AN EXPLICIT ANALYTIC FORMULA FOR MACDONALD POLYNOMIALS

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joint work with

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For each $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$ we denote by x^α the monomial

$$x^\alpha = x_1^{\alpha_1} \dots x_n^{\alpha_n}.$$

Monomial symmetric functions:

Let λ be any partition of length $\leq n$. The polynomial

$$m_\lambda(x_1, \dots, x_n) = \sum x^\alpha,$$

where the sum is running over all distinct permutations α of $\lambda = (\lambda_1, \dots, \lambda_n)$, is clearly symmetric, and the m_λ (as λ runs through all partitions of length $\leq n$) form a $\mathbb{Z}$-basis of Λ_n. Hence, the m_λ such that $l(\lambda) \leq n$ and $|\lambda| = k$ form a $\mathbb{Z}$-basis of Λ_n^k.

Let $X = \{x_1, x_2, x_3, \ldots\}$ be an infinite set of indeterminates, and $\Lambda_{\mathbb{Q}}$ the corresponding ring of symmetric functions with coefficients in $\mathbb{Q}$.

Elementary symmetric functions:

$$e_0(X) := 1,$$
$$e_k(X) := \sum_{i_1<i_2<\cdots<i_k} x_{i_1} x_{i_2} \ldots x_{i_k}, \qquad k \geq 1.$$

Complete symmetric functions:

$$h_0(X) := 1,$$
$$h_k(X) := \sum_{i_1\leq i_2\leq\cdots\leq i_k} x_{i_1} x_{i_2} \ldots x_{i_k}, \qquad k \geq 1.$$

Power sums:

$$p_k(X) := \sum_i x_i^k, \qquad k \geq 1.$$

The e_k, h_k and p_k form three algebraic bases of $\Lambda_{\mathbb{Q}}$.

For any $\lambda = (\lambda_1, \ldots, \lambda_n)$, we define

$$e_\lambda := e_{\lambda_1} \ldots e_{\lambda_n}.$$

Analogously, one defines h_λ and p_λ.

The e_λ, h_λ and p_λ form three vector space bases of $\Lambda_{\mathbb{Q}}$.

Shifted factorials:

$$\begin{aligned}(a)_k &:= \frac{\Gamma(a+k)}{\Gamma(a)} \\ &= a(a+1)\cdots(a+k-1) \quad \text{for } k = 0, 1, 2 \ldots.\end{aligned}$$

Standard q-notation:

Let $0 < |q| < 1$.

$$(a;q)_\infty := \prod_{j\geq 0}(1-aq^j).$$

q-Shifted factorials:

$$\begin{aligned}(a;q)_k &:= \frac{(a;q)_\infty}{(aq^k;q)_\infty} \\ &= (1-a)(1-aq)\cdots(1-aq^{k-1}) \quad \text{for } k = 0, 1, 2 \ldots.\end{aligned}$$

Occasionally, we employ the compact notation

$$(a_1, \ldots, a_m; q)_k := (a_1;q)_k \ldots (a_m;q)_k.$$

A_{n-1} Macdonald polynomials.

Let q, t be independent indeterminates and let $F = \mathbb{Q}(q,t)$ be the field of rational functions in q and t.

We consider now the scalar product defined by

$$< p_\lambda, p_\mu > = < p_\lambda, p_\mu >_{q,t} = \delta_{\lambda\mu} z_\lambda \prod_{i=1}^{l(\lambda)} \frac{1-q^{\lambda_i}}{1-t^{\lambda_i}}.$$

The Macdonald polynomials $P_\lambda = P_\lambda(q,t) = P_\lambda(X;q,t)$ are uniquely determined by

(A) $P_\lambda = m_\lambda + \sum\limits_{\mu<\lambda} u_{\lambda\mu} m_\mu,$

with coefficients $u_{\lambda\mu} \in F$;

(B) $< P_\lambda, P_\mu >_{q,t} = 0$ if $\lambda \neq \mu$.

