Yangian Algebras in Physics

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YBE $\longrightarrow$ RTT

RTT $\swarrow$ $\searrow$

Bethe Ansatz $\qquad$ Quantum Groups

Bethe Ansatz $\downarrow$

Solutions of (Nonlinear) Quantum Integrable Models

Quantum Groups $\downarrow$ $\downarrow$

Quantum Algebras (Trigonometric Solutions of YBE)

Yangian (Rational Solutions of YBE)

(I) Yangian (Drinfeld):

$$Y_\lambda : \{ I_\lambda, J_\lambda \}$$

$$[I_\lambda, I_\mu] = C_{\lambda\mu\nu} I_\nu \qquad \text{(Lie algebra)}$$

$$[I_\lambda, J_\mu] = C_{\lambda\mu\nu} J_\nu$$

For $sl(2)$:

$$[J_\lambda, [J_\mu, I_\nu]] - [I_\lambda, [J_\mu, J_\nu]] =$$

$$a_{\lambda\mu\nu\alpha\beta\gamma} \{ I_\alpha, I_\beta, J_\gamma \}$$

$$a_{\lambda\mu\nu\alpha\beta\gamma} = \frac{1}{4!} C_{\lambda\alpha\sigma} C_{\mu\beta\tau} C_{\nu\gamma\rho} C_{\sigma\tau\rho}$$

$$\{ x_1, x_2, x_3 \} = \sum_{i \neq j \neq k} x_i x_j x_k$$

For $sl(n)$ $n > 2$:

$$[[J_\lambda, J_\mu], [I_\sigma, I_\tau]] + [[J_\sigma, J_\tau], [I_\lambda, J_\mu]] =$$

$$(a_{\lambda\mu\nu\alpha\beta\gamma} C_{\sigma\tau\nu} + a_{\sigma\tau\nu\alpha\beta\gamma} C_{\lambda\mu\nu}) \{ I_\alpha, I_\beta, J_\gamma \}$$

Not all of the relations are independent. Using Jacobi identities and the particular forms of $C_{\lambda\mu\nu}$ we can find the independent ones for

$SU(2)$, $SU(3)$ and $SO(5)$.

The co-product reads:

$$\Delta(AB) = (\Delta A)(\Delta B)$$

$$\Delta(I_i) = I_i \otimes 1 + 1 \otimes I_i$$

$$\Delta(J_i) = J_i \otimes 1 + 1 \otimes J_i + \frac{h}{2} C_{ijk} I_j \otimes I_k$$

For instance for $SU(2)$ we have

$$[I_i, I_j] = i\,\epsilon_{ijk} I_k \qquad (i,j,k = 1,2,3)$$

$$[I_i, J_j] = i\,\epsilon_{ijk} J_k$$

$$[J_3, [J_+, J_-]] = \left(\frac{h^2}{4}\right)(I_+ J_- - J_+ I_-) I_3$$

$$(A_\pm = A_1 \pm i A_2)$$

(II) Simple Realizations of $Y(SU(2))$

- For any $SU(2)$-operators $\vec{S}_i$ at i-th site

$$\vec{I} = \sum_{i=1}^{N} \vec{S}_i$$

$$\vec{J} = \sum_{i=1}^{N} \mu_i \vec{S}_i + \frac{i\hbar}{2} \epsilon_{ijk} (\vec{S}_j \times \vec{S}_k)$$

