An approach to the hypergeometric transformations through the Cauchy kernel

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2004.8.2, Nankai Institute(Tianjin)

based on:

1. Y.K: Adv. in Math. 187 (2004), p53–97.
2. Y.K, M.Noumi: Indag. Math. 14 (2003), p395–421.
3. Y.K: in preparation

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★ Multiple basic hypergeometric series (BHS) of type A

♡ Origin (for ordinary hypergeometric series)

- **W.Holman, L.Biedenharn, J.Louck**
  Quantum mechanics (Ired. repr. of $SU(n)$)

♠ Derivation

- **S.Milne** A certain algebraic invariants and q-difference equation
- **S.Milne, G.Lily, G.Bhatnager, C.Krattenthaler, M.Schlosser**
  Multidimensional matrix inversion
  (Multidimensional Bailey lattice)
- **H.Rosengren, M.Schlosser** Karlsson-Minton type reduction formula
- **M.Noumi-Y.K** Cauchy kernel and Macdonald's q-difference operators
  (This talk)

♡ Main theme of my talk

$$\sum_{l(\lambda)\leq \min(n,m)} S_\lambda(x)S_\lambda(y) = \prod_{1\leq i\leq n, 1\leq k\leq m} \frac{1}{1-x_iy_k}$$

(Cauchy Kernel)

$\Downarrow$ in this talk.

Various (multiple) hypergeometric transformations in Bailey hierarchy

Example

The Bailey transformation formula for ${}_{10}W_9$ series

$$
\begin{aligned}
&{}_{10}W_9\left[a; b, c, d, e, f, \lambda a q^{N+1}/ef, q^{-N}; q; q\right] \\
&= \frac{(aq)_N (aq/ef)_N (\lambda q/e)_N (\lambda q/f)_N}{(aq/e)_N (aq/f)_N (\lambda q)_N (\lambda q/ef)_N} \\
&\qquad {}_{10}W_9\left[\lambda; \lambda b/a, \lambda c/a, \lambda d/a, e, f, \right. \\
&\qquad\qquad \left. \lambda a q^{N+1}/ef, q^{-N}; q; q\right]
\end{aligned}
$$

where $\lambda = a^2 q/bcd$.

§Preliminaries and multiple Euler transformation

♡ Cauchy reproducing kernel for Schur function

$$\prod(x;y) := \sum_{l(\lambda)\le \min(n,m)} S_\lambda(x)S_\lambda(y) = \prod_{1\le i\le n}\prod_{1\le k\le m}\frac{1}{1-x_iy_k}$$

for the variables of $x = (x_1, \cdots x_n)$ and $y = (y_1, \cdots, y_m)$.

Where $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_n)$: partition

$l(\lambda)$: the length of the partition λ

$S_\lambda(x)$: Schur function of the partition λ defined by

$$S_\lambda(x) := \frac{\det\left(x_i^{\lambda_j+n-j}\right)_{i,j=1}^{n}}{\det\left(x_i^{n-j}\right)_{i,j=1}^{n}}$$

The Schur functions $S_\lambda(x)$ are a family of homogeneous polynomials that are parametrized by partitions λ with $l(\lambda) \le n$.

★ The third Heine transformation formula for basic hypergeometric series (BHS)

Here, we assume $0 < q < 1$.

$$ {}_2\phi_1\left[\begin{matrix} a, b \\ c \end{matrix}; q; u\right] = \frac{(abu/c)_\infty}{(u)_\infty} {}_2\phi_1\left[\begin{matrix} c/a, c/b \\ c \end{matrix}; q; abu/c\right] $$

where we denote the basic hypergeometric series ${}_{r+1}\phi_r$ as

$$ {}_{r+1}\phi_r\left[\begin{matrix} a_0, a_1, \ldots a_r \\ c_1, \ldots c_r \end{matrix}; q; u\right] := \sum_{n\in\mathbb{N}} \frac{(a_0)_n (a_1)_n \ldots (a_r)_n}{(c_1)_n \ldots (c_r)_n (q)_n} u^n. $$

and

$$ (a)_\infty := \prod_{n\in\mathbb{N}} (1-aq^n), \quad (a)_k := \frac{(a)_\infty}{(aq^k)_\infty} \quad \text{for } k \in \mathbb{C} $$

is a q-shifted factorial.

