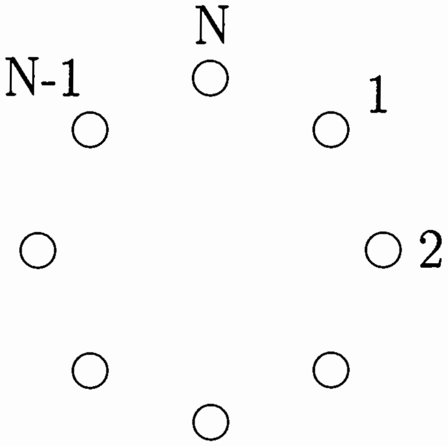
Magnetic interpretation of the Robinson - Schensted - Knuth algorithm

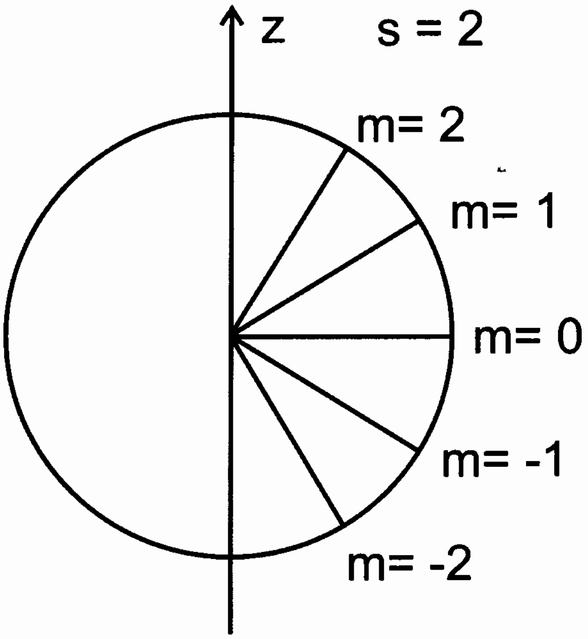
1. Introduction
2. Kinematics of the Heisenberg linear ring

$$\tilde{\boldsymbol{N}} = \{\boldsymbol{j} = \mathbf{1}, \mathbf{2}, \ldots, \boldsymbol{N}\}$$



- the set of nodes of a magnet
- a magnetic chain
- *the alphabet* of nodes

$$\tilde{n} = \{i = 1, 2, ..., n\}, \quad n = 2s + 1$$



- the set of basis states of a single node
- the alphabet of spins

The combinatorial construction

$$\tilde{n}^{\tilde{N}} = \{f : \tilde{N} \to \tilde{n}\}$$

- the set of all magnetic configurations of (N, s)
- or the set of all words of the length N in the alphabet $\tilde{n}$ of spins

$$|f >= |i_1, i_2, ..., i_N >, \quad i_j \in \tilde{n}, \quad j \in \tilde{N}$$

Kinematics of the Bethe Ansatz, rarefied Brillouin zones, and the duality of Weyl

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Abstract

Kinematics of the Heisenberg magnetic ring of N nodes, each with the spin s, has been discussed within the frame of the Weyl duality between actions of the symmetric group on N objects and the unitary group $U(n)$ of quantum symmetry of the single-node space ($n = 2s + 1$), in the quantum space of the magnet. The basis of wavelets is presented, and rarefied Brillouin zones, corresponding to orbits of the translation group with non-trivial stabilizers, are pointed out. The combinatorial Robinson - Schensted - Knuth algorithm, the Jucys-Murphy operators and Kerov - Kirillov - Reshetikhin bijection are applied to partial diagonalization of the Heisenberg Hamiltonian, as well as to classification of finite analogues of string solutions.

1. Introduction

We dedicate this report to Prof. James D. Louck, and deal with applications of unitary group representation techniques in magnetism

Monographs:

L. C. Biedenharn and J. D. Louck, Angular Momentum in Quantum Physics, Encycl. Math. Vol. 8

L. C. Biedenharn and J. D. Louck, The Racah-Wigner Algebra in Quantum Theory, Encycl. Math. Vol. 9, Cambridge Univ. Press 1981

Articles in proceedings of "Symmetry and Structural Properties of Condensed Matter" (SSPCM):

1990 J. D. Louck and L. C. Biedenharn, Special functions associated with SU(3) Wigner-Clebsh-Gordan coefficients

1994 L. C. Biedenharn, W. Y. C. Chen, M. A. Kohe, and J. D. Louck, The role of SU(2) 3n-j coefficients in SU(3)

1996 J. D. Louck, Combinatorial aspects of representations of the unitary group

1998 J. D. Louck, W. Y. C. Chen and W. H. Galbraith, Generating functions for SU(2) binary rec~~c~~oupling coefficients

2000 J. D. Louck, New perspectives on the unitary group and its tensor operators

2002 J. D. Louck, Skew Gelfand-Tsetlin patterns, lattice permutations, and skew pattern polynomials

We hope to continue this series...

