Some Basic Vector symmetric Functions

The elementary symmetric functions:

$$1+\sum a_{i_1\cdots i_n}\, x_1^{i_1}\cdots x_n^{i_n}$$

$$=\left(1+\alpha_{11}\, x_1+\cdots+\alpha_{1n}\, x_n\right)\left(1+\alpha_{21}\, x_1+\cdots+\alpha_{2n}\, x_n\right)\cdots$$

The homogeneous symmetric functions

$$1+\sum h_{i_1\cdots i_n}\, x_1^{i_1}\cdots x_n^{i_n}$$

$$=\left(\frac{1}{1-\left[\alpha_{11}\, x_1+\cdots+\alpha_{1n}\, x_n\right]}\right)\left(\frac{1}{1-\left[\alpha_{21}\, x_1+\cdots+\alpha_{2n}\, x_n\right]}\right)\cdots$$

The power sum symmetric functions

$$\sum \binom{k}{i_1\cdots i_n} p_{i_1\cdots i_n}\, x_1^{i_1}\cdots x_n^{i_n}$$

$$=\left(\alpha_{11}\, x_1+\cdots+\alpha_{1n}\, x_n\right)^{k}+\left(\alpha_{21}\, x_1+\cdots+\alpha_{2n}\, x_n\right)^{k}+\cdots$$

The monomial symmetric functions are defined by

$$m_{\left(1^{h_1}\cdots n^{h_n}\right)^{r_1}\cdots\left(1^{k_1}\cdots n^{k_n}\right)^{r_p}}$$

$$=\sum_{j\cdots k\cdots s\cdots t}\left[\alpha_{j1}^{h_1}\cdots\alpha_{jn}^{h_n}\right]\cdots\left[\alpha_{k1}^{h_1}\cdots\alpha_{kn}^{h_n}\right]\cdots\left[\alpha_{s1}^{k_1}\cdots\alpha_{sn}^{k_n}\right]\cdots\left[\alpha_{t1}^{k_1}\cdots\alpha_{tn}^{k_n}\right]$$

sum over all *distinct* monomials with $r = r_1+\cdots+r_p$ *different* indices $j\cdots k\cdots s\cdots t$.

Example

Consider for example the case of four vectors in three dimensions.

Let $X = (x, y, z)$ denote a vector of three scalar variables.
Let X_1, X_2, X_3, X_4 be four values of X with

$$X_i = (\alpha_i, \beta_i, \gamma_i).$$

An example of a monomial symmetric function in the vectors is (three notations)

$m_{(210)(010)} =$	$\sum$	$\left[x_1^2 y_1\right] x_2$
$m_{(210)(010)} =$	$\sum_{i \neq j}$	$\left[\alpha_i^2 \beta_i\right] \alpha_j$
$m_{(210)(010)} =$	$\sum_{\{i,j\} \subseteq \{1,2,3,4\}}$	$\left[\alpha_i^2 \beta_i\right] \alpha_j + \left[\alpha_j^2 \beta_j\right] \alpha_i.$

Note the following properties

$m_{(210)(010)} =$	Sum Over Sequences	$\sum_{i \neq j}$	One Monomial for each Sequence	$\left[\alpha_i^2 \beta_i\right] \alpha_j$
$m_{(210)(010)} =$	Sum Over Sets	$\sum_{\{i,j\} \subseteq \{1,2,3,4\}}$	One Symmetric Monomial Sum for each Set	$+\left[\alpha_i^2 \beta_i\right] \alpha_j$ $+\left[\alpha_j^2 \beta_j\right] \alpha_i.$

The ij –Symmetric Monomial Sum is called a *permanent*

We will adopt a variant of MacMahon's notation

We will write

$$\left(x_1^{h_1}\cdots x_n^{h_n}\right)^{(r_1)}\cdots\left(x_1^{k_1}\cdots x_n^{k_n}\right)^{(r_p)} = m_{\left(1^{h_1}\cdots n^{h_n}\right)^{r_1}\cdots\left(1^{k_1}\cdots n^{k_n}\right)^{r_p}}$$

