4_1 & 4_1 & 3_1 & 2_1 & 1_{\frac{3}{2}} & 0_{\frac{3}{2}} & 2_2 & 3_2 & 4_2
\end{array}$$

0 1/2 1 3/2 2

$D_5^{(2)}$:

$$\begin{array}{cc|ccccccccccccccc}
2_0 & 1_0 & 0_1 & 1_2 & 2_2 & 3_2 & 4_2 & 3_2 & 2_2 & 1_2 & 0_3 & 1_4 & 2_4 & 3_4 & 4_4 & 3_4 & 2_4 \\
3_0 & 2_0 & 1_0 & 0_1 & 1_2 & 2_2 & 3_2 & 4_2 & 3_2 & 2_2 & 1_2 & 0_3 & 1_4 & 2_4 & 3_4 & 4_4 & 3_4 \\
4_0 & 3_0 & 2_0 & 1_0 & 0_1 & 1_2 & 2_2 & 3_2 & 4_2 & 3_2 & 2_2 & 1_2 & 0_3 & 1_4 & 2_4 & 3_4 & 4_4 \\
3_0 & 4_0 & 3_0 & 2_0 & 1_0 & 0_1 & 1_2 & 2_2 & 3_2 & 4_2 & 3_2 & 2_1 & 1_2 & 0_3 & 1_4 & 2_4 & 3_4
\end{array}$$

0 1 2 3 4

Coset decomposition of affine Weyl groups

- Affine Weyl group $W = \langle s_0, s_1, \ldots, s_\ell \rangle$
- Finite Weyl group $\overline{W} = \langle s_1, s_2, \ldots, s_\ell \rangle$
- Let W_s be the set of all minimal right coset representatives of W with respect to $\overline{W}$, then

$$W_s = \{ w \in W \mid \ell(\overline{w}\, w) > \ell(w) \text{ for all } \overline{w} \in \overline{W} \}$$

$$= \{ w \in W \mid \overline{w(\rho) - \rho} \in \overline{P}^+ \}$$

$$= \{ w \in W \mid \overline{w(\lambda + \rho) - \rho} \in \overline{P}^+ \text{ for all } \lambda \in P^+ \}$$

Bruhat graph

- Vertices: w for all $w \in W_s$
- Directed edges: $w' \longrightarrow w$ labelled by s_k for $w, w' \in W_s$ and $k \in I$ if and only if $w = w'\, s_k$ with $\ell(w) = \ell(w') + 1$

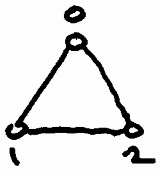
Grading with respect to length

$$W_s = \cup_{d=0}^{\infty} W_s^{(d)} \quad \text{where} \quad W_s^{(d)} = \{ w \in W_s \mid \ell(w) = d \}$$

Lemma [Kang] Given $w' \in W_s^{(d)}$ and $k \in I$ then $w = w'\, s_k \in W_s^{(d+1)}$ if and only if $w'(\alpha_k) \in \Delta^+ \backslash \overline{\Delta}^+$.

Poincare polynomial $P = \sum_{w \in W_s} q^{\ell(w)}$

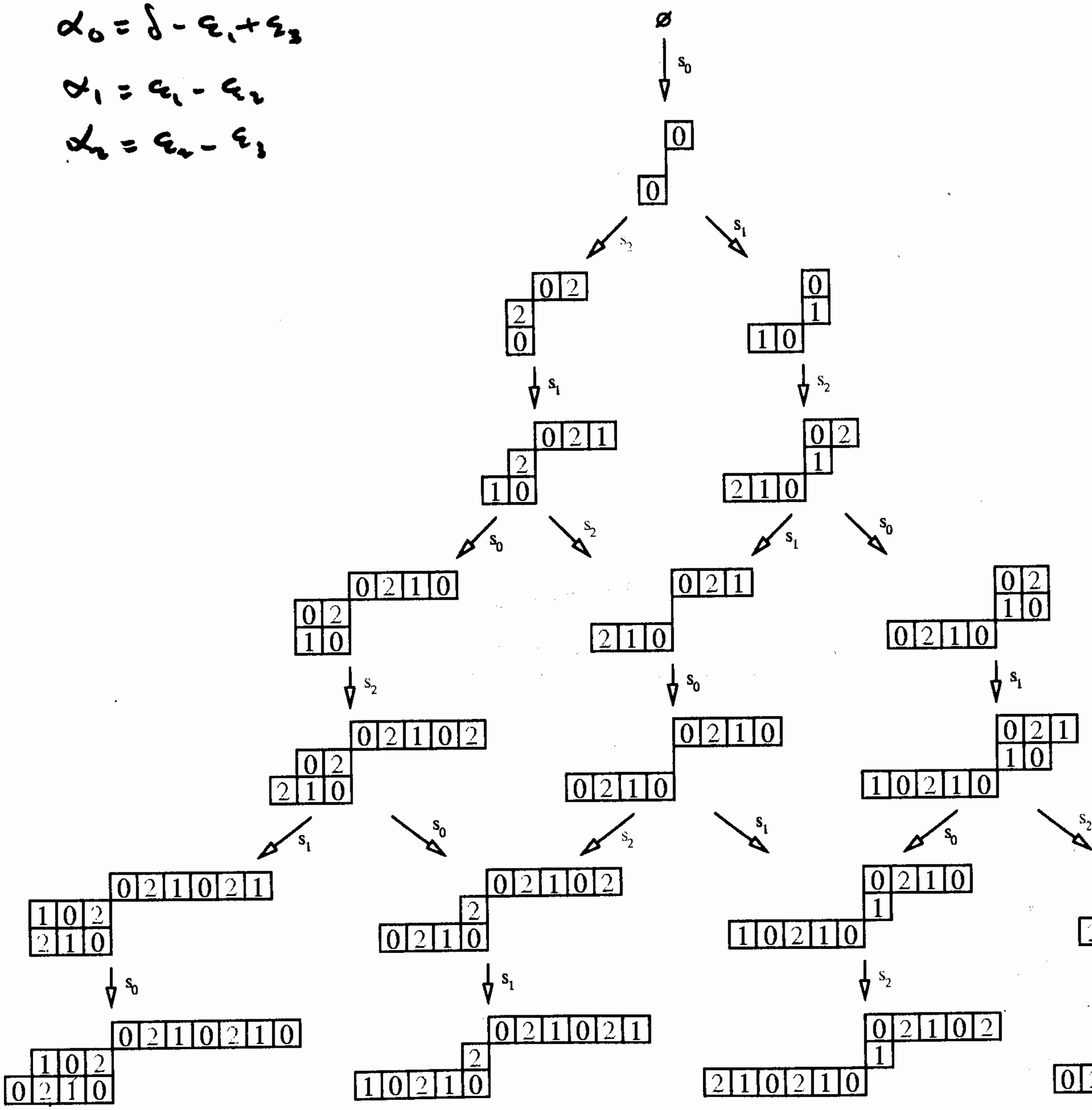
$A_2^{(1)}$



$\alpha_0 = \delta - \varepsilon_1 + \varepsilon_3$

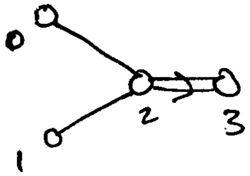
$\alpha_1 = \varepsilon_1 - \varepsilon_2$