3. The linear space of the quantum states of a magnet

$$\mathcal{H} = lc_{\mathbb{C}}\ \tilde{n}^{\tilde{N}} \cong h_1 \otimes h_2 \otimes \cdots \otimes h_N \cong h^{\otimes N}$$

for the whole magnet

$h = lc_{\mathbb{C}}\ \tilde{n}$ - for a single node

$dim\ h = n$

$< i_1 | i_2 >= \delta_{i_1 i_2}$ - the unitary structure of h

$< f_1 | f_2 >= \delta_{f_1 f_2}$ - the unitary structure of $\mathcal{H}$

$dim\ \mathcal{H} = n^N$

4. The duality of Weyl

Two dual actions on $\mathcal{H}$

$$A : \Sigma_N \times \mathcal{H} \to \mathcal{H} \qquad B : U(n) \times \mathcal{H} \to \mathcal{H}$$

$$A(\sigma) = \begin{pmatrix} f \\ f \circ \sigma^{-1} \end{pmatrix}, \quad f \in \tilde{n}^{\tilde{N}}, \quad \sigma \in \Sigma_N$$

$$B(a)\ |f\rangle = B(a)\ |\underbrace{i_1, i_2, \ldots, i_N}_{f}\rangle =$$

$$= \sum_{(i'_1, i'_2, \ldots, i'_N) \in \tilde{n}^{\tilde{N}}} a_{i'_1\, i_1} a_{i'_2\, i_2} \ldots a_{i'_N\, i_N} | \underbrace{i'_1, \ldots, i'_j, \ldots, i'_N}_{f'} \rangle,$$

$f \in \tilde{n}^{\tilde{N}}$

for

$$a = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{13} \\ a_{21} & a_{22} & \ldots & a_{23} \\ \ldots & \ldots & \ldots & \ldots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{pmatrix} \in U(n),$$

define the actions $\boldsymbol{A}$ and $\boldsymbol{B}$ on the set $\tilde{n}^{\tilde{N}}$ and - by the linear extension - on the space $\mathcal{H}$.

Actions $\boldsymbol{A}$ and $\boldsymbol{B}$ mutually commute

$$\underbrace{[A(\sigma), B(a)] = 0, \quad \sigma \in \Sigma_N, \quad a \in U(n)}_{\Downarrow}$$

there exist a basis in $\mathcal{H}$, in which irreps of each dual group Σ_N and $U(n)$, can be simultaneously determined.

Actions $\boldsymbol{A}$ and $\boldsymbol{B}$ decompose into appropriate irreps

$$\boldsymbol{A} = \sum_{\lambda \vdash N,\, |\lambda| \leq n} \oplus (dim\, D^{\lambda})\, \Delta^{\lambda} \text{ in } \Sigma_N,$$

$$\boldsymbol{B} = \sum_{\lambda \vdash N,\, |\lambda| \leq n} \oplus (dim\, \Delta^{\lambda})\, D^{\lambda} \text{ in } U(n),$$

so that

$$n^N = \sum_{\lambda \vdash N,\, |\lambda| \leq n} (dim\, D^{\lambda})(dim\, \Delta^{\lambda})$$

and

$\mathcal{H} = \sum_{\lambda \vdash N,\, |\lambda| \leq n} \oplus\, \mathcal{H}^{\lambda}$ decomposition of $\mathcal{H}$ into sectors λ with a definite permutational symmetry Δ^{λ}

$\lambda = \{N\}$ bosons	$\lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ parabosons
$\lambda = \{1^N\}$ fermions	$\lambda^{Tr} = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ parafermions

Note: in the Heisenberg model, each sector $\mathcal{H}^{\lambda}$, $|\lambda| \leq n$, is realised as a physically admited space (no superselection rules). Irreducible bases of the Weyl duality

$$\mathcal{H}^{\lambda} = U^{\lambda} \otimes V^{\lambda},$$

U^{λ} - a standard carrier space of the irrep D^{λ} of $U(n)$
V^{λ} - a standard carrier space of the irrep Δ^{λ} of Σ_N

so

$$U^{\lambda} = lc_{\mathbb{C}} WT(\lambda, \tilde{n}),$$
$$V^{\lambda} = lc_{\mathbb{C}} SYT(\lambda),$$

where

$WT(\lambda, \tilde{n})$	the set of all semistandard Young tableaux of the shape λ in the alphabet $\tilde{n}$ of spins or the set of all *Weyl tableaux* of the shape λ in the alphabet $\tilde{n}$ or a standard basis of U^{λ}
$SYT(\lambda)$	the set of all standard *Young tableaux* of the shape λ and the weight $\mathbf{1}^{N}$ in the alphabet N of nodes or a standard basis of V^{λ}