Multiplication of Monomial Symmetric Functions

The rule for multiplication of monomial functions is we obtain a sum of monomial functions corresponding to all ways of joining any number of parts of each monomial function.

$$(u)(v)\circ(w) = (u\,w)(v) + (u)(v\,w) + (u)(v)(w)$$

$$\left[\Pi_i\,(u_i)^{(r_i)}\right]\circ\left[\Pi_j\,(v_j)^{(s_j)}\right] = \sum$$
$$\Pi_{ij}\,(u_i)^{(r_i')}\,(v_j)^{(s_j')}\left(\Pi_i\,(u_i)^{(r_i'')}\,\|\,\Pi_j\,(v_j)^{(s_j'')}\right)$$

1) Permanents

$$\left(\Pi_i\,(u_i)^{(r_i)}\,\|\,\Pi_j\,(v_j)^{(s_j)}\right) = \sum\Pi_{ij}\,(u_i\,v_j)^{(m_{ij})}$$

2) Divided Powers

$$(w)^{(p)}(w)^{(q)} = \binom{p+q}{q}(w)^{(p+q)}$$

Peter Doubilet's Theory of Symmetric Functions

1. Doubilet defines a vector space with elements m_π , p_π , a_π , h_π and f_π indexed by the partitions of a set,
analogous to the classical symmetric functions m_λ , p_λ , a_λ , h_λ and f_λ indexed by the partitions of an integer.

2. Doubilet suspected that there was a symmetric function theory associated with his vector space but did not discover the theory.

3. Our main point today is that the elements m_π , p_π , a_π , h_π and f_π are **actual symmetric functions** of **vectors**.

4. Ordinary symmetric functions can be considered vector symmetric functions where the indices are all **the same**,
Doubilet's symmetric functions correspond to vector symmetric functions where the indices are all **different**.

5. The classical case and the Doubilet case are the two extreme cases of vector symmetric functions!

6. By setting various sets of indices equal to each other, we can obtain from the Doubilet theory all of vector symmetric function theory!

Let π denote a partition of the set $A = \{x_1, \cdots, x_n\}$
with r blocks $B_1 \cdots B_r$.
$\pi = \{\{x_{i_{11}}, \cdots, x_{i_{1n_1}}\}, \cdots, \{x_{i_{r1}}, \cdots, x_{i_{rn_r}}\}\} = \{B_1, \cdots, B_r\}$
$B_j = \{x_{i_{j1}}, \cdots, x_{i_{jn_j}}\} \qquad (1 \le j \le r).$

We can associate to π the following vector symmetric functions

Monomial Functions m_π	$(x_{i_{11}} \cdots x_{i_{1n_1}}) \cdots (x_{i_{r1}} \cdots x_{i_{rn_r}})$
Power Sum Functions p_π	$(x_{i_{11}} \cdots x_{i_{1n_1}}) \circ \cdots \circ (x_{i_{r1}} \cdots x_{i_{rn_r}})$
Elementary Functions a_π	$[(x_{i_{11}}) \cdots (x_{i_{1n_1}})] \circ \cdots \circ [(x_{i_{r1}}) \cdots (x_{i_{rn_r}})]$

Because there are no repeats, the rule for circle product can be cast in a suggestive form, as a sum over the lattice Π_A of partitions of the set A ordered by refinement.

$\sigma \le \pi$ if every block of σ is contained in a block of π.

Doubilet's Formulas

Some of the formulas found by Doubilet which can be easily derived by vector symmetric function theory are the following:

1. $p_\pi = \sum_{\sigma \geq \pi} m_\sigma.$

2. $m_\pi = \sum_{\sigma} \mu(\pi,\sigma)\, p_\sigma.$

3. $a_\pi = \sum_{\sigma \wedge \pi = 0} m_\sigma$

4. $a_\pi = \sum_{0 \leq \sigma \leq \pi} \mu(0,\sigma)\, p_\sigma\, \zeta(\sigma,\pi)$