Especially, for $N=2$ $[\vec{L}, \vec{K}]=0$

$$\vec{I} = \vec{L} + \vec{K}$$

$$\vec{J} = a\vec{L} + b\vec{K} + i\hbar\, \vec{L} \times \vec{K}$$

- For spin $\frac{1}{2}$ (Haldane - Shastry)

$$\vec{I} = \sum_{i=1}^{N} \vec{S}_i$$

$$\vec{J} = \sum_{j<k}^{N} \cot\left[\pi\left(\frac{j-k}{\pi}\right)\right] \vec{S}_j \times \vec{S}_k$$

that commute with Haldane-Shastry model:

$$H = \sum_{i \neq j} \left(z_i z_j / z_{ij} z_{ji} \right) (P_{ij} - 1)$$

$$z_i - z_j = z_{ij}, \quad z_m = e^{im(2\pi/N)}$$

- $\vec{I} = \sum_{i=1}^{N} \vec{S}_i$

$$J_+ = \sum_{\langle i,j \rangle}^{N} \theta_{i,j}\, a_i^\dagger b_j - U \sum_{i \neq j} \epsilon_{ij}\, I_i^+ I_j^3$$

$$J_- = \sum_{\langle i,j \rangle}^{N} \theta_{i,j}\, b_i^\dagger a_j + U \sum_{i \neq j} \epsilon_{ij}\, I_i^- I_j^3$$

$$J_3 = \frac{1}{2} \left\{ \sum_{\langle i,j \rangle}^{N} \theta_{i,j} (a_i^\dagger a_j - b_i^\dagger b_j) + U \sum_{i \neq j} \epsilon_{ij}\, I_i^+ I_j^- \right\}$$

$\theta_{i,j} = \delta_{i,j-1} - \delta_{i,j+1}$

$\epsilon_{i,j} = 1\ (i<j),\ -1\ (i>j),\ 0\ (i=j)$

that commute with 1-dimensional Hubbard model (Korepin - Uglov).

$$a_i^\dagger = c_{i\uparrow}^\dagger\ ,\quad a_i = c_{i\uparrow}$$

$$b_i^\dagger = c_{i\downarrow}^\dagger\ ,\quad b_i = c_{i\downarrow}$$

(fermionic operators with spin at i-th site).

- $SO(4,2)$

$M_{\mu\nu} = -M_{\nu\mu}$ $\quad (\mu, \nu = 0, 1, 2, 3)$

P_μ

K_μ $\quad$ special conformal

D $\quad$ delitation

In 3 dimensions there are

$\vec{R}$: Rotion in 3 dimensional space

$\vec{B}$: Lorentz boost

D : Delitation

Def. $\quad \vec{I} = \vec{R}$

$\quad \vec{J} = \vec{R} \times \vec{B} + i(D-1)\vec{B}$

then

$\vec{I}$ and $\vec{J}$ satisfy $Y(su(2))$.

namely, the finite W-algebra formed by $\vec{I}$ and $\vec{J}$ is nothing but Yangian.

有2个Casimirs，例如

$$C_2 = \frac{1}{2}\left(M_{\mu\nu} M^{\mu\nu} + p^\mu K_\mu + K^\mu p_\mu \right) - D^2$$

SO(4,2) 具体实现：

$$M_{\mu\nu} = x_\mu p_\nu - x_\nu p_\mu$$

$$K_\mu = 2x_\mu (x \cdot p) - x^2 p_\mu$$

$$D = (x \cdot p) \quad \Rightarrow \quad C_2 = 0$$

The above forms classical SO(4,2) group. What is the quantum correction?

$$\mathcal{L} = \frac{1}{g_{YM}^2} \mathrm{Tr}\Big\{ \frac{N}{2} F_{\mu\nu} F^{\mu\nu} + D_\mu \phi^I D^\mu \phi^I - \frac{1}{2}[\phi^I, \phi^J][\phi^I, \phi^J] \pm \text{fermions} \Big\}$$