Note that the third Heine transformation is a q-analogue of the Euler transformation formula for the Gauss' hypergeometric function ${}_2F_1$.

★ Definition of multiple BHS

We call the multiple BHS the formal power series of the following form with the condition that is invariant under the permutation of the subscript

$$\sum_{\beta\in\mathbb{N}^n} \frac{\Delta(xq^\beta)}{\Delta(x)} \times \text{(basic hypergeometric stuff)}$$

where $\beta = (\beta_1, \ldots, \beta_n) \in \mathbb{N}^n$, $|\beta| = \sum_{i=1}^n \beta_i$,

$$\Delta(x) = \prod_{1\le i<j\le n} (x_i - x_j)$$

and

$$\Delta(xq^\beta) = \prod_{1\le i<j\le n} (x_i q^{\beta_i} - x_j q^{\beta_j})$$

are Vandermonde determinants of $x = (x_1, \ldots, x_n)$ and $xq^\beta = (x_1 q^{\beta_1}, \ldots, x_n q^{\beta_n})$ respectively.

♡ Euler transformation formula for multiple BHS of type A

Theorem (Y.K) Suppose that none of denominators vanish. Then we have the Euler transformation formula between basic hypergeometric series in $U(n+1)$ and in $U(m+1)$

$$\sum_{\gamma\in\mathbb{N}^n} u^{|\gamma|}\frac{\Delta(xq^{\gamma})}{\Delta(x)} \prod_{1\le i,j\le n}\frac{(a_jx_i/x_j)_{\gamma_i}}{(qx_i/x_j)_{\gamma_i}}$$

$$\prod_{1\le i\le n,1\le k\le m}\frac{(b_kx_iy_k/x_ny_m)_{\gamma_i}}{(cx_iy_k/x_ny_m)_{\gamma_i}}$$

$$=\frac{(a_1\cdots a_nb_1\cdots b_mu/c^m)_\infty}{(u)_\infty}$$

$$\sum_{\delta\in\mathbb{N}^m}(a_1\cdots a_nb_1\cdots b_mu/c^m)^{|\delta|}\frac{\Delta(yq^{\delta})}{\Delta(y)}$$

$$\prod_{1\le k,l\le m}\frac{((c/b_l)y_k/y_l)_{\delta_k}}{(qy_k/y_l)_{\delta_k}}$$

$$\prod_{1\le i\le n,1\le k\le m}\frac{((c/a_i)x_iy_k/x_ny_m)_{\delta_k}}{(cx_iy_k/x_ny_m)_{\delta_k}}$$

for $a_1^{-1},\ldots,a_n^{-1},b_1/c,\ldots,b_m/c\in\mathbb{C}$.

§From Cauchy kernel to multiple Euler transformation

Step1 Action of (Macdonald type) q-difference operator on each variables

(Special case of) Macdonald's q-difference operator $D_x(u)$ (acting on Schur functions) is defined by

$$D_x(u) = \sum_{K\subset[1,\ldots,n]} (-u)^{|K|} q^{\binom{|K|}{2}} \prod_{i\in K, j\notin K} \frac{1-qx_i/x_j}{1-x_i/x_j} \prod_{i\in K} T_{x_i,q}$$
$$= \frac{1}{\Delta(x)} \prod_{1\leq i\leq n} (1-uT_{x_i,q})\Delta(x)$$

where T_{q,x_i} are the q-shift operators in x_i

The Schur function $S_\lambda(x)$ are the joint eigenfunctions of $D_x(u)$.

$$D_x(u)S_\lambda(x) = S_\lambda(x)\prod_{i=1}^{n}(1-uq^{\lambda_i+n-i}).$$

♡ A source identity due to Mimachi-Noumi, Kirillov-Noumi

Suppose that $n \geq m$. Then

$$D_x(u)\prod(x;y) = (u)_{n-m} D_y(uq^{n-m})\prod(x;y).$$

This is a self duality for the Schur function and the identity for rational functions of the following form:

$$\sum_{K\subset[1,\ldots,r]} (-u)^{|K|} q^{\binom{|K|}{2}} \prod_{i\in K, j\notin K} \frac{1-qz_i/z_j}{1-z_i/z_j}$$
$$\prod_{i\in K, 1\leq k\leq p} \frac{1-z_i w_k}{1-qz_i w_k}.$$