5. $m_{[\rho,\pi]} = \sum \mu(\rho,\sigma)\, p_\sigma\, \zeta(\sigma,\pi) = \sum_{\sigma \wedge \pi = \rho} m_\sigma$

$$1.\ p_\pi = \sum_{\sigma \geq \pi} m_\sigma.$$

Proof : When π has two parts we have

$$p_\pi = \left(x^{I}\right) \circ \left(x^{J}\right) = \left(x^{I}\right)\left(x^{J}\right) + \left(x^{I+J}\right)$$

When π has three parts we have

$$p_\pi = \left(x^{I}\right) \circ \left(x^{J}\right) \circ \left(x^{K}\right)$$

$$= \left(x^{I}\right)\left(x^{J}\right) \circ \left(x^{K}\right) + \left(x^{I+J}\right) \circ \left(x^{K}\right)$$

and

$$\left(x^{I}\right)\left(x^{J}\right) \circ \left(x^{K}\right) = \left(x^{I}\right)\left(x^{J}\right)\left(x^{K}\right) + \left(x^{I+K}\right)\left(x^{J}\right) + \left(x^{I}\right)\left(x^{J+K}\right)$$

$$\left(x^{I+J}\right) \circ \left(x^{K}\right) = \left(x^{I+J}\right)\left(x^{K}\right) + \left(x^{I+J+K}\right)$$

so that

$$p_\pi = \left(x^{I}\right) \circ \left(x^{J}\right) \circ \left(x^{K}\right)$$

$$= \left(x^{I}\right)\left(x^{J}\right)\left(x^{K}\right)$$
$$+\left(x^{I+K}\right)\left(x^{J}\right) + \left(x^{I}\right)\left(x^{J+K}\right) + \left(x^{I+J}\right)\left(x^{K}\right)$$
$$+\left(x^{I+J+K}\right)$$

Continuing in this way we can formally prove the result stated.

$$2.\ m_\pi = \sum_{\sigma} \mu(\pi,\sigma)\, p_\sigma.$$

Proof : By Mobius inversion.

$$3.\ a_\pi = \sum_{\sigma\wedge\pi=0} m_\sigma$$

Proof : The function a_π is of the form

$$\left[\left(x_{i_{11}}\right)\cdots\left(x_{i_{1n_1}}\right)\right]\circ\cdots\circ\left[\left(x_{i_{r1}}\right)\cdots\left(x_{i_{rn_r}}\right)\right]$$

which by the rules for multiplicaton will be a sum over all partitions σ such that no block of σ contains more than one element from any block of π. This is the same as $\sigma \wedge \pi = 0$.

Note that $[(\sigma \le \tau) \wedge (\sigma \le \pi) \Leftrightarrow (\sigma \le \tau \wedge \pi)] \Rightarrow \zeta(\sigma,\tau)\zeta(\sigma,\pi) = \zeta(\sigma,\tau\wedge\pi)$

$$5.\quad m_{[\rho,\pi]} = \sum \mu(\rho,\sigma)\ p_\sigma\ \zeta(\sigma,\pi) = \sum\nolimits_{\sigma\wedge\pi=\rho} m_\sigma$$

Proof : $\sum_\sigma \mu(\rho,\sigma)\ p_\sigma\ \zeta(\sigma,\pi) = \sum_\sigma \mu(\rho,\sigma)\left[\sum_\tau \zeta(\sigma,\tau)\ m_\tau\right]\zeta(\sigma,\pi)$

$$= \sum_\tau\sum_\sigma \mu(\rho,\sigma)\ [\ \zeta(\sigma,\tau)\ \zeta(\sigma,\pi)]\ m_\tau = \sum_\tau\sum_\sigma\ \mu(\rho,\sigma)\ \zeta(\sigma,\tau\wedge\pi)\ m_\tau$$

$$= \sum_\tau\ \delta(\rho,\tau\wedge\pi)\ m_\tau = \sum\nolimits_{\tau\wedge\pi=\rho}\ m_\tau$$

$$4.\ a_\pi = \sum_{0\le\sigma\le\pi} \mu(0,\sigma)\ p_\sigma\ \zeta(\sigma,\pi)$$

Proof Apply the proposition to $a_\pi = \sum_{\sigma\wedge\pi=0} m_\sigma$.