Super YM meets YB

I) spinor derivative and anti-commutative c-number

引入指标 $A = {}^{\alpha}{}_{i}$, $\dot{A} = {}_{\dot{\alpha} j}$

$\alpha, \dot{\alpha} = 1, 2$ — dot and without dot

$i, j = 1, 2, \cdots N$ super indices

$\alpha, \dot{\alpha} \in SL(2, C)$

∵ ~~Vector~~ ~ Lorentz group ~ $SL(2, C)$:

$$P_\mu = i \partial_\mu \quad (\mu = 0, 1, 2, 3)$$

Decomposing momentum into "half":

$$\{Q_A, \bar{Q}_{\dot{A}}\} = 2 \sigma^{\mu}_{A\dot{A}} P_\mu \quad , (\{\, ,\} = [\, ,]_+)$$

$$\sigma^0 = I, \quad \bar{\sigma}^{m\, \dot{\beta}\beta} = \epsilon^{\dot{\beta}\dot{\gamma}} \epsilon^{\beta\rho} \sigma^{m}_{\rho\dot{\gamma}} \quad (m = 1, 2, 3)$$

(即带尖与不带尖分量分别用带尖、不带尖 ∈ 矩阵)

则 $\{\sigma^n, \sigma^m\}_{\alpha\beta} = 2\eta^{nm} \delta_{\alpha\beta}$

$$\eta^{mn} = (-1, +1, +1, +1) \quad (n, m = 0, 1, 2, 3)$$

$$Q_A = \frac{\partial}{\partial\theta^A} - i\,\not\partial_{A\dot B}\,\bar\theta^{\dot B}$$

$$\bar Q_{\dot A} = -\frac{\partial}{\partial\bar\theta^{\dot A}} + i\,\theta^B \not\partial_{B\dot A}$$

$$\left(\not\partial_{A\dot B} = \sigma^\mu_{A\dot B}\,\partial_\mu\right)$$

θ ($\bar\theta$) 间均反互易（对任意分量）

$$Q_A\bar Q_{\dot A} = \left(\frac{\partial}{\partial\theta^A} - i\not\partial_{A\dot B}\bar\theta^{\dot B}\right)\left(-\frac{\partial}{\partial\bar\theta^{\dot A}} + i\,\theta^B\not\partial_{B\dot A}\right)$$

$$= -\frac{\partial^2}{\partial\theta^A\partial\bar\theta^{\dot A}} + i\,\bar\theta^{\dot B}\frac{\partial}{\partial\bar\theta^{\dot A}}\not\partial_{A\dot B} + \bar\theta^{\dot B}\theta^B\not\partial_{A\dot B}\not\partial_{B\dot A} + i\,\delta_A^C\not\partial_{C\dot A} - i\,\theta^B\frac{\partial}{\partial\theta^A}\not\partial_{B\dot A}$$

$$\left(+\frac{\partial^2}{\partial\bar\theta^{\dot A}\partial\theta^A}\right) \qquad (i\not\partial_{A\dot A})$$

$$\Rightarrow \{Q_A, \bar Q_{\dot A}\} = 2i\not\partial_{A\dot A} + \theta^B\bar B^{\dot B}\left[\not\partial_{B\dot A}\not\partial_{A\dot B} - \not\partial_{A\dot B}\not\partial_{B\dot A}\right]$$

但 $\not\partial_{B\dot A}\not\partial_{A\dot B} - \not\partial_{A\dot B}\not\partial_{B\dot A} = (\sigma^\mu_{B\dot A}\sigma^\nu_{A\dot B} - \sigma^\mu_{A\dot B}\sigma^\nu_{B\dot A})\partial_\mu\partial_\nu = 0$

$\because \partial_\mu\partial_\nu = \partial_\nu\partial_\mu$

$$\therefore \{Q_A, \bar Q_{\dot A}\} = 2i\not\partial_{A\dot A} = 2\sigma^\mu_{A\dot A}P_\mu$$

ie. 一切均以 $\sigma^\mu_{A\dot A}$ 为"基"换算到超空间.

$\alpha, \dot\alpha$ — Lorentz 指标，$i, j = 1, \cdots, N$ — 超指标.