<u>Step2</u> Multiple principal specialization to the source identity

Define the specialization $y = (y_1, \ldots, y_r)$ to the points $p_\alpha(v;x)$ by

$$y = (y_1, \ldots, y_r) \rightarrow$$
$$p_\alpha(v;x) = (-v/x_i, -v/x_i q, \ldots, -v/q^{\alpha_1 - 1}, -v/x_2, \ldots$$
$$-v/x_n, \ldots, -v/x_n q^{\alpha_n - 1}).$$

By specializing $z \to p_\alpha(1;x)$ and $w \to p_\beta(q^{-1};y)$ to the source identity, we have :

♠ The terminating version of Euler transformation for multiple BHS

$$\sum_{\gamma\in\mathbb{N}^n,\gamma\le\alpha} (q^{|\alpha|-|\beta|}u)^{|\gamma|}\frac{\Delta(xq^\gamma)}{\Delta(x)} \prod_{1\le i,j\le n}\frac{(q^{-\alpha_j}x_i/x_j)_{\gamma_i}}{(qx_i/x_j)_{\gamma_i}} \prod_{1\le i\le n,1\le k\le m}\frac{(q^{\beta_k}x_iy_k)_{\gamma_i}}{(x_iy_k)_{\gamma_i}}$$

$$= (u)_{|\alpha|-|\beta|}\sum_{\delta\in\mathbb{N}^m,\delta\le\beta}(u)^{|\delta|}\frac{\Delta(yq^\delta)}{\Delta(y)} \prod_{1\le k,l\le m}\frac{(q^{-\beta_l}y_k/y_l)_{\delta_k}}{(qy_k/y_l)_{\delta_k}} \prod_{1\le i\le n,1\le k\le m}\frac{(q^{\alpha_i}x_iy_k)_{\delta_k}}{(x_iy_k)_{\delta_k}}.$$

Step 3 Analytic continuation and appropriate change of parameters

So we arrive at multiple Euler transformation!!!

§From multiple Euler transformation formula to (multiple) hypergeometric transformations

Multiple Euler transformation in some cases

$$\Downarrow$$

<u>Step</u> Taking the coeff.of u^N and appropriate change of parameters

$$\Downarrow$$

Various types of hypergeometric transformations (including very well-poised hypergeometric series)

In sake of $m = 1$ case of our Euler transformation formula, we can give a elementary proof of multiple q-Pfaff-Saalschütz summation formula due to S.C.Milne including the ordinary q-Pfaff-Saalscütz summation formula:

$$
{}_3\phi_2\left[\begin{matrix} a, b, q^{-l} \\ c, q^{1-l}ab/c \end{matrix}; q; q\right] = \frac{(c/a)_l(c/b)_l}{(c)_l(c/ab)_l}.
$$

As is well-known, the ordinary q-Pfaff-Saalschütz summation formula is derived from the third Heine transformations

♡ Beyond the q-Pfaff-Saalschütz summation

Hereafter, I shall present some results on transformation formula for multiple very well-poised BHS arising from our Euler transformation. In fact, some useful and interesting special cases are included (especially $m = 1$ case of each formula).

★ Very-well-poised BHS

The BHS ${}_{n+1}\phi_n$ is "well-poised" if $a_0 q = a_1 c_1 = \cdots = a_n c_n$. It is called very-well-poised if it is well-poised and if $a_1 = q\sqrt{a_0}$ and $a_2 = -q\sqrt{a_0}$. Namely, the very-well-poised ${}_{n+1}\phi_n$ is expressed as the following form

$$\sum_{k \in \mathbb{N}} \frac{1 - a_0 q^{2k}}{1 - a_0} \frac{(a_0)_k (a_3)_k \cdots (a_n)_k}{(q)_k (a_0 q/a_3)_k \cdots (a_0 q/a_n)_k} u^k$$
$$:= {}_{n+1}W_n \left[a_0; a_3, \ldots, a_n; q; u\right]$$