Thus

$$\{|\lambda\, t\, y\rangle \mid \lambda \vdash N, |\lambda| \leq n, t \in WT(\lambda, \tilde{n}), y \in SYT(\lambda)\}$$

is the irreducible standard basis in $\mathcal{H}$ along the duality of Weyl, or

$$\mathcal{H}^{\lambda} = lc_{\mathbb{C}}\ WT(\lambda, \tilde{n}) \times SYT(\lambda)$$

each sector $\mathcal{H}^{\lambda}$ is spanned on pairs $(t, y) \in WT(\lambda, \tilde{n}) \times SYT(\lambda)$

$$A(\sigma)|\lambda\, t\, y\,\rangle = \sum_{y' \in SYT(\lambda)} \Delta^{\lambda}_{y'y}(\sigma)\, |\lambda\, t\, y'\,\rangle$$

$$B(a)|\lambda\, t\, y\,\rangle = \sum_{t' \in WT(\lambda,\tilde{n})} D^{\lambda}_{t't}(a)\, |\lambda\, t'\, y\,\rangle$$

so that

$$\begin{array}{lll}
\text{sector} & \text{basis} & \text{label}\\
\downarrow & & \\
\lambda & t & y\\
 & \uparrow & \uparrow\\
 & \text{in } \Sigma_N & \text{in } U(n)\\
 & \text{repetition} & \text{label}
\end{array}$$

5. Kostka decomposition

The action $\boldsymbol{A} : \Sigma_N \times \mathcal{H} \rightarrow \mathcal{H}$, can be restricted to the set $\tilde{\boldsymbol{n}}^{\tilde{N}}$, on which it acts purely permutationally. Let

$$|f_o\rangle = |\underbrace{1\ldots1}_{\mu_1}\underbrace{2\ldots2}_{\mu_2}\underbrace{\ldots}_{\ldots}\underbrace{n\ldots n}_{\mu_n}\rangle$$

$\Rightarrow \boldsymbol{\mu} = (\mu_1, \mu_2, \ldots, \mu_n)$ - the *weight* of the magnetic configuration $f_o \in \tilde{\boldsymbol{n}}^{\tilde{N}}$, $\sum_{i \in \tilde{n}} \mu_i = N$, or $\boldsymbol{\mu} \models \boldsymbol{N}$ (a composition of N).

$\mathcal{O}_\mu = \{f_o \circ \sigma^{-1} | \sigma \in \Sigma_N\}$ - the orbit generated from $f_o \in \tilde{\boldsymbol{n}}^{\tilde{N}}$ by the action $\boldsymbol{A}$.

$$|\mathcal{O}(\nu)| = \frac{N!}{\prod_{i \in \tilde{n}} \nu_i!}$$ - the length of the orbit $\mathcal{O}(\nu)$

$\Sigma^\mu = \Sigma_{\mu_1} \times \Sigma_{\mu_2} \times \ldots \times \Sigma_{\mu_n}$ - the Young subgroup of Σ_N, the stabiliser of f_o

$\boldsymbol{A}\big|_{\mathcal{O}_\mu} = \boldsymbol{R}^{\Sigma_N:\Sigma^\mu}$ - the transitive representation of Σ_N, acting on the orbit $\mathcal{O}_\mu$.

$$\boldsymbol{R}^{\Sigma_N:\Sigma^\nu} = \begin{array}{lll} \sum_{\lambda \trianglerighteq \mu} & K_{\lambda\mu} & \Delta^\lambda \\ \uparrow & \uparrow & \\ \text{the order} & \text{Kostka} & \\ \text{of dominance} & \text{numbers} & \end{array}$$

decomposition into irreps of Σ_N.

It implies the equality

$$|\mathcal{O}_\mu| = \sum_{\lambda \trianglerighteq \mu} K_{\lambda\,\mu} dim\, \Delta^\lambda = \sum_{\lambda \trianglerighteq \mu} K_{\lambda\,\mu} K_{\lambda\,1^N}$$

where

$|\mathcal{O}_\mu|$ - the length of the orbit $\mathcal{O}_\mu$,

$K_{\lambda\,\mu} K_{\lambda\,1^N}$ - the number of pairs (t, y) of tableaux with

1^o $sh\, t = sh\, y = \lambda$,

2^o weight $t = \mu$,

3^o - weight $y = \{1^N\}$.