$\therefore$ $\{Q_A, \bar{Q}_{\dot{A}}\} = 2\,\sigma^{\mu}_{A\dot{A}}\, P_\mu$

$\{Q_A, Q_B\} = \{\bar{Q}_{\dot{A}}, \bar{Q}_{\dot{B}}\} = [P_\mu, P_\nu] = 0$

$[Q_A, P_\mu] = [\bar{Q}_{\dot{A}}, P_\mu] = 0$

(α $\dot{A} = \dot{\alpha}\, j$) (以上是通常的对指标记号)

Super[illegible]: $\Phi(x, \theta, \bar{\theta})$

Cov[illegible]t [illegible]or derivative:

$$\begin{cases} D_A = \dfrac{\partial}{\partial\theta^A} + i\,\partial\!\!\!/_{A\dot{A}}\,\bar{\theta}^{\dot{A}} \\ \bar{D}_{\dot{A}} \equiv -\dfrac{\partial}{\partial\bar{\theta}^{\dot{A}}} - i\,\theta^A\,\partial\!\!\!/_{A\dot{A}} \end{cases}$$

$$\Rightarrow \quad \underline{\{D_A, \bar{D}_{\dot{B}}\} = -2i\,\sigma^{\mu}_{A\dot{B}}\,\partial_\mu}$$

$$\mathcal{D}_A = D_A + i\,\mathcal{A}_A$$

$$\bar{\mathcal{D}}_{\dot{A}} = \bar{D}_{\dot{A}} + i\,\bar{\mathcal{A}}_{\dot{A}}$$

Gauge transformation:

$$\mathcal{A}_A \to \mathcal{A}_A' = U\mathcal{A}_A U^{-1} - iUD_A U^{-1}$$

$$\bar{\mathcal{A}}_{\dot{A}} \to \mathcal{A}_{\dot{A}}' = U\bar{\mathcal{A}}_{\dot{A}} U^{-1} - iU\bar{D}_{\dot{A}} U^{-1}$$

Curvatures:

$$\{\mathcal{D}_A , \mathcal{D}_B\}_+ = iF_{AB}$$
$$= i\left[D_A\mathcal{A}_B + D_B\mathcal{A}_A + i\{\mathcal{A}_A , \mathcal{A}_B\}\right]$$

$$\{\bar{\mathcal{D}}_{\dot{A}} , \bar{\mathcal{D}}_{\dot{B}}\} = i\bar{F}_{\dot{A}\dot{B}}$$
$$= i\left[\bar{D}_{\dot{A}}\bar{\mathcal{A}}_{\dot{B}} + \bar{D}_{\dot{B}}\bar{\mathcal{A}}_{\dot{A}} + i\{\bar{\mathcal{A}}_{\dot{A}} , \bar{\mathcal{A}}_{\dot{B}}\}\right]$$

$$\{\mathcal{D}_A , \bar{\mathcal{D}}_{\dot{B}}\}_+ = iF_{A\dot{B}} - 2i\sigma^\mu_{A\dot{B}}\mathcal{D}_\mu$$

$$F_{A\dot{B}} = D_A\bar{\mathcal{A}}_{\dot{B}} + \bar{D}_{\dot{B}}\mathcal{A}_A + i\{\mathcal{A}_A , \bar{\mathcal{A}}_{\dot{B}}\}_+ + 2i\sigma^\mu_{A\dot{B}}\mathcal{A}_\mu$$

$$[\mathcal{D}_\mu , \mathcal{D}_A] = iF_{\mu A}$$
$$= i\left[\partial_\mu\mathcal{A}_A - D_A\mathcal{A}_\mu + i[\mathcal{A}_\mu , \mathcal{A}_A]\right]$$