★ Definition of the multiple very well-poised BHS $W^{n,m}$

$$W^{n,m}\left(\begin{matrix} \{a_i\}_n \\ \{x_i\}_n \end{matrix} \middle| s; \{u_k\}_m; \{v_k\}_m; z \right)$$

$$= \sum_{\gamma \in \mathbb{N}^n} z^{|\mu|} \prod_{1 \le i < j \le n} \frac{\Delta(xq^\gamma)}{\Delta(x)}$$

$$\prod_{1 \le i \le n} \frac{1 - q^{|\gamma| + \gamma_i} s x_i / x_n}{1 - s x_i / x_n}$$

$$\prod_{1 \le j \le n} \frac{(s x_j / x_n)_{|\gamma|}}{((sq/a_j) x_j / x_n)_{|\gamma|}} \left(\prod_{1 \le i \le n} \frac{(a_j x_i / x_j)_{\gamma_i}}{(q x_i / x_j)_{\gamma_i}} \right)$$

$$\prod_{1 \le k \le m} \frac{(v_k)_{|\gamma|}}{(sq/u_k)_{|\gamma|}} \left(\prod_{1 \le i \le n} \frac{(u_k x_i / x_n)_{\gamma_i}}{((sq/v_k) x_i / x_n)_{\gamma_i}} \right),$$

where $\{u_i\}_n$ means $u_1, \ldots, u_n$ according to this order.

Note that in the case when $n = 1$, $W^{1,m}$ reduces ${}_{2m+4}W_{2m+3}$ series.

★ Duality transformation formula

Here, we give a multiple transformation formula of Watson type for very-well-poised basic hypergeometric series including as special case,

$$
{}_8W_7\left[a; b, c, d, e, q^{-N}; q; \frac{a^2q^{N+2}}{bcde}\right]
$$

$$
= \frac{(a^2q^2/bcde)_N(e)_N(aq)_N}{(aq/b)_N(aq/c)_N(aq/d)_N}
$$

$$
{}_4\phi_3\left[\begin{matrix} q^{-N}, aq/be, aq/ce, aq/de \\ q^{1-N}/e, a^2q^2/bcde, aq/e \end{matrix}; q; q\right]
$$

One can check that this formula can be obtained by combining a Watson transformation

$$
{}_8W_7\left[a; b, c, d, e, q^{-N}; q, \frac{a^2q^{2+N}}{bcde}\right]
$$

$$
= \frac{(aq)_N(aq/de)_N}{(aq/d)_N(aq/e)_N} {}_4\phi_3\left[\begin{matrix} q^{-N}, d, e, aq/bc \\ aq/b, aq/c, deq^{-N}/a \end{matrix}; q, q\right]
$$

and Sears transformation. (Same results including balanced case is rediscovered by H.Rosengren (to appear in Constr. Approx,))

♡ Duality transformation formula for multiple BHS)

$$W^{m,n+1}\left(\begin{matrix}\{b_k\}_m\\ \{y_k\}_m\end{matrix}\middle| a; c, \{d_i x_i/x_n\}_n; q^{-N}, \{e x_n/x_i\}_n; \frac{a^{n+1}q^{N+n+1}}{BcDe^n}\right)$$

$$= \frac{(a^{n+1}q^{n+1}/b_1\cdots b_m c d_1\cdots d_n e^n)_N}{(aq/c)_N} \prod_{1\le k\le m}\frac{(aqy_k/y_m)_N}{((aq/b_k)y_k/y_m)_N}\prod_{1\le i\le n}\frac{(ex_n/x_i)_N}{((aq/d_i)x_n/x_i)_N}$$

$$\sum_{\gamma\in\mathbb{N}^n} q^{|\gamma|}\frac{\Delta(xq^\gamma)}{\Delta(x)}\prod_{1\le i,j\le n}\frac{((aq/d_je)x_i/x_j)_{\gamma_i}}{(qx_i/x_j)_{\gamma_i}}$$

$$\prod_{1\le i\le n, 1\le k\le m}\frac{((aq/b_ke)x_iy_k/x_ny_m)_{\gamma_i}}{((aq/e)x_iy_k/x_ny_m)_{\gamma_i}}$$

$$\prod_{1\le i\le n}\frac{((aq/ce)x_i/x_n)_{\gamma_i}}{(q^{1-N}e^{-1}x_i/x_n)_{\gamma_i}}$$

$$\frac{(q^{-N})_{|\gamma|}}{(a^{n+1}q^{n+1}/b_1\cdots b_m c d_1\cdots d_n e^n)_{|\gamma|}}$$

where $B = b_1\cdots b_m$ and $D = d_1\cdots d_n$.