Robinson, Schensted and Knuth algorithm (RSK):
this equality is implied by the KKR bijection

$$\mathcal{O}_\mu \ni f \xrightarrow{RSK} (t, y) \equiv (P(f), Q(f)) \in \bigcup_{\lambda \trianglerighteq \mu} ST(\lambda,\, \mu) \times ST(\lambda, 1^N)$$

since each magnetic configuration $f \in \mathcal{O}_\mu$ can be looked at as a rectangular $n \times N$ matrix $m(f)$ with $\{0, 1\}$ entries

$$m_{ij}(f) = \begin{cases} 1 & \text{if } f(j) = i,\ j \in \tilde{N} \\ 0 & \text{otherwise} \end{cases}$$

E.g. let $N = 6, n = 3 (s = 1)$, and $|f\rangle = |212213\rangle (\mu = (2, 3, 1))$ then

$$m(f) = \begin{array}{c|cccccc} & 1 & 2 & 3 & 4 & 5 & 6 \\ \hline 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 2 & 1 & 0 & 1 & 1 & 0 & 0 \\ 3 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}$$

6. The Robinson-Schensted-Knuth algorithm

The correspondence

$$\mathcal{O}_\mu \ni f \longrightarrow m(f) \in M_{n\times N}(\mu, 1^N)$$

with

$$m(f) = \begin{array}{c|cccc|c} & 1 & 2 & \dots & N & \text{row sums} \\ \hline 1 & & & & & \mu_1 \\ 2 & & & & & \mu_2 \\ \vdots & & & & & \vdots \\ n & & & & & \mu_n \\ \hline \text{column sums} & 1 & 1 & \dots & 1 & N \end{array} \in M_{n\times N}(\mu, 1^N)$$

- establishes a bijection between elements of an orbit $\mathcal{O}_\mu$ of Σ_N on $\tilde{n}^{\tilde{N}}$ and $n \times N$ rectangular $(0, 1)$ matrices $M_{n\times N}(\mu, 1^N)$ with the row sums μ and the column sums $\{1^N\}$
- gives rise to sorting procedures for a magnetic configuration f, looked at as a word of the length N in the alphabet $\tilde{n}$ of spins; in this way one arrives at *the Robinson-Schensted-Knuth algorithm*

$$\sum_{i\in\tilde{n}} m_{ij}(f) = 1, \quad j \in \tilde{N} \quad \text{column sums} \Rightarrow \{1^N\}$$

$$\sum_{j\in\tilde{N}} m_{ij}(f) = \nu_i, \quad i \in \tilde{N} \quad \text{row sums} \Rightarrow \mu$$

$m_{i\bullet}$ - the i - th row of the matrix $m(f)$ it is a sequence of the length N of $\{0, 1\}$; each '1" points the node occupied by the spin $i \in \tilde{n}$

$m_{\bullet j}$ - the j - th column of $m(f)$ it is a sequence of the length n with one "1" and other zeros; the "1" points the spin which occupies the node $j \in \tilde{N}$

Let

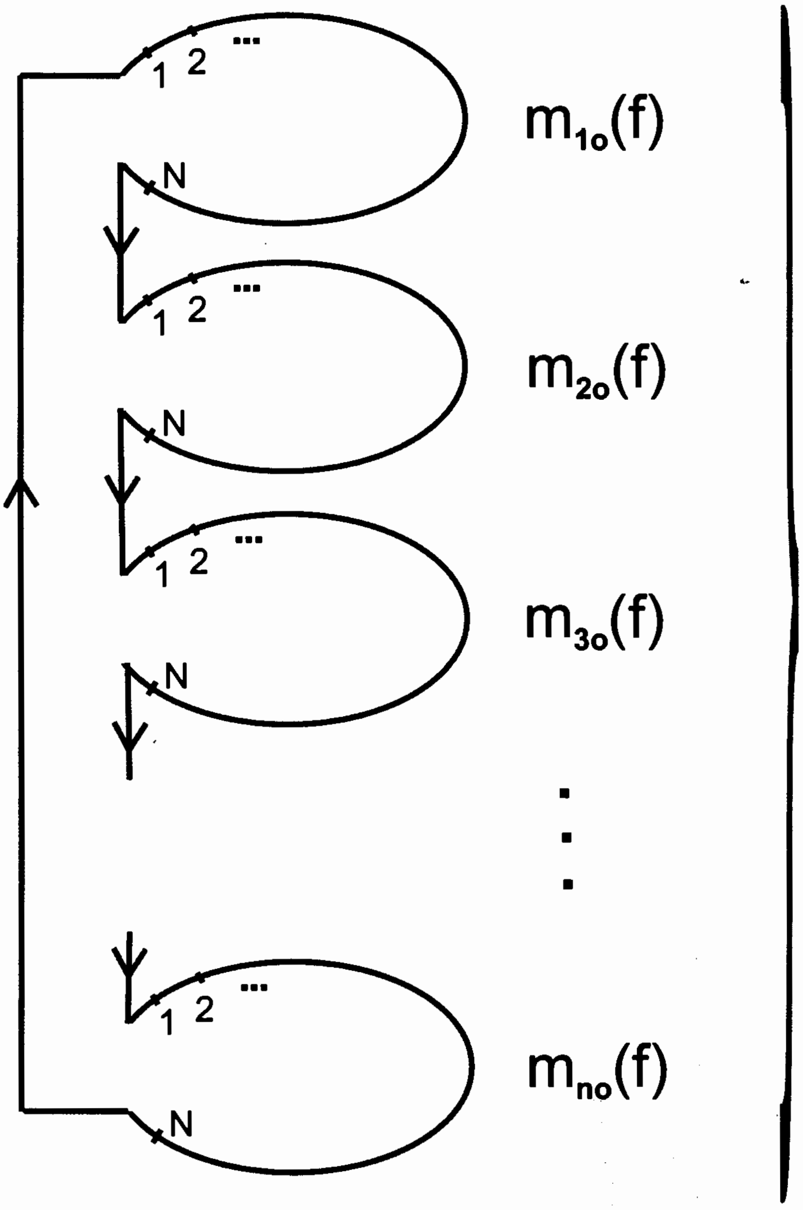
$$z(f) = m_{1\bullet}(f) m_{2\bullet}(f) \cdots m_{n\bullet}(f) \ \ (m_{1\bullet}(f))$$