$\alpha_2 = \varepsilon_2 - \varepsilon_3$



$$P(q) = 1 + q + 2q^2 + 2q^3 + 3q^4 + 3q^5 + \dots = \prod_{k=1}^{2} (1-q^k)^{-1}$$

$B_3^{(1)}$

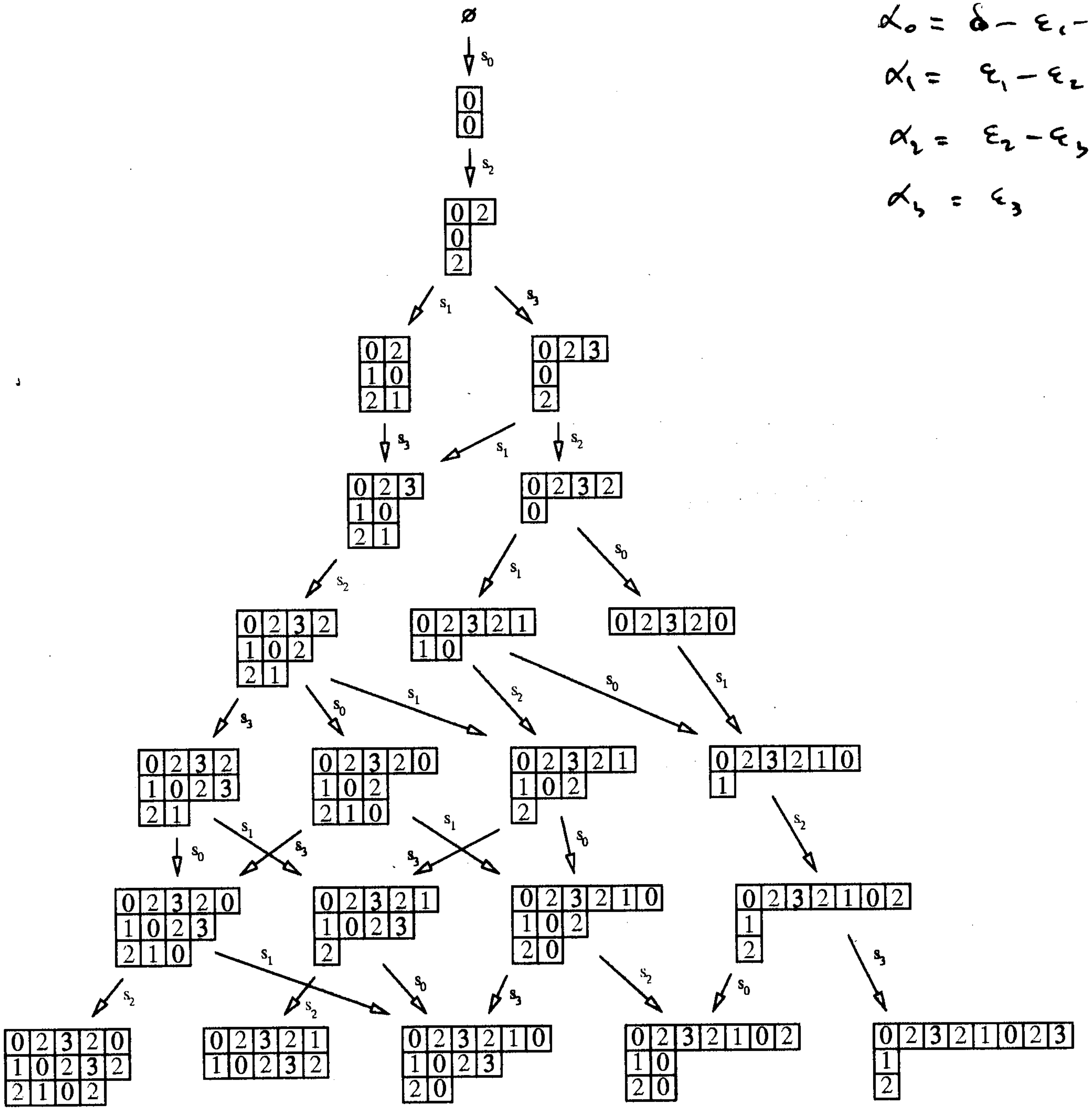


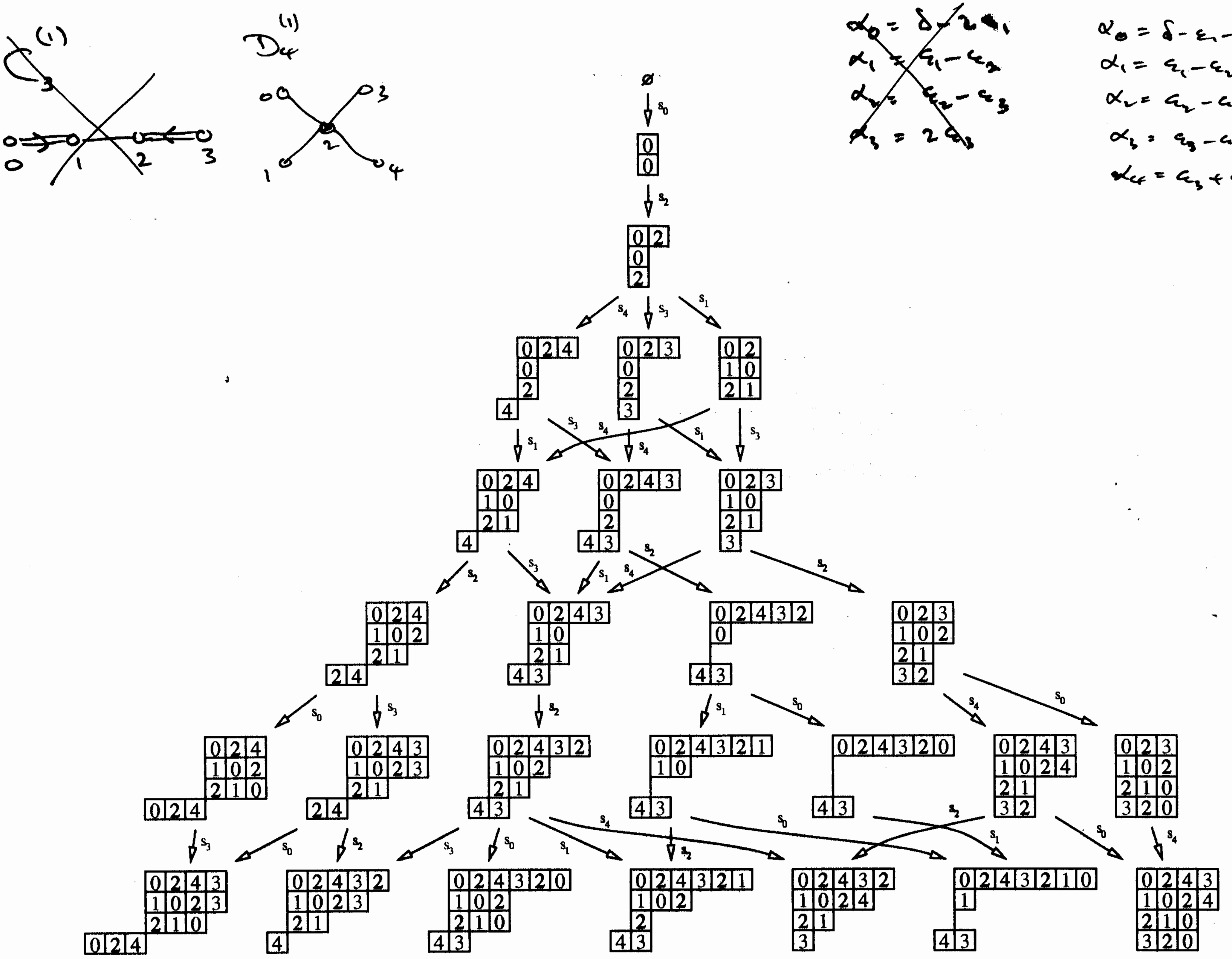
$\alpha_0 = \delta - \varepsilon_1 - \varepsilon_2$