Special cases:

(1) For $q = t$ the P_λ reduce to the Schur functions s_λ.

(2) For $q = 0$ they reduce to the Hall–Littlewood functions $P_\lambda(X;t)$.

(3) To obtain Jack's symmetric functions, we set $q = t^\alpha$ and let $t \to 1$, so that $q \to 1$ also. Then
$$\frac{1-q^m}{1-t^m} = \frac{1-t^{\alpha m}}{1-t^m} \to \alpha,$$
as $t \to 1$, for all m.

(4) For $t = 1$ (and q arbitrary) we have $P_\lambda(q,1) = m_\lambda$.

(5) For $q = 1$ (and t arbitrary) we have $P_\lambda(1,t) = e_{\lambda'}$.

One column case:

$$P_{(1^r)} = e_r,$$

which follows from (A) (since $m_{(1^r)} = e_r$).

One row case:

$$P_{(r)} = \frac{(q;q)_r}{(t;q)_r}\, g_r,$$

where the $g_r = g_r(X;q,t)$ are the **modified complete** symmetric functions, given by the generating function

$$\prod_{i\geq 1} \frac{(tux_i;q)_\infty}{(ux_i;q)_\infty} = \sum_{r\geq 0} u^r\, g_r(X;q,t).$$

Clearly,

$$g_r = \sum_{|m|=r} \prod_{j\geq 1} \frac{(t;q)_{m_j}}{(q;q)_{m_j}} x_j^{m_j},$$

which follows from the q-binomial theorem,

$$\frac{(tx;q)_\infty}{(x;q)_\infty} = \sum_{m\geq 0} \frac{(t;q)_m}{(q;q)_m} x^m, \qquad |x| < 1.$$

The symmetric functions g_r form an algebraic basis of Λ_F. They may be expanded in terms of any classical basis. Their explicit development in terms of power sums and monomial symmetric functions is given by [**Macdonald**, 1995], in terms of other classical bases by [**Lassalle**, 2001].

For $n = 1$ we have $P_{(r)} = x_1^r$.

In the $n = 2$, i.e. the A_1 case, if we set $x_1 x_2 = 1$ and write $x_1 = e^{-i\theta}$ (hence $x_2 = e^{i\theta}$), we have

$$P_{(r)} = \frac{(q;q)_r}{(t;q)_r} \sum_{m=0}^{r} \frac{(t;q)_m (t;q)_{r-m}}{(q;q)_m (q;q)_{r-m}} e^{(r-2m)i\theta}$$

which is, up to a factor, a **continuous *q*-ultraspherical polynomial** of degree r in $\cos\theta$.

Principal specialization formula.

$$P_\lambda(1,t,\dots,t^{n-1};q,t) = t^{n(\lambda)} \prod_{s\in\lambda} \frac{1-q^{a'(s)}t^{n-l'(s)}}{1-q^{a(s)}t^{l(s)+1}},$$

where

$$n(\lambda) = \sum_{i\geq 1} \binom{\lambda'_i}{2},$$

$$a(s) = a_\lambda(s) = \lambda_i - j, \qquad a'(s) = j-1,$$

$$l(s) = l_\lambda(s) = \lambda'_j - i, \qquad l'(s) = i-1,$$

so that $l'(s)$, $l(s)$, $a(s)$ and $a'(s)$ are respectively the numbers of squares in the Ferrers diagram of λ to the north, south, east and west of the square s. The numbers $a(s)$ and $a'(s)$ are called respectively the arm-length and arm-colength of s, and $l(s)$, $l'(s)$ the leg-length and leg-colength. The hook-length at s is $a(s)+l(s)+1$.

Now let

$$b_\lambda = b_\lambda(q,t) = < P_\lambda, P_\lambda >_{q,t}^{-1} \in F.$$

Explicitly,

$$b_\lambda(q,t) = \prod_{s\in\lambda} \frac{1-q^{a(s)}t^{l(s)+1}}{1-q^{a(s)+1}t^{l(s)}}.$$

Duality.