be the sequence of the length nN, with the last element at $m_{n\bullet}(f)$ connected with the first element of $m_{1\bullet}(f)$. $z(f)$ can be looked at as an n - tuple cover $(\tilde{n}, \tilde{N}, f)$ of the chain $\tilde{N}$, corresponding to magnetic configuration f.

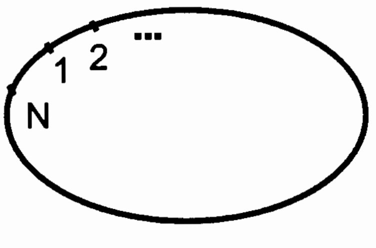
In the cyclic order of the cover $(\tilde{n}, \tilde{N}, f)$, each node of the chain $\tilde{N}$ acquires a new number, given by

$$j = \sum_{i'=1}^{i-1} \mu_{i'} + l, \ \ 1 \leq l \leq \mu_i$$

for the consecutive l - the node in the convolution $m_{i\bullet}(f)$
For example, for $|f\rangle = |212213\rangle (\mu = (2, 3, 1))$
the initial label on the chain $\tilde{N}$ $\mathbf{123456}$
the label on the cover $(\tilde{3}, \tilde{6}, f)$ $\mathbf{314526}$
the permutation $\sigma = \binom{123456}{251346} = (12543)(6)$
word $(\sigma(f)) = \mathbf{251346}$



the cover
$(\tilde{n},\tilde{N},f)$
of the magneti
configuration f
the rows $m_{io}(f)$
are consecutiv
convolutions
of this cover
$m_{ij}(f) = \delta_{i,f(j)}$



$\tilde{N}$ - the base $\tilde{N}$ of the cover $(\tilde{n},\tilde{N},f)$

Arrows indicate the cyclic order in the cover $(\tilde{n},\tilde{N},f)$

The natural cyclic order on the cover $(\tilde{n}, \tilde{N}, f)$ defines uniquely a permutation $\sigma(f) \in \Sigma_N$ by

$$\sigma(f) = \begin{pmatrix} j \\ \sigma(f,j) \end{pmatrix}, \quad j \in \tilde{N},$$

where $\sigma(f,j)$, $j \in \tilde{N}$ denotes the label on the chain $\tilde{N}$ of that node which has the number j in the cover $(\tilde{n}, \tilde{N}, f)$. The word

$$\boldsymbol{word}\,(\sigma(f)) = \sigma(f,1)\sigma(f,2)\ldots\sigma(f,N)$$

in the alphabet of nodes serves as the starting point for the Robinson-Schensted-Knuth algorithm

Let

$$\tilde{n}^* = \bigcup_{N=0}^{\infty} \tilde{n}^{\tilde{N}} \text{ -- the free monoid on the alphabet of spins}$$

$$RSK : \tilde{n}^* \to \bigcup_{N=0}^{\infty} \quad \bigcup_{\lambda \vdash N,\, |\lambda| \leq n} \quad WT(\lambda, \tilde{n}) \times SYT(\lambda)$$

the irreducible basis of the Weyl duality

$$RSK(f) = (P(f), Q(f))$$

such that $sh\,P(f) = sh\,Q(f) = \lambda$, $P(f) \in WT(\lambda, \tilde{n})$, $Q(f) \in SYT(\lambda)$

The RSK algorithm is defined in terms of the famous Schensted mapping

$$\boldsymbol{P : \tilde{n}^* \to Tab(\tilde{n})} \quad \begin{array}{l}\text{the set of all semistandard}\\ \text{tableaux in the alphabet}\\ \boldsymbol{\tilde{n}} \text{ of spins}\end{array}$$

$$\boldsymbol{Tab(\tilde{n}) = \bigcup_{N=0}^{\infty} \bigcup_{\lambda \vdash N,\, |\lambda| \leq n} WT(\lambda, \tilde{n})}$$

involving the *insertion procedure*, well known from the literature.