$\alpha_1 = \varepsilon_1 - \varepsilon_2$

$\alpha_2 = \varepsilon_2 - \varepsilon_3$

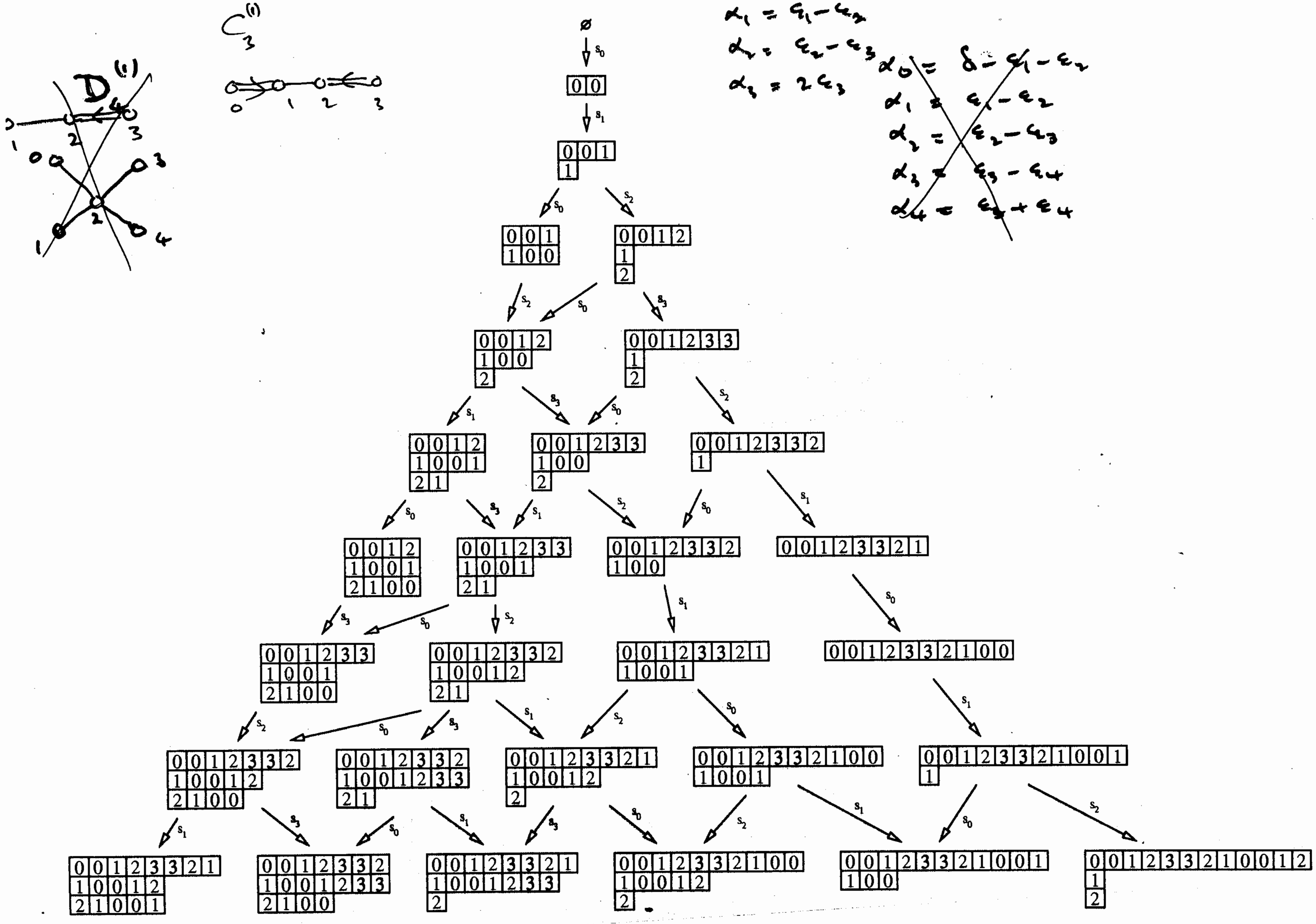
$\alpha_3 = \varepsilon_3$





~~$\alpha_0 = \delta - 2\varepsilon_1$~~
~~$\alpha_1 = \varepsilon_1 - \varepsilon_2$~~
~~$\alpha_2 = \varepsilon_2 - \varepsilon_3$~~
~~$\alpha_3 = 2\varepsilon_3$~~

$\alpha_0 = \delta - \varepsilon_1 - \varepsilon_2$
$\alpha_1 = \varepsilon_1 - \varepsilon_2$
$\alpha_2 = \varepsilon_2 - \varepsilon_3$
$\alpha_3 = \varepsilon_3 - \varepsilon_4$
$\alpha_4 = \varepsilon_3 + \varepsilon_4$