We define

$$Q_\lambda := b_\lambda P_\lambda$$

so that

$$< P_\lambda, Q_\mu >_{q,t} = \delta_{\lambda\mu},$$

i.e., (P_λ), (Q_λ) are dual bases of Λ_F for the scalar product $<, >_{q,t}$.

We also define an automorphism

$$\omega_{q,t} : \Lambda_F \to \Lambda_F$$

by

$$\omega_{q,t}(p_r) = (-1)^{r-1}\frac{1-q^r}{1-t^r}p_r.$$

Clearly, $\omega_{q,t}^{-1} = \omega_{t,q}$, and $\omega_{t,t} = \omega$.

For all partitions λ we have

$$\omega_{q,t}P_\lambda(q,t) = Q_{\lambda'}(t,q),$$

or equivalently

$$\omega_{q,t}Q_\lambda(q,t) = P_{\lambda'}(t,q).$$

Note that

$$Q_{(r)} = g_r.$$

(Jing and Jósefiak, 1992) Two row formula.

Let $\lambda = (\lambda_1, \lambda_2)$ be a partition of two **parts** λ_1 and λ_2. Then

$$Q_{(\lambda_1,\lambda_2)} = \sum_{m=0}^{\lambda_2} c_m^{(q,t)}(q^{\lambda_1-\lambda_2})\, Q_{(\lambda_1+m)}\, Q_{(\lambda_2-m)},$$

where

$$c_m^{(q,t)}(u) = \frac{(1-uq^{2m})}{(1-u)} \frac{(1/t;q)_m}{(q;q)_m} \frac{(u;q)_m}{(qtu;q)_m} t^m.$$

J. & J. derived this result by applying **inverse relations.** Specifically, they inverted the following relation.

Pieri formula for one row Macdonald polynomials.

Let $\lambda_1, \lambda_2 \in \mathbb{N}$. Then

$$Q_{(\lambda_1)}\, Q_{(\lambda_2)} = \sum_{r=0}^{\lambda_2} d_r^{(q,t)}(q^{\lambda_1-\lambda_2})\, Q_{(\lambda_1+r,\lambda_2-r)},$$

where

$$d_r^{(q,t)}(u) = \frac{(t;q)_r}{(q;q)_r} \frac{(q^{r+1}u;q)_r}{(q^r tu;q)_r}.$$

As a matter of fact, the infinite lower-triangular matrices

$$\left(c_{m-r}^{(q,t)}(uq^{2r})\right)_{m,r\in\mathbb{Z}} \qquad \text{and} \qquad \left(d_{r-l}^{(q,t)}(uq^{2l})\right)_{r,l\in\mathbb{Z}}$$

are **inverses** of each other, i.e., the orthogonality relation

$$\sum_{r=l}^{m} c_{m-r}^{(q,t)}(uq^{2r})\, d_{r-l}^{(q,t)}(uq^{2l}) = \delta_{ml},$$

for all $m, l \in \mathbb{Z}$, holds.

The above orthogonality relation (first observed, in an independent context, by **Bressoud** in 1983) is a special case of the terminating ${}_6\phi_5$ summation:

$$\sum_{k=0}^{m} \frac{(1-aq^{2k})}{(1-a)} \frac{(a,b,c,q^{-m};q)_k}{(q,aq/b,aq/c,aq^{m+1};q)_k} \left(\frac{aq^{m+1}}{bc}\right)^k \\ = \frac{(aq,aq/bc;q)_m}{(aq/b,aq/c;q)_m}.$$