Suppose $\sigma = \{\sigma_1, \cdots, \sigma_k\}$ is a set partition. Define $\|\sigma\| = \sigma_1 \cup \cdots \cup \sigma_k$.

Proposition : Let $\rho_1, \cdots, \rho_k$ be set partitions. Define

$\pi = \{\|\rho_1\|, \cdots, \|\rho_k\|\}$
$\rho = \rho_1 \cup \cdots \cup \rho_k$

Then

$$m_{\rho_1} \circ m_{\rho_2} \circ \cdots \circ m_{\rho_k} = \sum_{\sigma \wedge \pi = \rho} m_\sigma = m_{[\rho, \pi]}$$

Proof :

$$m_{\rho_1} \circ m_{\rho_2} \circ \cdots \circ m_{\rho_k} = \sum_\sigma m_\sigma$$

sum over all σ for which each block is a union containing at most one block from each set partition ρ_i. These correspond to partitions σ satisfying $\sigma \wedge \pi = \rho$. Therefore

$m_{\rho_1} \circ m_{\rho_2} \circ \cdots \circ m_{\rho_k}$
$= \sum_{\sigma \wedge \pi = \rho} m_\sigma$
$= \sum_{\rho \leq \sigma \leq \pi} m_\sigma$

as follows from (5.) above.

Analogues of the Segment Monomials

Suppose $G = \{ g_\pi \mid \pi \in \Pi(A) \}$ is a family of *vsfl's*

We can define for study some segment analogues of G by

$$g_{[\rho,\pi]} = \sum_\sigma \mu(\rho,\sigma)\ g_\sigma\ \zeta(\sigma,\pi)$$

$$g_{(\rho,\pi)} = \sum_\sigma \zeta(\rho,\sigma)\ g_\sigma\ \mu(\sigma,\pi)$$

$$g_{\{\rho,\pi\}} = \sum_\sigma |\mu(\rho,\sigma)|\ g_\sigma\ \zeta(\sigma,\pi)$$

$$g_{\|\rho,\pi\|} = \sum_\sigma \zeta(\rho,\sigma)\ g_\sigma\ |\mu(\sigma,\pi)|$$

Proposition : We have $a_{(0,\pi)} = \mu(0,\pi)\ p_\pi$

Proof

$$\begin{aligned} a_{(0,\pi)} &= \sum_{\sigma\le\pi} \zeta(0,\sigma)\ a_\sigma\ \mu(\sigma,\pi) \\ &= \sum_{\rho\le\sigma\le\pi} \zeta(0,\sigma)\ \big[\mu(0,\rho)\ p_\rho\ \zeta(\rho,\sigma)\big]\ \mu(\sigma,\pi) \\ &= \sum_{\rho\le\sigma\le\pi} \mu(0,\rho)\ p_\rho\ \big[\zeta(0,\sigma)\ \zeta(\rho,\sigma)\ \mu(\sigma,\pi)\big] \\ &= \sum_{\rho\le\sigma\le\pi} \mu(0,\rho)\ p_\rho\ \big[\zeta(\rho,\sigma)\ \mu(\sigma,\pi)\big] \\ &= \sum_{\rho\le\pi} \mu(0,\rho)\ p_\rho\ \delta(\rho,\pi) \\ &= \mu(0,\pi)\ p_\pi \end{aligned}$$

The Vector Frobenius Map

Recall that for any permutation $(x_{11} \cdots x_{1r_1}) \cdots (x_{k1} \cdots x_{1r_k})$ the classical (scalar) Frobenius Map satisfies

$$\phi\ [(x_{11} \cdots x_{1r_1}) \cdots (x_{k1} \cdots x_{1r_k})] = p_{r_1} \ \cdots \ p_{r_k}.$$

Suppose D_λ is a Young Tableau of shape λ.