It $\boldsymbol{f}$ is a row, that is $\boldsymbol{sh\, P(f) = \{N\}}$, or

$$\boldsymbol{P(f) = \boxed{f(1)}\boxed{f(2)}\boxed{\ldots}\boxed{f(N)}}$$

which implies, by standardness, $\boldsymbol{f(1) \leq f(2) \leq \ldots \leq f(N)}$, then

$$\boldsymbol{P(fx) = \begin{cases} \boxed{f(1)}\boxed{f(2)}\boxed{\ldots}\boxed{f(N)}\boxed{x} & \text{for } x \geq f(N) \\ \begin{array}{l}\boxed{f(1)}\boxed{\ldots}\boxed{x}\boxed{\ldots}\boxed{f(N)} \\ \boxed{y}\end{array} & \begin{array}{l}\text{for } x < f(N) \\ y = f(j) \text{ such that} \\ f(j-1) \leq x < f(j\end{array} \end{cases}}$$

In general,

$$\boldsymbol{P(fx) = P(P(f)x), \quad f \in \tilde{n}^*, x \in \tilde{n}}$$

Thus $\boldsymbol{P(f)}$ is constructed consecutively, and $\boldsymbol{Q(f)}$ is formed paralelly.

In our construction

$$f \to m(f) \to z(f) \to \sigma(f) \to word(\sigma(f))$$

we have

$$Q(f) = P(word(\sigma(f))$$

and $P(f)$ is constructed paralelly

The word

$$std(f) = word(\sigma(f)^{-1})$$

(in the alphabet $\tilde{N}$ of nodes) is known as the standardization of the word f. Lascoux, Leclerc and Thibon shown recently that the notion of standardization allows us to generalize the result of Frobenius algebra $\mathbb{C}[\Sigma_N]$ for the case $N = n,\ \mu = \{1^N\}$

$$P(\sigma) = Q(\sigma^{-1}), \text{ for } \sigma \in \Sigma_N \subset \tilde{N}^{\tilde{N}}$$

$$\sigma = \begin{pmatrix} 1 & \dots & N \\ \sigma(1) & \dots & \sigma(N) \end{pmatrix} \to word(\sigma) = \sigma(1)\sigma(2)\dots\sigma(N$$

The generalization is

$$Q(f) = P(std(f^{-1}))$$

which coincides with our description.

An example, $N = 6, n = 3, |f\rangle = |babbac\rangle$ $\mu = (2,3,1)$

$$m(f) = \begin{array}{c|cccccc|c} & 1 & 2 & 3 & 4 & 5 & 6 & \\ \hline a & 0 & 1 & 0 & 0 & 1 & 0 & 2 \\ b & 1 & 0 & 1 & 1 & 0 & 0 & 3 \\ c & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ \hline & 1 & 1 & 1 & 1 & 1 & 1 & \end{array}$$

$$\begin{array}{rl} z(f) = & \overbrace{(010010)}^{m_{a\bullet}(f)}\overbrace{(101100)}^{m_{b\bullet}(f)}\overbrace{(000001)}^{m_{c\bullet}(f)}[(01\ldots \\ j = & \quad 1 \quad 2 \qquad 3 \;\; 4 \, 5 \qquad\qquad\qquad 6 \\ \sigma(f,j) = & \quad 2 \quad 5 \qquad 1 \;\; 3 \, 4 \qquad\qquad\qquad 6 \end{array}$$

$$\sigma(f) = \begin{pmatrix} 123456 \\ 251346 \end{pmatrix}, \quad \sigma(f^{-1}) = \begin{pmatrix} 123456 \\ 314256 \end{pmatrix}$$

word $(\sigma(f)) = 251346$ $word(\sigma(f)^{-1}) \equiv std(f) = 314256$ $P(word(\sigma(f))) = Q(f) = \begin{array}{|c|c|c|c|}\hline 1 & 3 & 4 & 6 \\ \hline 2 & 4 \\ \cline{1-2} \end{array}$