In general, if the infinite matrices $(f_{mr})_{m,r\in\mathbb{Z}}$ and $(g_{rl})_{r,l\in\mathbb{Z}}$ are **inverses** of each other, i.e.,

$$\sum_{r\in\mathbb{Z}} f_{mr} g_{rl} = \delta_{ml} \qquad \text{for all } m, l \in \mathbb{Z},$$

and

$$\sum_{r\in\mathbb{Z}} g_{mr} f_{rl} = \delta_{ml} \qquad \text{for all } m, l \in \mathbb{Z},$$

then it is immediate that the following **inverse relations** hold (subject to convergence of the sums):

$$\sum_{r\in\mathbb{Z}} f_{mr} a_r = b_m \qquad \text{for all } m,$$

if and only if

$$\sum_{l\in\mathbb{Z}} g_{rl} b_l = a_r \qquad \text{for all } r.$$

Similarly (*"rotated inversion"*):

$$\sum_{m\in\mathbb{Z}} f_{mr} a_m = b_r \qquad \text{for all } r,$$

if and only if

$$\sum_{r\in\mathbb{Z}} g_{rl} b_r = a_l \qquad \text{for all } l.$$

Defining

$$\begin{aligned} a_l &:= Q_{(\lambda_1+l)}\, Q_{(\lambda_1-l)}, \\ b_r &:= Q_{(\lambda_1+r,\lambda_2-r)}, \\ f_{mr} &:= c^{(q,t)}_{m-r}(uq^{2r}), \end{aligned}$$

and

$$g_{rl} := d^{(q,t)}_{r-l}(uq^{2l}),$$

for all $m, r, l \in \mathbb{Z}$, we can see (after shifting the summation index by l) that $\sum_{r\in\mathbb{Z}} g_{rl} b_r = a_l$ holds, for $u = q^{\lambda_1-\lambda_2}$ and the above values of a_l, b_r and g_{rl}, by the Pieri formula for one row Macdonald polynomials.

Thus, by "rotated inversion", we immediately establish $\sum_{m\in\mathbb{Z}} f_{mr} a_m = b_r$, for $u = q^{\lambda_1-\lambda_2}$ and the above values of a_l, b_r and f_{mr}. Setting now $r = 0$ gives Jing and Jósefiak's result. □

(Lassalle, 2001) Three row formula.

Let $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ be a partition of three parts. Then

$$Q_{(\lambda_1,\lambda_2,\lambda_3)} = \sum_{m_1,m_2\geq 0} c^{(q,t)}_{(m_1,m_2)}(q^{\lambda_1-\lambda_3}t, q^{\lambda_2-\lambda_3}) \times Q_{(\lambda_1+m_1,\lambda_2+m_2)}\, Q_{(\lambda_3-m_1-m_2)},$$

where

$$\begin{aligned} c^{(q,t)}_{(m_1,m_2)}(u_1,u_2) &= \frac{(1-u_1q^{2m_1})(1-u_2q^{2m_2})}{(1-u_1)(1-u_2)} \\ &\times \frac{(1/t;q)_{m_1}(1/t;q)_{m_2}}{(q;q)_{m_1}(q;q)_{m_2}} \frac{(u_1;q)_{m_1}(u_2;q)_{m_2}}{(qtu_1;q)_{m_1}(qtu_2;q)_{m_2}} \\ &\times \frac{(qu_1/tu_2;q)_{m_1}}{(qu_1/u_2;q)_{m_1}} \frac{(q^{-m_2}tu_1/u_2;q)_{m_1}}{(q^{-m_2}u_1/u_2;q)_{m_1}} t^{m_1+m_2} \\ &\times \Bigg[1 + \frac{(1-q^{m_1})(1-q^{m_2})}{(q^{m_1}-tu_2/u_1)(q^{m_2}-tu_1/u_2)} \\ &\times \left(t - (u_1q^{m_1} - u_2q^{m_2})\frac{(t-q^{m_1})(t-q^{m_2})}{(1-u_1q^{2m_1})(1-u_2q^{2m_2})}\right)\Bigg]. \end{aligned}$$