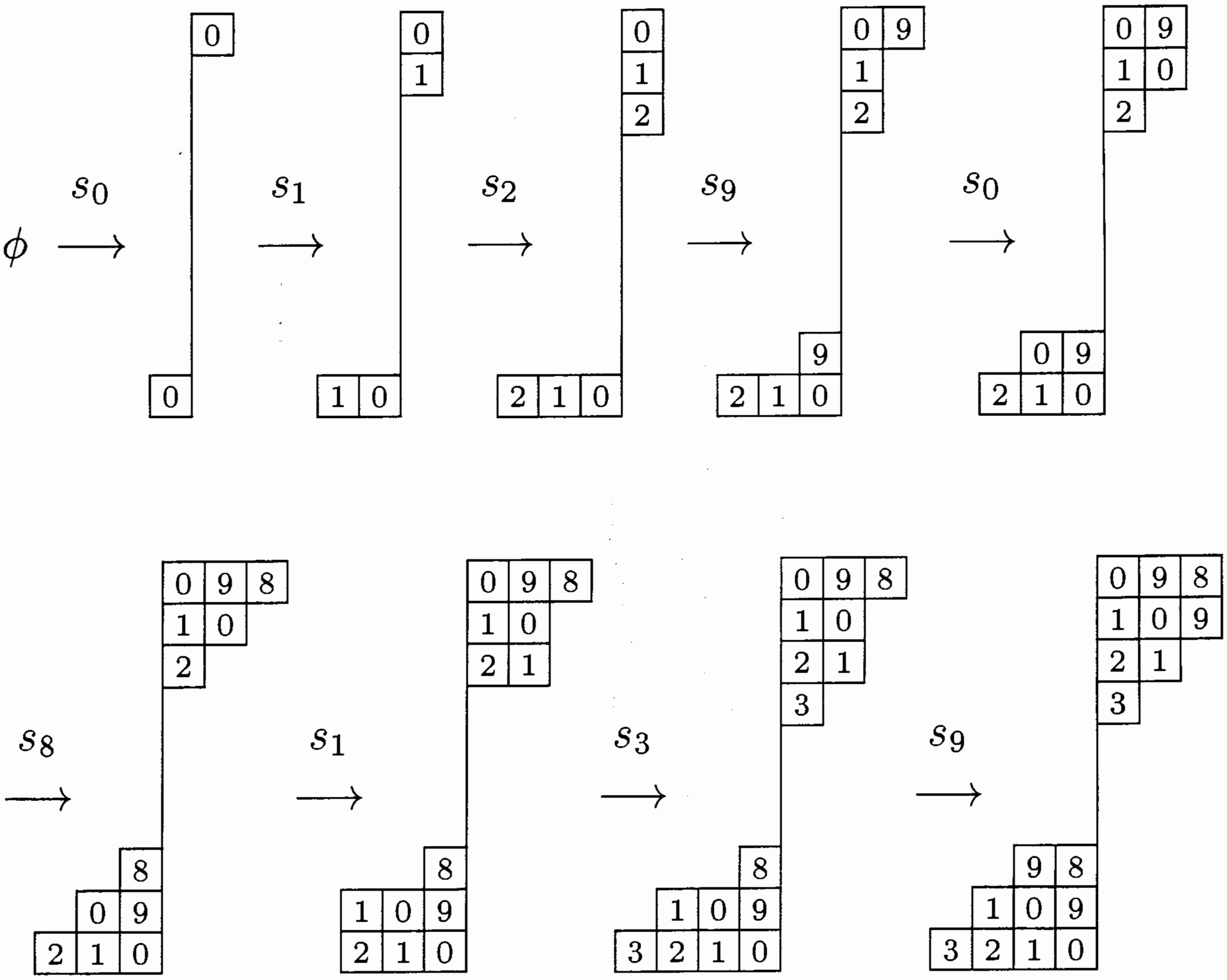
$\alpha_0 = \delta - 2\varepsilon_1$
$\alpha_1 = \varepsilon_1 - \varepsilon_2$
$\alpha_2 = \varepsilon_2 - \varepsilon_3$
$\alpha_3 = 2\varepsilon_3$

~~$\alpha_0 = \delta - \varepsilon_1 - \varepsilon_2$~~
~~$\alpha_1 = \varepsilon_1 - \varepsilon_2$~~
~~$\alpha_2 = \varepsilon_2 - \varepsilon_3$~~
~~$\alpha_3 = \varepsilon_3 - \varepsilon_4$~~
~~$\alpha_4 = \varepsilon_3 + \varepsilon_4$~~

Generation of the numbering scheme for $A_\ell^{(1)}$

Ex. For $A_\ell^{(1)}$ with $\ell = 9$ and $w = s_0 s_1 s_2 s_9 s_0 s_8 s_1 s_3 s_9$, proceeding recursively, we find

$$w(\rho) - \rho = 3\epsilon_1 + 3\epsilon_2 + 2\epsilon_3 + \epsilon_4 - 2\epsilon_8 - 3\epsilon_9 - 4\epsilon_{10} - 9\delta.$$



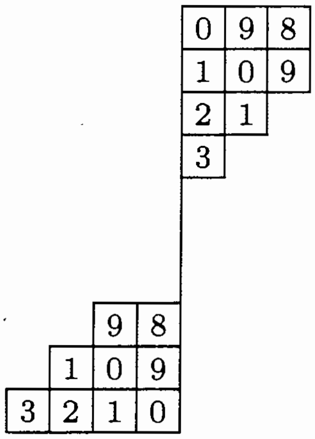
Note $w = w_\zeta$ with $\zeta = (3,3,2,1) \in \mathcal{P}$, $\zeta' = (4,3,2)$ and $|\zeta| = 9$.

Illustration for $A_\ell^{(1)}$ in the case $\ell = 9$

Let $w_\zeta = s_0 s_1 s_2 s_9 s_8 s_7 s_0 s_1 s_9$ then $\zeta = \left(\begin{smallmatrix} 2 & 1 \\ 3 & 1 \end{smallmatrix}\right) = (3321) \in \mathcal{F}$,

$w_\zeta(\rho) - \rho = 3\epsilon_1 + 3\epsilon_4 + 2\epsilon_3 + \epsilon_4 - 2\epsilon_8 - 3\epsilon_9 - 4\epsilon_{10} - 9\delta$ and

$F^{\zeta;\overline{\zeta'}} =$



$$M_{w_\zeta} = e^{-9\delta}\ \mathrm{ch}\, \overline{V}^{\{3321;\overline{234}\}}.$$

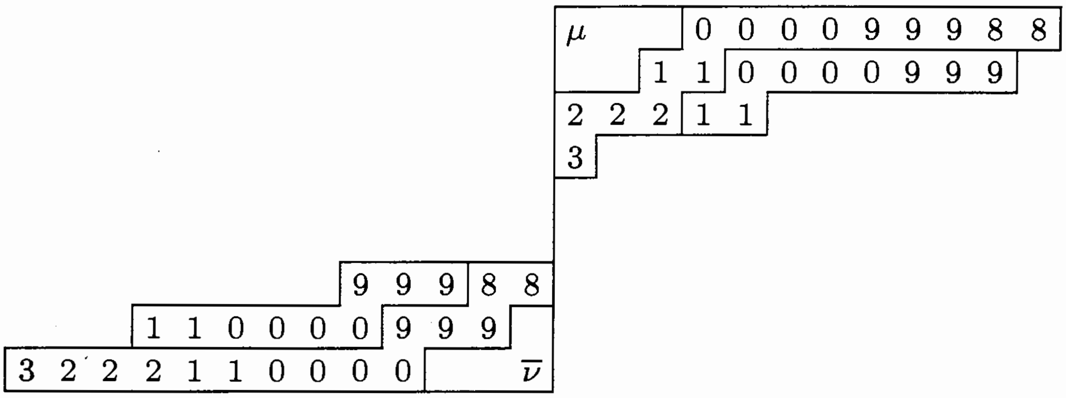
To obtain the corresponding contribution, $M^\lambda_{w_\zeta}$, to the numerator expansion first note that:

$$w_\zeta(\lambda + \rho) - \rho = \lambda + (w_\zeta(\rho) - \rho) + (w_\zeta(\lambda) - \lambda).$$

Thus we add three contributions: one from λ itself corresponding in general to a composite Young diagram of the form $F^{\mu;\overline{\nu}}$, one obtained, as above, from the row lengths of $F^{\zeta;\overline{\zeta'}}$, and one obtained by stretching the rows of $F^{\zeta;\overline{\zeta'}}$ by factors of $\lambda_{\eta_{ij}}$.