1. $Pos(D_\lambda) = \sum_\theta \ \theta \cdots$

Sum over all permutations fixing the row position of every entry

2. $Neg(D_\lambda) = \sum_\theta \ (-1)^\theta \ \theta \cdots$

Alternating Sum of all permutations fixing the row position of every entry

The most important properties of the Frobenius map are

$$\phi\left[\ Pos\left(D_\lambda\right)\ \right] = h_\lambda \qquad \phi\left[\ Neg\left(D_\lambda\right)\ \right] = a_\lambda$$

The Vector Frobenius Map is the linear map Φ with

$$\Phi\ [(x_{11} \cdots x_{1r_1}) \cdots (x_{k1} \cdots x_{1r_k})] = (x_{11} \cdots x_{1r_1}) \circ \cdots \circ (x_{k1} \cdots x_{1r_k})$$

The most important properties of the vector Frobenius map are

$$\Phi\left[\ Pos\left(D_\pi\right)\ \right] = h_\pi \qquad \Phi\left[\ Neg\left(D_\pi\right)\ \right] = a_\pi.$$

Mobius Algebra and Kronecker Inner Product

The Mobius Algebra $A(L, K)$ of a finite lattice L is defined to be the the semi-group algebra of L over a field K with $x \bullet y = x \wedge y$. Then $\delta_x = \sum_{y \le x} y\,\mu(y, x)$ are a system of orthogonal idempotents.

Furthermore $x = \sum_{y \le x} \delta_y$ and $x \bullet y = \sum_{t \le x \wedge y} \delta_t$.

Theorem

$A(\Pi_n, K) \approx Kron(\Pi_n, K)$

where $Kron(\Pi_n, K)$ is the Doubilet algebra of vector symmetric functions indexed by partitions of a set, with multiplication given by the Kronecker inner product.

Proof

From Doubilet's computations we have

$$[h_\pi, h_\sigma] = n!\; h_{\pi \wedge \sigma}.$$

Consider the linear map from the Mobius Algebra $A(\Pi_n, K)$ to the algebra $Kron(\Pi_n, K)$

$$\pi \longrightarrow \tfrac{1}{n!}\, h_\pi \,.$$

From Doubilet's Formula we see that we have an isomorphism.

Corollary

In the Kronecker Algebra $Kron(\Pi_n, K)$ the elements

$$\delta_\pi = \left(\tfrac{1}{n!}\right)\sum_{\sigma \le \pi} h_\sigma \, \mu(\sigma, \pi)$$

are a system of orthogonal idempotents. Furthermore we have

$$\left(\tfrac{1}{n!}\right)h_\pi = \sum_{\sigma \le \pi} \delta_\sigma .$$

It is also shown by Doubilet that

$$[p_\pi, p_\sigma] = \delta_{\pi\sigma} \frac{n!}{|\mu(0, \pi)|}$$

Since the system of orthogonal idempotents is unique we can identify the δ_π functions with the p_π functions and use this to express the h_σ functions in terms of the p_π functions and visa versa from the above formulas!

A Supersymmetric Doubilet Theory

We wish to consider a supersymmetric extension of Doubilet's theory. To do this we will first introduce the concept of vector symmetric functions over a signed alphabet. Suppose L is an alphabet of negative letters $L = \{x_1, \cdots, x_n\}$. We can think of *Super* L as the algebra of anticommuting coordinate functions of a set of vectors $V = \{v_1, \cdots, v_r\}$ with $v_j = (x_{1j}, \cdots, x_{nj})$ where all the x_{ij} are in L. The coordinate functions satisfy $(x_i)[v_j] = x_{ij}$. A symmetric function Φ of the vectors in V is a polynomial $\Phi(x_{ij})$ which satisfies for all permutations θ of $1, \cdots, r$ that $\Phi(x_{ij}) = \Phi(x_{i\,\theta(j)})$.