Let $\mathbb{N}$ denote the set of nonnegative integers.
For $\theta = (\theta_1, \ldots, \theta_n) \in \mathbb{N}^n$ let $|\theta| = \sum_{i=1}^n \theta_i$ and define

$$d^{(q,t)}_{\theta_1,\ldots,\theta_n}(u_1, \ldots, u_n) := \prod_{k=1}^n \frac{(t;q)_{\theta_k}}{(q;q)_{\theta_k}} \frac{(q^{|\theta|+1}u_k;q)_{\theta_k}}{(q^{|\theta|}tu_k;q)_{\theta_k}}$$
$$\times \prod_{1\leq i<j\leq n} \frac{(tu_i/u_j;q)_{\theta_i}}{(qu_i/u_j;q)_{\theta_i}} \frac{(q^{-\theta_j+1}u_i/tu_j;q)_{\theta_i}}{(q^{-\theta_j}u_i/u_j;q)_{\theta_i}}.$$

(Macdonald, 1987; Koornwinder, 1988) Explicit Pieri formula for A_{n-1} Macdonald polynomials.

Let $\lambda = (\lambda_1, \ldots, \lambda_n)$ be an arbitrary partition of n parts and $\lambda_{n+1} \in \mathbb{N}$. Let $u_k := q^{\lambda_k - \lambda_{n+1}} t^{n-k}$, for all $1 \leq k \leq n$. We have

$$Q_{(\lambda_1,\ldots,\lambda_n)}\, Q_{(\lambda_{n+1})}$$
$$= \sum_{\theta\in\mathbb{N}^n} d^{(q,t)}_{\theta_1,\ldots,\theta_n}(u_1, \ldots, u_n)\, Q_{(\lambda_1+\theta_1,\ldots,\lambda_n+\theta_n,\lambda_{n+1}-|\theta|)}.$$

For $\theta = (\theta_1, \ldots, \theta_n) \in \mathbb{N}^n$ define

$$
\begin{aligned}
c^{(q,t)}_{\theta_1,\ldots,\theta_n}(u_1,\ldots,u_n) := {} & \prod_{k=1}^{n} t^{\theta_k} \frac{(q/t;q)_{\theta_k}}{(q;q)_{\theta_k}} \frac{(qu_k;q)_{\theta_k}}{(qtu_k;q)_{\theta_k}} \\
& \times \prod_{1\le i<j\le n} \frac{(qu_i/tu_j;q)_{\theta_i}}{(qu_i/u_j;q)_{\theta_i}} \frac{(q^{-\theta_j}tu_i/u_j;q)_{\theta_i}}{(q^{-\theta_j}u_i/u_j;q)_{\theta_i}} \\
& \times \prod_{1\le i<j\le n} (u_iq^{\theta_i} - u_jq^{\theta_j})^{-1} \det_{1\le i,j\le n} \Bigg[(u_iq^{\theta_i})^{n-j} \\
& \times \left(1 - t^{j-1} \frac{1-tu_iq^{\theta_i}}{1-u_iq^{\theta_i}} \prod_{k=1}^{n} \frac{u_k - u_iq^{\theta_i}}{tu_k - u_iq^{\theta_i}} \right) \Bigg].
\end{aligned}
$$

We have the following general result.

Recursion formula for Macdonald polynomials.

Let $\lambda = (\lambda_1, \ldots, \lambda_{n+1})$ be a partition of $n+1$ parts. Let $u_k := q^{\lambda_k - \lambda_{n+1}} t^{n-k}$, for all $1 \le k \le n$. We have

$$
\begin{aligned}
Q_{(\lambda_1,\ldots,\lambda_{n+1})} = {} & \sum_{\theta\in\mathbb{N}^n} c^{(q,t)}_{\theta_1,\ldots,\theta_n}(u_1,\ldots,u_n) \\
& \times Q_{(\lambda_{n+1}-|\theta|)}\, Q_{(\lambda_1+\theta_1,\ldots,\lambda_n+\theta_n)}.
\end{aligned}
$$

Idea of proof.

We invert the infinite transition matrix defined by the Pieri formula for Macdonald polynomials.

We accomplish this by applying a suitable extension (developed in our **thesis**, 1996) of **Krattenthaler**'s [1988] **operator method** for proving lower-triangular multidimensional matrix inversions.
(This method was already several times successfully utilized in the context of **multiple basic hypergeometric series associated to root systems**.