Example

If $\lambda = 3\Lambda_0 + \Lambda_1 + 2\Lambda_2 + \Lambda_8 + 2\Lambda_9 = 9\Lambda_0 + 3\epsilon_1 + 2\epsilon_2 - \epsilon_9 - 3\epsilon_{10}$, then the complete stretched diagram takes the form:

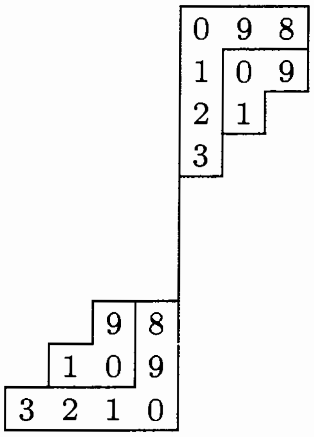


and

$$M_{w_\zeta}^{\lambda} = e^{9\Lambda_0}\, e^{-24\delta}\, \mathrm{ch}\, \overline{V}^{\{12,11,5,1;\overline{5,10,13}\}} .$$

For ease of comparison with the unstretched case, we note:

$F^{\zeta;\overline{\zeta'}} =$



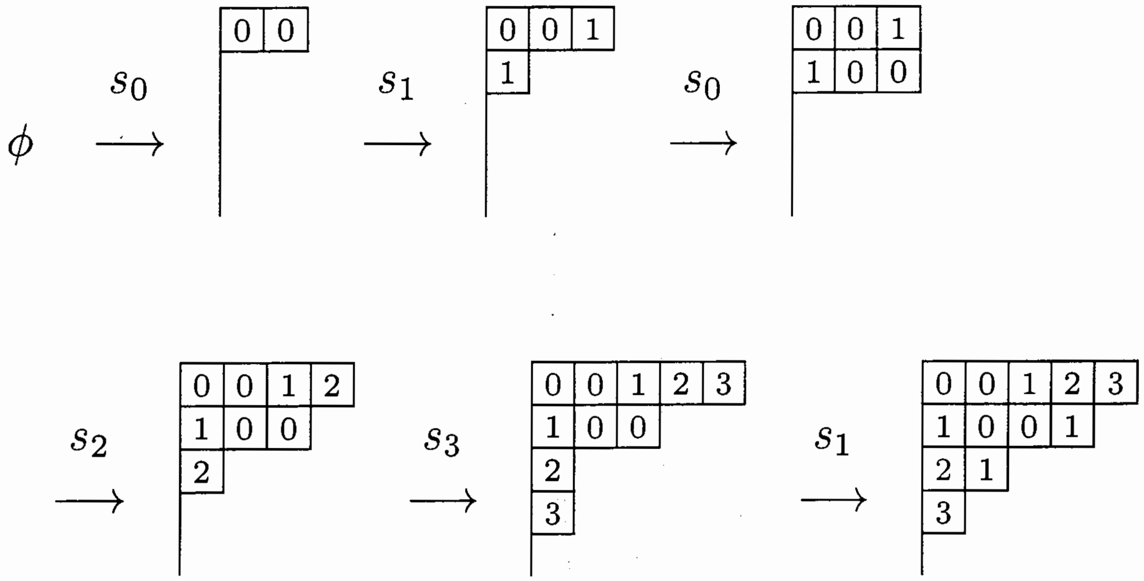
$$M_{w_\zeta} = e^{-9\delta}\, \mathrm{ch}\, \overline{V}^{\{3321;\overline{234}\}} .$$

Note the preservation of the nested hook or wrapped strip structure.

Generation of the numbering scheme for $C_\ell^{(1)}$

Ex. For $C_\ell^{(1)}$ with $\ell \geq 4$ and $w = s_0 s_1 s_0 s_2 s_3 s_1$, proceeding recursively, we find

$$w(\rho) - \rho = 5\epsilon_1 + 4\epsilon_2 + 2\epsilon_3 + \epsilon_4 - 6\delta.$$



Note $w = w_\gamma$ with $\gamma = (5, 4, 2, 1) = \left(\begin{smallmatrix} 4 & 2 \\ 3 & 1 \end{smallmatrix}\right) \in \mathcal{C}$ and $|\gamma|/2 = 6$.

Illustration for $C_\ell^{(1)}$

Let $w_\gamma = s_0 s_1 s_2 s_3 s_0 s_1$ then $\gamma = \binom{4\,2}{3\,1} = (5421) \in \mathcal{C}$, $w_\gamma(\rho) - \rho = 5\epsilon_1 + 4\epsilon_2 + 2\epsilon_3 + \epsilon_4$ and

$$F^\gamma = \begin{array}{|c|c|c|c|c|} \hline 0 & 0 & 1 & 2 & 3 \\ \hline 1 & 0 & 0 & 1 \\ \cline{1-4} 2 & 1 \\ \cline{1-2} 3 \\ \cline{1-1} \end{array} \qquad M_{w_\gamma} = e^{-6\delta}\ \mathrm{ch}\, \overline{V}^{\langle 5421 \rangle}.$$

To obtain the corresponding contribution, $M^\lambda_{w_\gamma}$, to the numerator expansion first note that:

$$w_\gamma(\lambda + \rho) - \rho = \lambda + (w_\gamma(\rho) - \rho) + (w_\gamma(\lambda) - \lambda).$$

Thus we add three contributions: one from λ itself, one obtained, as above, from the row lengths of F^γ, and one obtained by stretching the rows of F^λ by factors of $\lambda_{\eta_{ij}}$.

Example If $\lambda = 2\Lambda_0 + 3\Lambda_1 + 2\Lambda_3 = 7\Lambda_0 + 5\epsilon_1 + 2\epsilon_2 + 2\epsilon_3$, then the complete stretched diagram takes the form:

$$\begin{array}{ccccccccccccccccccc} \lambda & & & & & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 2 & 3 & 3 & 3 \\ & & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ & & 2 & 1 & 1 & 1 & 1 \\ 3 & 3 & 3 \end{array}$$

and

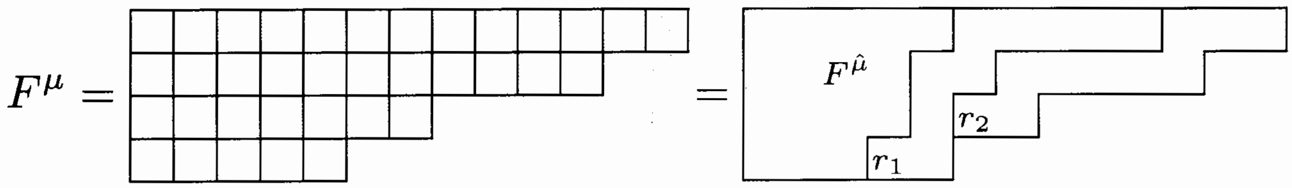
$$M^\lambda_{w_\gamma} = e^{7\Lambda_0}\ e^{-18\delta}\ \mathrm{ch}\, \overline{V}^{\langle 19,16,7,3 \rangle}.$$

Core diagrams and boundary strips

- The diagrams $F(w)$ that are generated by $w \in W_s$ consist of cores supplemented in all possible ways by strips of length $L = \tilde{h}^\vee$ starting in the first row.

Definition $\mu \equiv \hat{\mu}(\mathrm{mod}\, L)$ if and only if $F^{\hat{\mu}}$ is obtained from F^μ by the removal of all possible, say s, continuous boundary strips each of length L starting in the first row of F^μ.