$$\begin{array}{lccccccl} \text{step} & 1 & 2 & 3 & 4 & 5 & 6 & \\ Q(\text{step } l) & \begin{array}{|c|}\hline 2 \\ \hline \end{array} & \begin{array}{|c|c|}\hline 2 & 5 \\ \hline \end{array} & \begin{array}{|c|c|}\hline 1 & 5 \\ \hline 2 \\ \cline{1-1} \end{array} & \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 5 \\ \hline \end{array} & \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 & 5 \\ \cline{1-2} \end{array} & \begin{array}{|c|c|c|c|}\hline 1 & 3 & 4 & 6 \\ \hline 2 & 5 \\ \cline{1-2} \end{array} & = Q(f) \\ P(\text{step } l) & \begin{array}{|c|}\hline a \\ \hline \end{array} & \begin{array}{|c|c|}\hline a & a \\ \hline \end{array} & \begin{array}{|c|c|}\hline a & a \\ \hline b \\ \cline{1-1} \end{array} & \begin{array}{|c|c|}\hline a & a \\ \hline b & b \\ \hline \end{array} & \begin{array}{|c|c|c|}\hline a & a & b \\ \hline b & b \\ \cline{1-2} \end{array} & \begin{array}{|c|c|c|c|}\hline a & a & b & c \\ \hline b & b \\ \cline{1-2} \end{array} & = P(f) \end{array}$$

immediate RSK for $\boldsymbol{f} = \boldsymbol{babbac}$

$$\boldsymbol{P}(\text{step}\,\boldsymbol{l})\quad \begin{array}{|c|}\hline b\\\hline\end{array}\ \begin{array}{|c|}\hline a\\\hline b\\\hline\end{array}\ \begin{array}{|c|c|}\hline a&b\\\hline b&\multicolumn{1}{c}{}\\\cline{1-1}\end{array}\ \begin{array}{|c|c|c|}\hline a&b&b\\\hline b&\multicolumn{2}{c}{}\\\cline{1-1}\end{array}\ \begin{array}{|c|c|c|}\hline a&a&b\\\hline b&b&\multicolumn{1}{c}{}\\\cline{1-2}\end{array}\ \begin{array}{|c|c|c|c|}\hline a&a&b&c\\\hline b&b&\multicolumn{2}{c}{}\\\cline{1-2}\end{array}\quad = \boldsymbol{P}(\boldsymbol{f})$$

$$\boldsymbol{Q}(\text{step}\,\boldsymbol{l})\quad \begin{array}{|c|}\hline 1\\\hline\end{array}\ \begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}\ \begin{array}{|c|c|}\hline 1&3\\\hline 2&\multicolumn{1}{c}{}\\\cline{1-1}\end{array}\ \begin{array}{|c|c|c|}\hline 1&3&4\\\hline 2&\multicolumn{2}{c}{}\\\cline{1-1}\end{array}\ \begin{array}{|c|c|c|}\hline 1&3&4\\\hline 2&5&\multicolumn{1}{c}{}\\\cline{1-2}\end{array}\ \begin{array}{|c|c|c|c|}\hline 1&3&4&6\\\hline 2&5&\multicolumn{2}{c}{}\\\cline{1-2}\end{array}\quad = \boldsymbol{Q}(\boldsymbol{f})$$

7. Conclusions

- The Heisenberg model of a magnetic chain $(\tilde{\boldsymbol{n}}, \tilde{\boldsymbol{N}})$ is a natural physical realization of a huge amount of combinatorial theory of representations of the symmetric and unitary groups
- Within this model, all sectors $\mathcal{H}^{\lambda}$ of the duality of Weyl between the symmetric and unitary group are realised without any superselection rules
- The algorithm of Robinson-Schensted-Knuth, applied to each Σ_{N} - orbit $\mathcal{O}_{\mu}$ on the set of magnetic configurations yields a complete classification scheme of the irreducible basis of the Weyl duality
- Presentation of a magnetic configuration $\boldsymbol{f} \in \tilde{\boldsymbol{n}}^{\tilde{\boldsymbol{N}}}$ as an $\boldsymbol{n}$ - tuple cover $(\tilde{\boldsymbol{n}}, \tilde{\boldsymbol{N}}, \boldsymbol{f})$ of the magnetic chain $\tilde{\boldsymbol{N}}$ demonstrates in a transparent way the magnetic interpretation of combinatorics (in particular - sorting procedures) involved in the Robinson-Schensted-Knuth algorithm, and reflect the algebraic structure of the group algebra $\mathbb{C}[\Sigma_{N}]$
- It is somehow suprising that the bijection of RSK on an orbit $\mathcal{O}_{\mu}$ of Σ_{N} yields a complete irreducible basis. Moreover, the bijection of Kerov, Kirillov and Reshetikhin implies hitherto a complete classification of eigenstates of the Heisenberg Hamiltonian. In which way such a tiny and sophisticated information on dynamics of the magnetic chain (involving solutions of highly non-linear spectral problems) is expressible in terms of purely combinatorial operations on Σ_{N} - orbits ?