Having the explicit multidimensional matrix inversion at hand (in fact, we are able to prove an even more general result than here needed), it is straightforward to apply **multidimensional inverse relations** to deduce the desired result. □

The Jack limit. ($q = t^\alpha$, then $t \to 1$)

Fix some positive real number α. We note

$$P_\lambda = \lim_{t\to 1} P_\lambda(t^\alpha, t) \quad \text{and} \quad Q_\lambda = \lim_{t\to 1} Q_\lambda(t^\alpha, t).$$

For $\theta = (\theta_1, \ldots, \theta_n) \in \mathbb{N}^n$ and any indeterminate a define

$$c_\theta^{(a)}(u_1, \ldots, u_n) := \prod_{k=1}^{n} \frac{(1-a)_{\theta_k}}{\theta_k!} \frac{(u_k+1)_{\theta_k}}{(u_k+1+a)_{\theta_k}}$$

$$\times \prod_{1\le i<j\le n} \frac{(u_i - u_j + 1 - a)_{\theta_i}}{(u_i - u_j + 1)_{\theta_i}} \frac{(u_i - u_j - \theta_j + a)_{\theta_i}}{(u_i - u_j - \theta_j)_{\theta_i}}$$

$$\times \prod_{1\le i<j\le n} (u_i + \theta_i - u_j - \theta_j)^{-1} \det_{1\le i,j\le n} \Bigg[(u_i + \theta_i)^{n-j}$$

$$- (u_i + \theta_i - a)^{n-j} \frac{u_i + \theta_i + a}{u_i + \theta_i} \prod_{k=1}^{n} \frac{u_i + \theta_i - u_k}{u_i + \theta_i - u_k - a} \Bigg].$$

We have the following general result.

Recursion formula for Jack polynomials.

Let $\lambda = (\lambda_1, \ldots, \lambda_{n+1})$ be a partition of $n+1$ parts.
Let $u_k := \lambda_k - \lambda_{n+1} + (n-k)/\alpha$, for all $1 \le k \le n$. Then

$$Q_{(\lambda_1,\ldots,\lambda_{n+1})} = \sum_{\theta\in\mathbb{N}^n} c_{\theta_1,\ldots,\theta_n}^{(1/\alpha)}(u_1, \ldots, u_n)$$

$$\times\, Q_{(\lambda_{n+1}-|\theta|)}\, Q_{(\lambda_1+\theta_1,\ldots,\lambda_n+\theta_n)}.$$

The Hall–Littlewood limit.

($q \to 0$ in $P_\lambda(q,t)$, or $q \to 0$ in $Q_\lambda(t,q)$)

Let $\lambda = (\lambda_1, \ldots, \lambda_{n+1})$ be a partition of length $n+1$. For $\theta = (\theta_1, \ldots, \theta_n) \in \mathbb{N}^n$ define

$$c_\theta^{(t)}(\lambda) := (-1)^{|\theta|} t^{\sum_{i=1}^n \binom{\theta_i}{2}} \prod_{i=1}^n \frac{(t^{1+\lambda_i-\lambda_{i+1}};t)_{\theta_i}}{(t;t)_{\theta_i}} \\ \times \sum_{k=0}^n \prod_{i=k+1}^n \frac{t^{\theta_i}-1}{1-t^{\lambda_{i+1}-\lambda_i-\theta_i}}.$$

We have the following general result.

Recursion formula for Hall–Littlewood polynomials.

Let $\lambda = (\lambda_1, ..., \lambda_{n+1})$ be a partition of length $n+1$. Then

$$Q_{(\lambda_1,\ldots,\lambda_{n+1})} = \sum_{\theta \in \mathbb{N}^n} c_\theta^{(t)}(\lambda)\, Q_{(\lambda_{n+1}-|\theta|)}\, Q_{(\lambda_1+\theta_1,\ldots,\lambda_n+\theta_n)}.$$