Generalized Jacobi identities:

$$[\{\mathcal{D}_A, \mathcal{D}_B\}, \mathcal{D}_C] + [\{\mathcal{D}_C, \mathcal{D}_A\}, \mathcal{D}_B]$$
$$+ [\{\mathcal{D}_B, \mathcal{D}_C\}, \mathcal{D}_A] = 0$$

$$\Rightarrow \begin{cases} \mathcal{D}_C F_{AB} + \mathcal{D}_B F_{CA} + \mathcal{D}_A F_{BC} = 0 & (1) \\ \bar{\mathcal{D}}_{\dot{C}} \bar{F}_{\dot{A}\dot{B}} + \bar{\mathcal{D}}_{\dot{B}} \bar{F}_{\dot{C}\dot{A}} + \bar{\mathcal{D}}_{\dot{A}} \bar{F}_{\dot{B}\dot{C}} = 0 & (2) \end{cases}$$

$$[\{\mathcal{D}_A, \bar{\mathcal{D}}_{\dot{B}}\}, \bar{\mathcal{D}}_{\dot{C}}] + [\{\bar{\mathcal{D}}_{\dot{C}}, \mathcal{D}_A\}, \bar{\mathcal{D}}_{\dot{B}}]$$
$$+ [\{\bar{\mathcal{D}}_{\dot{B}}, \bar{\mathcal{D}}_{\dot{C}}\}, \mathcal{D}_A] = 0$$

$$\Rightarrow$$

$$\bar{\mathcal{D}}_{\dot{C}} F_{A\dot{B}} + \bar{\mathcal{D}}_{\dot{B}} F_{A\dot{C}} + \mathcal{D}_A \bar{F}_{\dot{B}\dot{C}}$$
$$+ 2i\sigma^{\mu}_{A\dot{B}} \bar{F}_{\mu\dot{C}} + 2i\sigma^{\mu}_{A\dot{C}} \bar{F}_{\mu\dot{B}} = 0 \qquad (3)$$

$\bar{\mathcal{D}}_{\dot{C}}$ 换为 $\mathcal{D}_C \Rightarrow$

$$\bar{\mathcal{D}}_{\dot{B}} F_{CA} + \mathcal{D}_C F_{A\dot{B}} + \mathcal{D}_A F_{C\dot{B}} + 2i\sigma^{\mu}_{A\dot{B}} F_{\mu C}$$
$$+ 2i\sigma^{\mu}_{C\dot{B}} F_{\mu A} = 0 \qquad (4)$$

$$[[\mathcal{D}_\mu, \mathcal{D}_A], \mathcal{D}_B] + [[\mathcal{D}_B, \mathcal{D}_\mu], \mathcal{D}_A] + [[\mathcal{D}_A, \mathcal{D}_B], \mathcal{D}_\mu] = 0$$

$$\Rightarrow \begin{cases} \mathcal{D}_\mu F_{AB} - \mathcal{D}_A F_{\mu B} + \mathcal{D}_B F_{\mu A} = 0 \\ \mathcal{D}_\mu \bar{F}_{\dot{A}\dot{B}} - \bar{\mathcal{D}}_{\dot{A}} \bar{F}_{\mu\dot{B}} + \bar{\mathcal{D}}_{\dot{B}} \bar{F}_{\mu\dot{A}} = 0 \end{cases}$$

$$\because \mathrm{Tr}(\sigma^m \bar{\sigma}^n) = -2\eta^{mn} \quad (0,1,2,3)$$

$$\Rightarrow$$

$$8iN F_{\mu\nu} = \bar{\sigma}_\nu{}^{\dot{B}A}(\mathcal{D}_\mu F_{A\dot{B}} - \mathcal{D}_A \bar{F}_{\mu\dot{B}} - \bar{\mathcal{D}}_{\dot{B}} F_{\mu A}) - (\mu \leftrightarrow \nu)$$

(Ⅲ) SO(4,2) Symmetry

一般 SYM 的 Lagrangian 仍为

$$\mathcal{L} \sim \frac{1}{4}\,\mathrm{Tr}(F_{\mu\nu}F^{\mu\nu}) + \cdots$$

可证这种 $\mathcal{L}$ 存在 SO(4,2) 对称.