A) We can define the analogue of an elementary symmetric function by the equivalent forms

$$(x_1) \wedge \cdots \wedge (x_p)[v_1 \cdots v_m] = \sum_{i_1 < \cdots < i_p} \sum_{\sigma} (-1)^{\theta} x_{\sigma(1), i_1} \cdots x_{\sigma(p), i_p}$$

$$(x_1) \wedge \cdots \wedge (x_p)[v_1 \cdots v_m] = \sum_{i_1 < \cdots < i_p} \sum_{\theta} (+1)^{\theta} x_{1, i_{\theta(1)}} \cdots x_{p, i_{\theta(p)}}$$

B) The polarized power-sum symmetric functions are

$$(x_1 \cdots x_p)[v_1 \cdots v_m] = \sum_{i} x_{1,i} \cdots x_{p,i}.$$

C) The monomial symmetric functions of any shape $\lambda = (1^p 2^q \cdots)$ are, with $degree(w_i) = \|w_i\|$ also have two equivalent forms

$$(w_1) \wedge \cdots \wedge (w_p)[v_1 \cdots v_m]$$

$$= \sum_{i_1 < \cdots < i_p} \sum_{\sigma} (-1)^{\left[\sum_{[p<q][\sigma(p)>\sigma(q)]} \|w_{\sigma(p)}\| \, \|w_{\sigma(q)}\|\right]} w_{\sigma(1)}[v_{i_1}] \cdots w_{\sigma(p)}[v_{i_p}]$$

$$(w_1) \wedge \cdots \wedge (w_p)[v_1 \cdots v_m]$$

$$= \sum_{i_1 < \cdots < i_p} \sum_{\sigma} (+1)^{\sigma} w_1[v_{i_{\sigma(1)}}] \cdots w_p[v_{i_{\sigma(p)}}]$$

D) The homogeneous symmetric function are defined by

$$(x_1) \vee \cdots \vee (x_p)[v_1 \cdots v_m] = \sum_{i_1 \le \cdots \le i_p} \sum_{\sigma} (-1)^{\theta} x_{\sigma(1), i_1} \cdots x_{\sigma(p), i_p}$$

$$(x_1) \vee \cdots \vee (x_p)[v_1 \cdots v_m] = \sum_{i_1 \le \cdots \le i_p} \sum_{\theta} (+1)^{\theta} x_{1, i_{\theta(1)}} \cdots x_{p, i_{\theta(p)}}$$

Here the inner sum is over all distinct sequences $i_{\theta(1)} \cdots i_{\theta(p)}$ we can get by permuting the possibly repeated indices $i_1 \le \cdots \le i_p$. Then we will find

$$(x_1) \vee \cdots \vee (x_p) = S_{\circ}\left[\ S_{\wedge}\left[\ (S_{\bullet}[x_1]) \wedge \cdots \wedge (S_{\bullet}[x_p])\ \right]\right]$$

$S_{\circ}$ $S_{\wedge}$ $S_{\bullet}$ denote respectively the circle antipode, the monomial product antipode and the antipode of the underlying (exterior) algebra of the $x's$.

E) The forgotten symmetric functions of any shape $\lambda = (1^p 2^q \cdots)$.satisfy

$$(w_1) \vee \cdots \vee (w_p)[v_1 \cdots v_m]$$

$$= \sum_{i_1 \leq \cdots \leq i_p} \sum_{\sigma} (-1)^{\sum_{[p<q] \wedge [\sigma(p)>\sigma(q)]} \|w_{\sigma(p)}\| \, \|w_{\sigma(q)}\|} w_{\sigma(1)}[v_{i_1}] \cdots w_{\sigma(p)}[v_{i_p}]$$

$$(w_1) \vee \cdots (w_p)[v_1 \cdots v_m]$$

$$= \sum_{i_1 \leq \cdots \leq i_p} \sum_{\sigma} (+1)^{\sigma} w_1\left[v_{i_{\sigma(1)}}\right] \cdots w_p\left[v_{i_{\sigma(p)}}\right]$$

Examples of vector supersymmetric functions

The three multiplications used in symmetric function algebra all anticommute by degree.