★ Balanced duality transformation formula

Now, we give a multiple transformation-summation formula of very-well-poised basic hypergeometric series including, as special cases, Jackson's terminating ${}_8W_7$ summation formula

$$
{}_8W_7\left[a; b, c, d, e, q^{-N}; q; q\right] = \frac{(aq)_N (aq/bc)_N (aq/bd)_N (aq/cd)_N}{(aq/b)_N (aq/c)_N (aq/d)_N (aq/bcd)_N}
$$

provided $a^2q^{N+1} = bcde$ (equation (2.6.1) in BHS), and ${}_{10}W_9$ transformation formula

$$
\begin{aligned}
&\frac{(aq/b)_N (aq/c)_N (aq/d)_N (aq/e)_N (\mu q)_N (\mu f/a)_N}{(\mu bf/a)_N (\mu cf/a)_N (\mu df/a)_N (\mu ef/a)_N (aq)_N (f)_N} \\
&{}_{10}W_9\left[a; b, c, d, e, f, \mu f q^N, q^{-N}; q; q\right] \\
&= {}_{10}W_9\left[\mu; aq/bf, aq/cf, aq/df, aq/ef,\right. \\
&\qquad\qquad \left.\mu f/a, \mu f q^N, q^{-N}; q; q\right]
\end{aligned}
$$

One can check that both the Bailey transformation and our ${}_{10}W_9$ transformation can be obtained by iterating twice the another one.

♡ Balanced duality transformation formula

$$\sum_{\gamma\in\mathbb{N}^n,|\gamma|=N} \frac{\Delta(xq^\gamma)}{\Delta(x)} \prod_{1\le i,j\le n} \frac{(a_jx_i/x_j)_{\gamma_i}}{(qx_i/x_j)_{\gamma_i}}$$

$$\prod_{1\le i\le n,1\le k\le m} \frac{(b_kx_iy_k/x_ny_m)_{\gamma_i}}{(cx_iy_k/x_ny_m)_{\gamma_i}}$$

$$= \sum_{\delta\in\mathbb{N}^m,|\delta|=N} \frac{\Delta(yq^\delta)}{\Delta(y)} \prod_{1\le k,l\le m} \frac{((c/b_l)y_k/y_l)_{\delta_k}}{(qy_k/y_l)_{\delta_k}}$$

$$\prod_{1\le i\le n,1\le k\le m} \frac{((c/a_i)x_iy_k/x_ny_m)_{\delta_k}}{(cx_iy_k/x_ny_m)_{\delta_k}}$$

when $a_1\cdots a_nb_1\cdots b_m = c^m$.

♡ multiple Jackson summation

$$W^{n,2}\left(\begin{matrix}\{b_i\}_n\\ \{x_i\}_n\end{matrix} \,\middle|\, a;c,e;q^{-N},d;q\right)$$

$$= \frac{(aq/b_1\cdots b_nc)_N}{(aq/b_1\cdots b_ncd)_N}\frac{(aq/cd)_N}{(aq/c)_N}$$

$$\prod_{1\le i\le n} \frac{((aq/b_id)x_i/x_n)_N(aqx_i/x_n)_N}{((aq/b_i)x_i/x_n)_N((aq/d)x_i/x_n)_N}$$

provided $a^2q^{N+1} = b_1\cdots b_ncde$.

♡ Multiple ${}_{10}W_9$ transformation formula

$$W^{n,m+2}\left(\begin{matrix}\{b_i\}_n\\ \{x_i\}_n\end{matrix}\,\middle|\, a; \{c_k y_k/y_m\}_m, d, e; \right.$$

$$\left. \{f y_m/y_k\}_m, \mu f q^N, q^{-N}; q\right)$$

$$= \frac{(\mu df/a)_N(\mu ef/a)_N}{(aq/d)_N(aq/e)_N}$$

$$\prod_{1\le k\le m} \frac{((\mu c_k f/a)y_k/y_m)_N(f y_m/y_k)_N}{(\mu q y_k/y_m)_N((aq/c_k)y_m/y_k)_N}$$