SO(4,2): 共 15 个 generators:

		数目
$M_{\mu\nu} = x_\mu P_\nu - x_\nu P_\mu$	$(\mu,\nu = 0,1,2,3)$	6
K_μ :	special conformal	4
P_μ :	4-维平移	4
D :	Dilitation	1
	总计	15

$$[K_\mu, K_\nu] = 0$$

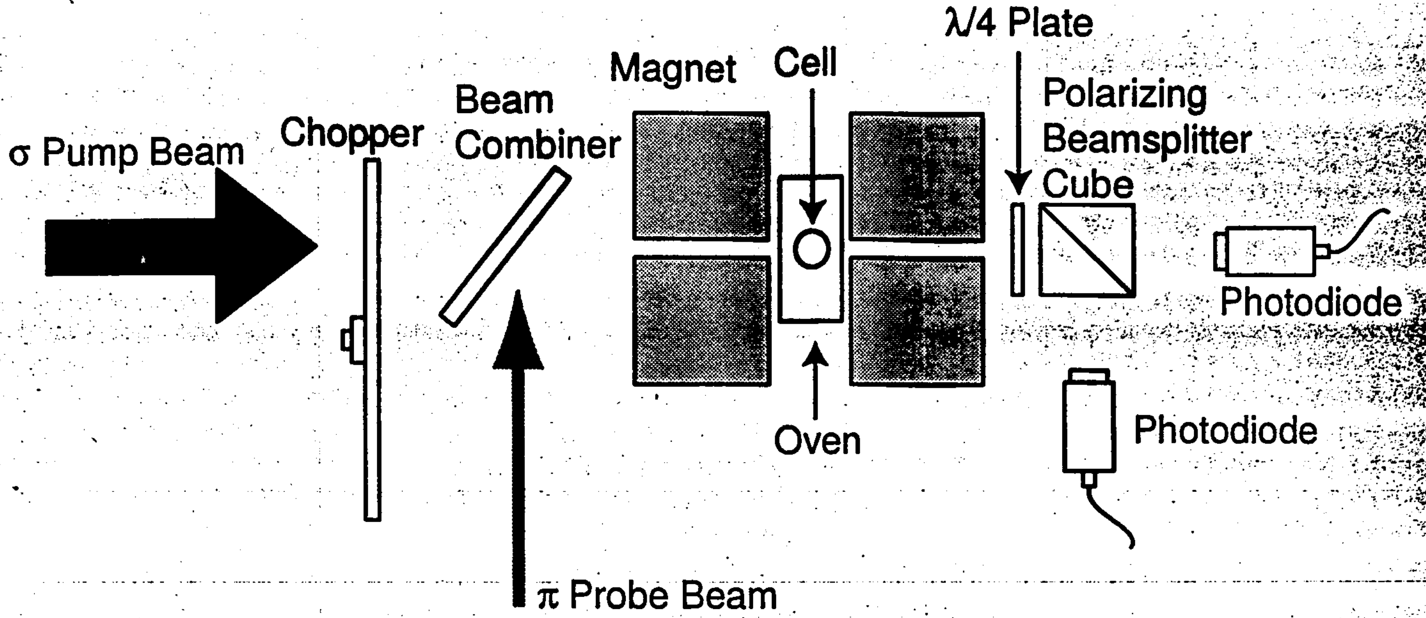
$$[P_\mu, K_\nu] = 2i\,(-M_{\mu\nu} + g_{\mu\nu} D)$$