(1) The letters factors within each parenthesis satisfy

$(\cdots x \bullet y \cdots) = -(\cdots y \bullet x \cdots)$

(2) The factors of the monomial functions satisfy

$$(w') \wedge (w'') = (-1)^{\|w'\| \, \|w''\|} (w'') \wedge (w')$$

(3) The circle product factors satisfy

$W' \circ W'' = (-1)^{\|W'\| \, \|W''\|} W'' \circ W'.$

In other words, because the x_{ij} anticommute $x \bullet x' = -x' \bullet x$ we have the commutation rules

$$(x_1) \wedge \cdots \wedge (x_p)[v_1 \cdots v_m] = (x_1) \cdots (x_p) \, [v_{\sigma 1} \cdots v_{\sigma m}]$$

because the functions are symmetric, while

$$(x_1) \wedge \cdots \wedge (x_p)[v_1 \cdots v_m] = (-1)^{\theta} (x_{\theta(1)}) \cdots (x_{\theta(p)}) \, [v_1 \cdots v_m]$$

because coordinates anticommute.

For homogeneous monomials of shape $\left(2^{p}\right)$, for example, we have

$$(x_1 x'_1)\cdots(x_p x'_p)[v_1\cdots v_m]$$

$$= \sum_{i_1<\cdots<i_p} \sum_{\sigma}(-1)^{\sigma}\, x_{\sigma(1),\, i_1} x'_{\sigma(1),\, i_1} \cdots x_{\sigma(p),\, i_p} x'_{\sigma(p),\, i_p}$$

$$(x_1 x'_1)\cdots(x_p x'_p)[v_1\cdots v_m]$$

$$= \sum_{i_1<\cdots<i_p} \sum_{\theta}(+1)^{\theta}\, x_{1,\, i_{\theta(1)}} x'_{1,\, i_{\theta(1)}} \cdots x_{p,\, i_{\theta(p)}} x'_{p,\, i_{\theta(p)}}$$

We see $(x_i x'_i) = (-)(x'_i x_i)$ while $(x_i x'_i)\,(x_j x'_j) = (+)(x_j x'_j)(x_i x'_i)$.

For homogeneous monomial functions of shape $\left(3^{p}\right)$ we have similarly

$$(x_1 x'_1 x''_1)\cdots(x_p x'_p x''_p)[v_1\cdots v_m]$$

$$= \sum_{i_1<\cdots<i_p} \sum_{\sigma}(-1)^{\sigma}\, x_{\sigma(1),\, i_1} x'_{\sigma(1),\, i_1} x''_{\sigma(1),\, i_1} \cdots x_{\sigma(p),\, i_p} x'_{\sigma(p),\, i_p} x''_{\sigma(p),\, i_p}$$

$$(x_1 x'_1 x''_1)\cdots(x_p x'_p x''_p)[v_1\cdots v_m]$$

$$= \sum_{i_1<\cdots<i_p} \sum_{\theta}(+1)^{\theta}\, x_{1,\, \theta(1)} x'_{1,\, \theta(1)} x''_{1,\, \theta(1)} \cdots x_{p,\, i_{\theta(p)}} x'_{p,\, i_{\theta(p)}} x''_{p,\, i_{\theta(p)}}$$

Note that $(x_i x'_i x''_i) = (-)(x'_i x_i x''_i)$ as before, but now
$(x_i x'_i x''_i)\,(x_j x'_j x''_j) = (-)(x_j x'_j x''_j)\,(x_i x'_i x''_i)$

Finally, we mentioned in the abstract of this talk some interesting facts about Mobius functions. While they did not occur in this talk, we record them here for the reader. They generalize the following well known facts.

Lemma The Mobius function and Zeta function of a lattice satisfies

$$\sum_{x \wedge b = 0} \mu(x,c) = \mu(0,c)\,\zeta(c,b).$$

$$\sum_{x \vee a = 1} \mu(c,x) = \zeta(a,c)\,\mu(c,1).$$

The generalizations are as follows.