$$\prod_{1\le i\le n} \frac{(aqx_i/x_n)_N((\mu b_i f/a)x_n/x_i)_N}{((aq/b_i)x_i/x_n)_N((\mu f/a)x_n/x_i)_N}$$

$$W^{m,n+2}\left(\begin{matrix}\{aq/c_k f\}_m\\ \{y_k\}_m\end{matrix}\,\middle|\, \mu; \{(aq/b_i f)x_i\}_n, \right.$$

$$\left. aq/df, aq/ef; \{(\mu f/a)/x_i\}_n, \mu f q^n, q^{-N}; q\right),$$

where $\mu = a^{m+2}q^{m+1}/b_1\cdots b_n c_1\cdots c_m d e f^{m+1}$.

§Symmetry of multiple BHS $W^{n,m}$

★ To the multiple Bailey transformation

In the particular case of $m = 1$ it transforms the balanced $W^{n,3}$ into $W^{1,n+2} \propto {}_{2n+8}W_{2n+7}$; the corresponding ${}_{2n+8}W_{2n+7}(s; u_1, \ldots, u_{2n+5})$ are completely symmetric with respect to the $2n+5$ parameters $u_1, \ldots, u_{2n+5}$. We can use this symmetry of ${}_{2n+8}W_{2n+7}$ to produce nontrivial transformation formulas for $W^{n,3}$ through the diagram:

$$\begin{array}{ccc} W^{n,3} & \xrightarrow{\text{Bailey}} & W^{n,3} \\ \downarrow & & \uparrow \\ {}_{2n+8}W_{2n+7} & \xrightarrow{\text{Symmetry}} & {}_{2n+8}W_{2n+7}. \end{array}$$

Thus we have two types of multiple Bailey transformation formulas for $W^{n,3}$

♡ Multiple Bailey transformation II (M.Noumi-Y.K)

$$
\begin{aligned}
&W^{n,3}\left(\begin{matrix} \{e_i\}_n \\ \{x_i\}_n \end{matrix} \middle| a; b, c, d; \right. \\
&\qquad\qquad \left. q^{-N}, f, a\lambda q^{1+N}/e_1 \cdots e_n f; q \right) \\
&= \prod_{1 \le i \le n} \left[\frac{(aqx_i)_N ((aq/e_i f)x_i)_N}{((aq/e_i)x_i)_N ((aq/f)x_i)_N} \right. \\
&\qquad\qquad \left. \frac{((\lambda q/e_i)z_i)_N ((\lambda q/f)z_i)_N}{(\lambda q z_i)_N ((\lambda q/e_i f)z_i)_N} \right] \\
&W^{n,3}\left(\begin{matrix} \{e_i\}_n \\ \{z_i\}_n \end{matrix} \middle| \lambda; aq/cd, aq/bd, aq/bc; \right. \\
&\qquad\qquad \left. q^{-N}, f, a\lambda q^{1+N}/e_1 \cdots e_n f; q \right).
\end{aligned}
$$

where $\lambda = a^2 q/bcd$ and $z_i = e_i/e_1 \cdots e_n x_i$.

♡ Multiple Bailey transformation I (Milne-Newcomb)

$$W^{n,3}\left(\begin{matrix} \{e_i\}_n \\ \{x_i\}_n \end{matrix} \middle| a; d, f, a\lambda q^{1+N}/e_1 \cdots e_n f; \right. \\ \left. q^{-N}, b, c; q \right)$$

$$= \frac{(aq/e_1 \cdots e_n f)_N (\lambda q/f)_N}{(aq/f)_N (\lambda q/e_1 \cdots e_n f)_N} \prod_{1 \le i \le n} \frac{(aqx_i)_N ((\lambda q/e_i) x_i)_N}{((aq/e_i) x_i)_N (\lambda q x_i)_N}$$

$$W^{n,3}\left(\begin{matrix} \{e_i\}_n \\ \{x_i\}_n \end{matrix} \middle| \lambda; aq/bc, f, a\lambda q^{1+N}/e_1 \cdots e_n; \right. \\ \left. q^{-N}, aq/cd, aq/bc; q \right)$$

where $\lambda = a^2 q/bcd$.

♡ Final comment

In addition, we have derived further multiple hypergeometric transformation formula (for example, some types of Sears trans, Bailey trans and etc.) and have described symmetry of them in references.

Thank you very much!!