5. The basis of wavelets

$$C_N \subset \Sigma_N$$

C$_N$ is the translation symmetry group of the chain $\tilde{N}$, and thus its irreps

$$\Gamma_k(j) = e^{2\pi ikj/N}$$

label the momenta, and the set of these labels

$$B = \{k = 0, \pm 1, \pm 2, \ldots, \begin{array}{ll} \pm(N/2-1), N/2 & \text{for N even} \\ \pm(N-1)/2, & \text{for N odd} \end{array}\Big\}$$

constitutes the Brillouin zone.

Under the subduction $A \downarrow C_N$, each Σ_N-orbit $\mathcal{O}_\mu$ decomposes into C_N-orbits

$$\mathcal{O}_\mu = \bigcup_{\kappa \in k(N)} m(\mu, \kappa)\mathcal{O}_{\mu\kappa}$$

$$m(\mu,\ \kappa) = \frac{\kappa}{N} \sum_{\kappa' \in K(\mathrm{gcd}(\mu/\kappa))} M(\kappa') \frac{(\frac{N}{\kappa\kappa'})!}{\prod_{i \in \tilde{n}} (\frac{\mu_i}{\kappa\kappa'})!}, \text{ or } 0$$

K(N) - the lattice of all divisors of N

$M(\kappa')$ - the Möebius function

gcd(μ/κ) - the greatest common divisor of integers $\mu_i/\kappa, i \in \tilde{n}$

Equivalently,

$$A|_{\mathcal{O}_\mu} = \sum_{\kappa \in k(N)} m(\mu, \kappa) R^{N:\kappa}$$

$R^{N:\kappa}$ - a transitive representation of C_N with the stabilizer $C_\kappa \lhd C_N$

The Fourier decomposition for the regular orbit ($\kappa = 1$)

$\boldsymbol{R}^{N:1} = \sum_{\boldsymbol{k}\in\boldsymbol{B}} \oplus \boldsymbol{\Gamma}_{\boldsymbol{k}}$, each quasimomentum k
appears exactly once

and for $\boldsymbol{\kappa}$-tuply rarefied orbits

$$\boldsymbol{R}^{N:\boldsymbol{\kappa}} = \sum_{\boldsymbol{k}\in\boldsymbol{B}/\boldsymbol{\kappa}} \oplus \boldsymbol{\Gamma}_{\boldsymbol{k}},$$

where

$\boldsymbol{B}/\boldsymbol{\kappa} = \{\boldsymbol{k} \in \boldsymbol{B} \mid \boldsymbol{k}/\boldsymbol{\kappa} \in \mathbb{Z}\}$ - a $\boldsymbol{\kappa}$-tuply rarefied
Brillouin zone

E.g., for N=6, n=2

j
1 − − − + + +
2 + − − − + +
3 + + − − − +
4 + + + − − −
5 − + + + − −
6 − − + + + −

a regular orbit

$$\boldsymbol{\kappa} = \mathbf{1}$$

$$\boldsymbol{C_{\kappa}} = \{\mathbf{6}\}$$

1 − + + − + +
2 + − + + − +
3 + + − + + −

a doubly rarefied orbit

$$\boldsymbol{\kappa} = \mathbf{2}$$

$$\boldsymbol{C_{\kappa}} = \{\mathbf{3}, \mathbf{6}\}$$

1 − + − + − +
2 + − + − + −

a triply rarefied orbit

$$\boldsymbol{\kappa} = \mathbf{3}$$

$$\boldsymbol{C_{\kappa}} = \{\mathbf{2}, \mathbf{4}, \mathbf{6}\}$$

Brillouin zone for N=6

$$\begin{array}{lcccccc}
B = \{0,\ \pm 1,\ \pm 2,\ 3\} & -2 & -1 & 0 & 1 & 2 & 3 \\
B/2 = \{0,\ \pm 2\} & -2 & & 0 & & 2 & \\
B/3 = \{0,\ 3\} & & & 0 & & & 3
\end{array}$$

One chooses *the basis of wavelets* on each linear $lc_{\mathbb{C}}\mathcal{O}_{\mu\kappa}$ of a C_N - orbit

$$|\mu\,\kappa\,t\,k >= \sqrt{\frac{\kappa}{N}} \sum_{j=1}^{N/\kappa} e^{2\pi i k j/N}\, |t\,j >$$

which provides the momentum representation in the space $\mathcal{H}$ of quantum states of the magnet