Lemma The Mobius function and Zeta function of a lattice satisfies

$$\sum_{x \wedge b = a} \zeta(a,x)\,\mu(x,c) = \mu(a,c)\,\zeta(c,b).$$

$$\sum_{x \vee a = b} \mu(c,x)\,\zeta(x,b) = \zeta(a,c)\,\mu(c,b).$$

Useful Formulas (I)

For $\Pi_A = \Pi_n$ the lattice of partitions of $A = \{x_1, \cdots, x_n\}$ we have
1. The type of a partition $\pi \in \Pi_A$ is $\lambda = (1^{r_1} 2^{r_2} \cdots)$ if π has r_i blocks of size i.
2. The sign of a set partition π is $(-1)^{\pi} = (-1)^{r_2+2r_3+3r_4+\cdots}$ if π is of type $(1^{r_1} 2^{r_2} \cdots)$.
3. A segment $[\sigma \leq \pi]$ of Π_n is of type $(1^{r_1} 2^{r_2} \cdots)$ if i blocks of π are composed of r_i blocks of σ and then $[\sigma \leq \pi] \approx \Pi_1^{r_1} \times \Pi_2^{r_2} \times \cdots$
4. The sign of a segment $[\sigma \leq \pi]$ of type $(1^{r_1} 2^{r_2} \cdots)$ is $sign(\sigma, \pi) = (-1)^{\sigma} (-1)^{\pi} = (-1)^{r_2+2r_3+3r_4+\cdots}$.
5. If $[\sigma \leq \pi]$ is of type $\lambda = (1^{r_1} 2^{r_2} \cdots)$ the Mobius function satisfies $\mu(\sigma \leq \pi) = (-1)^{r_2+2r_3+3r_4+\cdots} \Pi_i \, [(i-1)!^{r_i}]$.
6. The number of partitions in Π_n of type λ is $\binom{n}{\lambda} = \dfrac{n!}{(1!)^{r_1} r_1! \, (2!)^{r_2} r_2! \cdots}$.
7. We have $\sum_{\sigma \in \Pi_n} \lvert \mu[0,\sigma] \rvert = \sum_{\lambda \vdash n} \binom{n}{\lambda} \left[(0!)^{r_1} (1!)^{r_2} \cdots \right] = \sum_{\lambda \vdash n} \left[{n \atop \lambda} \right] = n!$ $\sum_{\sigma \in \Pi_n} \lvert \mu(\tau, \sigma) \rvert \, \zeta(\sigma, \pi) = type(\tau, \pi)!$

Useful Formulas (II)

For Λ_n the lattice of partitions of an integer n we have the notations
1. $\lambda \vdash n$ denotes that λ is a partition of the integer n.
2. $\lambda = (\lambda_1 \geq \cdots \geq \lambda_r)$ the parts of λ given in decending order.
3. $\lambda = (1^{r_1}2^{r_2}\cdots)$ λ has r_1 parts equal to 1, r_2 parts equal to 2, and so on.
4. $\lambda! = \lambda_1!\lambda_2!\cdots = 1!^{r_1}2!^{r_2}\cdots$
5. $\lvert\lambda\rvert = r_1!r_2!\cdots$ with $\lambda = (1^{r_1}2^{r_2}\cdots)$.
6. The sign of a partition of an integer is $(-1)^{\lambda} = (-1)^{r_2+2r_3+3r_4+\cdots}$.

For $S_A = S_n$ the permutations of $x_1, \cdots, x_n$ we have
1. The type of a permutation $\theta \in S_n$ is $\lambda = (1^{r_1}2^{r_2}\cdots)$ if θ has r_i cycles of length i.
2. The *sign* of θ is $(-1)^m$ if θ is a product of m transpositions. The map $\theta \to sign(\theta)\,\theta$ from $Z(S_n)$ to itself is called the sign involution.
4. The number of permutations in S_n of type λ is $\left[{n \atop \lambda} \right] = \dfrac{n!}{(1)^{r_1} r_1! \, (2)^{r_2} r_2! \cdots}$.
5. Since there are $n!$ permutations we must have $\sum_{\lambda \vdash n} \left[{n \atop \lambda} \right] = n!$.