Example For $L = 10$ and $\mu = (13, 11, 7, 5)$ we have $s = 2$ and $\hat{\mu} = (5, 4, 4, 3)$:

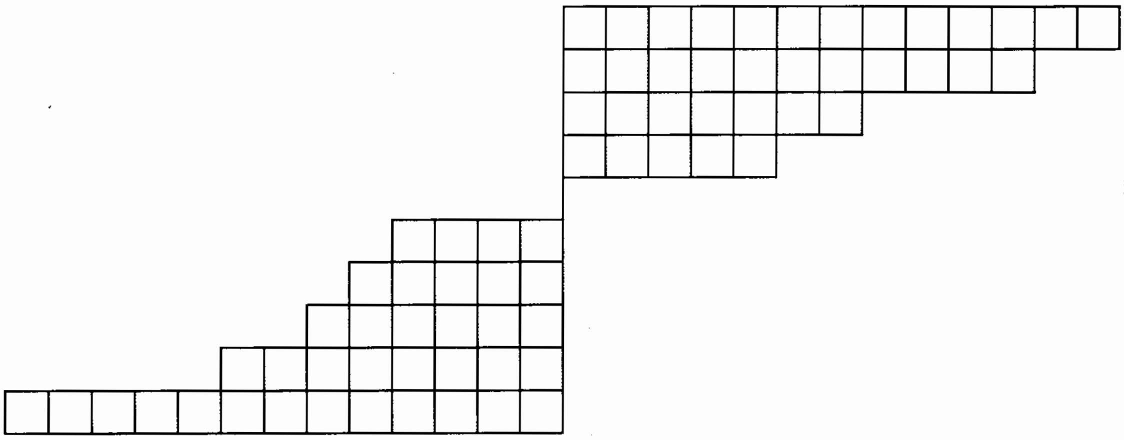


Core composite diagrams and pairs of boundary strips

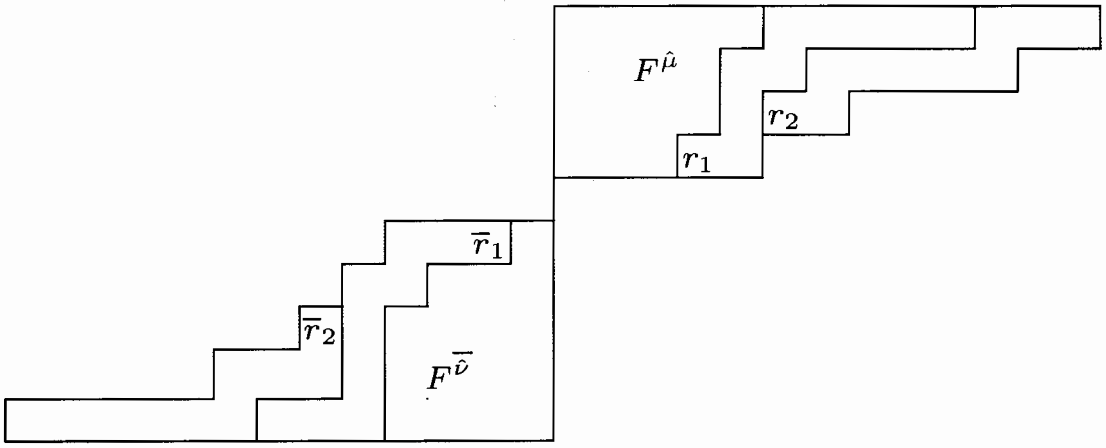
Definition $\mu;\overline{\nu} \equiv \hat{\mu};\overline{\hat{\nu}} (\mathrm{mod}\, L)$ if and only if $\mu \equiv \hat{\mu} (\mathrm{mod}\, L)$ and $\nu \equiv \hat{\nu} (\mathrm{mod}\, L)$ with $|\mu| - |\hat{\mu}| = |\nu| - |\hat{\nu}| = sL$ for some $s \in \boldsymbol{Z}^+$.

Example For $L = 10$ and $\mu;\overline{\nu} = (13,11,7,5;\overline{4,5,6,8,13})$ we have $s = 2$ and $\hat{\mu};\overline{\hat{\nu}} = (5,4,4,3;\overline{1,3,4,4,4})$:

$F^{\mu;\overline{\nu}} =$



$=$



Proposition For each $\mathbf{g} \supset \overline{\mathbf{g}}$ indexed by the rank ℓ, the denominator expansion takes the form:

$$A_\ell^{(1)} \supset A_\ell\colon\ M = \sum_{\substack{\mu;\overline{\nu} \equiv \zeta;\overline{\zeta}' \bmod (\ell+1),\, \zeta\in\mathcal{F} \\ \ell(\mu)+\ell(\nu),\, \ell(\zeta)+\ell(\zeta') \leq \ell+1}} (-1)^{|\zeta|+r+\overline{r}}\, e^{-d(\{\mu;\overline{\nu}\})\delta}\, \mathrm{ch}\, \overline{V}^{\{\mu;\overline{\nu}\}};$$

$$B_\ell^{(1)} \supset B_\ell\colon\ M = \sum_{\substack{\mu\equiv\alpha \bmod (2\ell-1),\, \alpha\in\mathcal{A} \\ \ell(\mu),\, \ell(\alpha)\leq\ell}} (-1)^{|\alpha|/2+r}\, e^{-d([\mu])\delta}\, \mathrm{ch}\, \overline{V}^{[\mu]};$$

$$C_\ell^{(1)} \supset C_\ell\colon\ M = \sum_{\substack{\mu\equiv\gamma \bmod (2\ell+2),\, \gamma\in\mathcal{C} \\ \ell(\mu),\, \ell(\gamma)\leq\ell}} (-1)^{|\gamma|/2+r+s}\, e^{-d(\langle\mu\rangle)\delta}\, \mathrm{ch}\, \overline{V}^{\langle\mu\rangle};$$

$$D_\ell^{(1)} \supset D_\ell\colon\ M = \sum_{\substack{\mu\equiv\alpha \bmod (2\ell-2),\, \alpha\in\mathcal{A} \\ \ell(\mu),\, \ell(\alpha)\leq\ell}} (-1)^{|\alpha|/2+r}\, e^{-d([\mu])\delta}\, \mathrm{ch}\, \overline{V}^{[\mu]};$$

$$A_{2\ell}^{(2)} \supset B_\ell\colon\ M = \sum_{\substack{\mu\equiv\gamma \bmod (2\ell+1),\, \gamma\in\mathcal{C} \\ \ell(\mu),\, \ell(\gamma)\leq\ell}} (-1)^{|\gamma|/2+r}\, e^{-d([\mu])\delta}\, \mathrm{ch}\, \overline{V}^{[\mu]};$$

$$A_{2\ell-1}^{(2)} \supset C_\ell\colon\ M = \sum_{\substack{\mu\equiv\alpha \bmod (2\ell),\, \alpha\in\mathcal{A} \\ \ell(\mu),\, \ell(\alpha)\leq\ell}} (-1)^{|\alpha|/2+r+s}\, e^{-d(\langle\mu\rangle)\delta}\, \mathrm{ch}\, \overline{V}^{\langle\mu\rangle};$$

$$D_{\ell+1}^{(2)} \supset B_\ell\colon\ M = \sum_{\substack{\mu\equiv\varepsilon \bmod (2\ell),\, \varepsilon\in\mathcal{E} \\ \ell(\mu),\, \ell(\varepsilon)\leq\ell}} (-1)^{(|\varepsilon|+p)/2+r}\, e^{-d([\mu])\delta}\, \mathrm{ch}\, \overline{V}^{[\mu]}.$$

The numerator expansions M^λ are determined by using the periodic grids to identify $k = \eta_{ij}$ for each (i,j) in the relevant diagram and scaling the contribution by $n_k(\lambda)$.

Example In the case of $A_9^{(1)} \supset A_9$ and

$$w = s_0 s_1 s_2 s_3 s_4 s_5 s_6 s_9 s_8 s_7 s_6 s_5 s_4 s_3 \cdot s_0 s_1 s_2 s_3 s_4 s_9 s_8 s_7 s_6 s_5 s_4$$

$$\cdot\, s_0 s_1 s_2 s_3 s_9 s_8 s_7 s_6 \cdot s_0 s_1 s_2 s_9 s_8 \cdot s_0 s_1 s_9$$

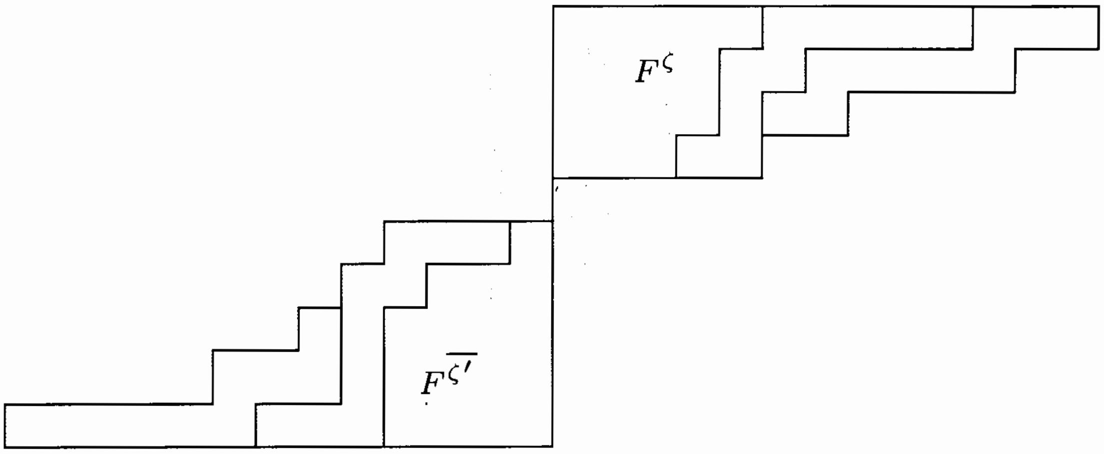
we have $\ell(w) = 41$ and

$$w \cdot 0 = w(\rho) - \rho = \{\mu; \overline{\nu}\} - d(w \cdot 0)\delta$$

$$= 13\epsilon_1 + 11\epsilon_2 + 7\epsilon_3 + 5\epsilon_4 - 4\epsilon_6 - 5\epsilon_7 - 6\epsilon_8 - 8\epsilon_9 - 13\epsilon_{10} - 57\delta.$$

Correspondingly, $\mu = (13, 11, 7, 5)$ and $\nu = (13, 8, 6, 5, 4)$, and $F^{\mu;\overline{\nu}}$ consists of the core $F^{\zeta;\overline{\zeta'}}$, with $\zeta = (5, 4, 4, 3)$, together with two pairs of continuous strips each of length $L = \ell + 1 = 10$.

$$F^{\mu;\overline{\nu}} =$$

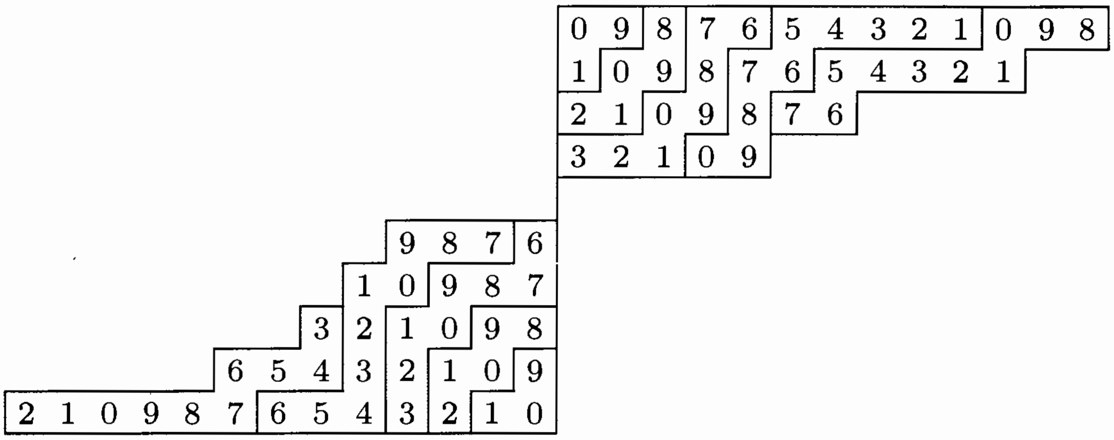


and the contribution to the denominator expansion is given by:

$$M_w = -e^{-57\delta}\ \mathrm{ch}\, \overline{V}^{\{13,11,7,5;\overline{4,5,6,8,13}\}}.$$

Numbering of diagrams consisting of core plus strips

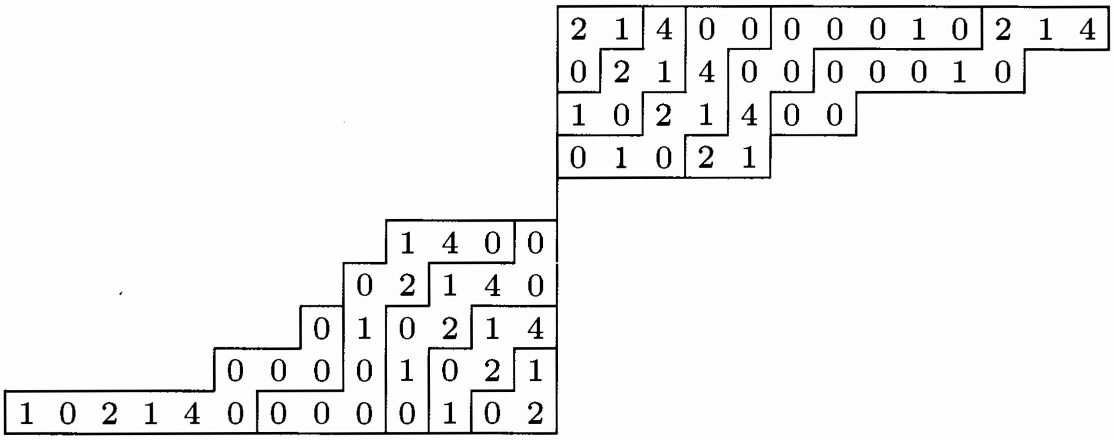
- In the case of $A_\ell^{(1)}$ the boxes of $F^{\mu;\overline{\nu}}$ are numbered by placing an entry $\eta_{ij} = (i-j) \bmod L$, with $L = \ell + 1$, in the box in the ith row and jth column. Typically, for $\ell = 9$:



- Rather than carrying out the stretching (which would give diagrams that are too wide to fit on the page) the above numbering of $F^{\{\mu;\overline{\nu}\}}$ can be used for any given dominant integral $\lambda \in \mathbf{h}^*$ to obtain $w(\lambda) - \lambda$ by introducing a new numbering in which each entry $\eta_{ij} = k$ is replaced by the stretching parameter $n_k(\lambda)$.

Calculation of $w(\lambda) - \lambda$

Example In our case of $A_\ell^{(1)}$ with $\ell = 9$ and w as before with $\ell(w) = 41$, if $\lambda = 2\Lambda_0 + \Lambda_2 + 4\Lambda_8 + \Lambda_9 = 8\Lambda_0 + \epsilon_1 + \epsilon_2 - 4\epsilon_9 - 5\epsilon_{10}$ then the re-numbering of $F^{\mu;\overline{\nu}}$ takes the form:



Adding the entries in each row gives $\overline{w \cdot \lambda}$. In fact

$$w(\lambda) - \lambda = -52\delta$$

$$+ 15\epsilon_1 + 8\epsilon_2 + 8\epsilon_3 + 4\epsilon_4 - 5\epsilon_6 - 7\epsilon_7 - 8\epsilon_8 - 4\epsilon_9 - 11\epsilon_{10}.$$

It is to be noted that $w(\lambda) - \lambda$ is not dominant integral. However,

$$w(\lambda + \rho) - \rho = \lambda + (w(\lambda) - \lambda) + (w(\rho) - \rho)$$

is dominant integral and the corresponding contribution to the numerator expansion is:

$$M_w^\lambda = -e^{8\Lambda_0} e^{-109\delta} \mathrm{ch}\, \overline{V}^{\{29,20,15,9;\overline{9,12,14,16,29}\}}.$$

Branching rules Given characters $\mathrm{ch}\, V^\lambda = M^\lambda / M$ of $\mathbf{g}$ with both M^λ and M expressed in terms of characters $\mathrm{ch}\, V_0^\nu$ of $\mathbf{g}_0$, it is only necessary to calculate M^{-1} and take products of characters of $\mathbf{g}_0$ in $M^\lambda M^{-1}$ to determine branching rules from $\mathbf{g}$ to $\mathbf{g}_0$ for any given λ.

In the case $A_\ell^{(1)} \supset A_\ell$ we have, setting $q = e^{-\delta}$,

$$
\begin{aligned}
M &= 1 - q\,\mathrm{ch}\, V_0^{\{1;\bar{1}\}} + q^2\big(\mathrm{ch}\, V_0^{\{1^2;\bar{2}\}} + \mathrm{ch}\, V_0^{\{2;\overline{1^2}\}}\big) \\
&\qquad - q^3\big(\mathrm{ch}\, V_0^{\{1^3;\bar{3}\}} + \mathrm{ch}\, V_0^{\{21;\overline{21}\}} + \mathrm{ch}\, V_0^{\{3;\overline{1^3}\}}\big) \ - \ \cdots
\end{aligned}
$$

This may be inverted up to any required depth by using the familiar product rules of A_ℓ to give

$$
\begin{aligned}
M^{-1} &= 1 + q\,\mathrm{ch}\, V_0^{\{1;\bar{1}\}} \\
&\quad + q^2\big(\mathrm{ch}\, V_0^{\{2;\bar{2}\}} + \mathrm{ch}\, V_0^{\{1^2;\overline{1^2}\}} + 2\mathrm{ch}\, V_0^{\{1;\bar{1}\}} + \mathrm{ch}\, V_0^{\{0;\bar{0}\}}\big) \\
&\quad + q^3\big(\mathrm{ch}\, V_0^{\{3;\bar{3}\}} + \mathrm{ch}\, V_0^{\{21;\overline{21}\}} + \mathrm{ch}\, V_0^{\{1^3;\overline{1^3}\}} \\
&\qquad + 2\mathrm{ch}\, V_0^{\{2;\bar{2}\}} + 2\mathrm{ch}\, V_0^{\{2;\overline{1^2}\}} + 2\mathrm{ch}\, V_0^{\{1^2;\bar{2}\}} \\
&\qquad + 2\mathrm{ch}\, V_0^{\{1^2;\overline{1^2}\}} + 5\mathrm{ch}\, V_0^{\{1;\bar{1}\}} + 2\mathrm{ch}\, V_0^{\{0;\bar{0}\}}\big) \ + \ \cdots
\end{aligned}
$$

Example In the case $\lambda = \Lambda_0 + \Lambda_1$ we have

$$\begin{aligned} M^\lambda = e^{2\Lambda_0}\bigl(&\operatorname{ch} V_0^{\{1;\bar{0}\}} - q^2 \operatorname{ch} V_0^{\{3;\bar{2}\}} + q^3 \operatorname{ch} V_0^{\{4;\overline{12}\}} \\ &+ q^4 \bigl(\operatorname{ch} V_0^{\{32;\bar{4}\}} - \operatorname{ch} V_0^{\{5;\overline{1^2 3}\}}\bigr) \; - \; \cdots \end{aligned}$$

It follows that for $A_\ell^{(1)}$ and $\lambda = \Lambda_0 + \Lambda_1$

$$\begin{aligned} \operatorname{ch} V^\lambda = e^{2\Lambda_0}\bigl(&\operatorname{ch} V_0^{\{1;\bar{0}\}} \\ &+ q\bigl(\operatorname{ch} V_0^{\{2;\bar{1}\}} + \operatorname{ch} V_0^{\{1^2;\bar{1}\}} + \operatorname{ch} V_0^{\{1;\bar{0}\}}\bigr) \\ &+ q^2\bigl(\operatorname{ch} V_0^{\{21;\bar{2}\}} + \operatorname{ch} V_0^{\{21;\overline{1^2}\}} + \operatorname{ch} V_0^{\{1^3;\overline{1^2}\}} \\ &\qquad + 3\operatorname{ch} V_0^{\{2;\bar{1}\}} + 3\operatorname{ch} V_0^{\{1^2;\bar{1}\}} + 3\operatorname{ch} V_0^{\{1;\bar{0}\}}\bigr) \\ &+ q^3\bigl(\operatorname{ch} V_0^{\{2^2;\overline{21}\}} + \operatorname{ch} V_0^{\{21^2;\overline{21}\}} + \operatorname{ch} V_0^{\{21^2;\overline{1^3}\}} \\ &\qquad + \operatorname{ch} V_0^{\{1^4;\overline{1^3}\}} + \operatorname{ch} V_0^{\{3;\bar{2}\}} + \operatorname{ch} V_0^{\{3;\overline{1^2}\}} \\ &\qquad + 4\operatorname{ch} V_0^{\{21;\bar{2}\}} + 5\operatorname{ch} V_0^{\{21;\overline{1^2}\}} + 2\operatorname{ch} V_0^{\{1^3;\bar{2}\}} \\ &\qquad + 3\operatorname{ch} V_0^{\{1^3;\overline{1^2}\}} + 8\operatorname{ch} V_0^{\{2;\bar{1}\}} + 9\operatorname{ch} V_0^{\{1^2;\bar{1}\}} \\ &\qquad + 7\operatorname{ch} V_0^{\{1;\bar{0}\}}\bigr) \\ &+ \; \cdots\bigr). \end{aligned}$$