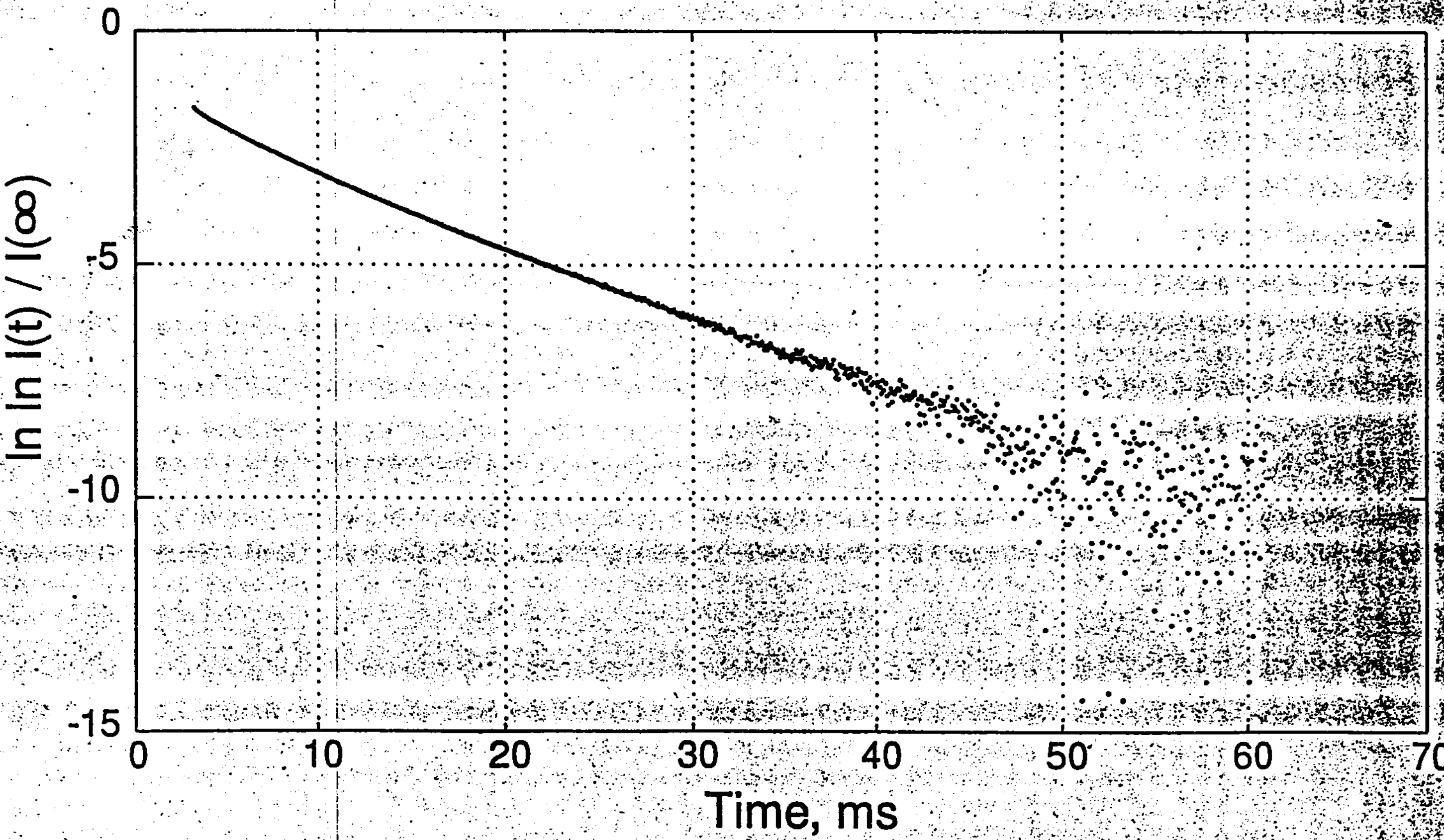
$$[M_{\mu\nu}, K_\rho] = i(g_{\nu\rho}K_\mu - g_{\mu\rho}K_\nu)$$

$$[D, K_\mu] = iK_\mu, \quad [D, P_\mu] = -iP_\mu$$

$$[D, M_{\mu\nu}] = 0$$

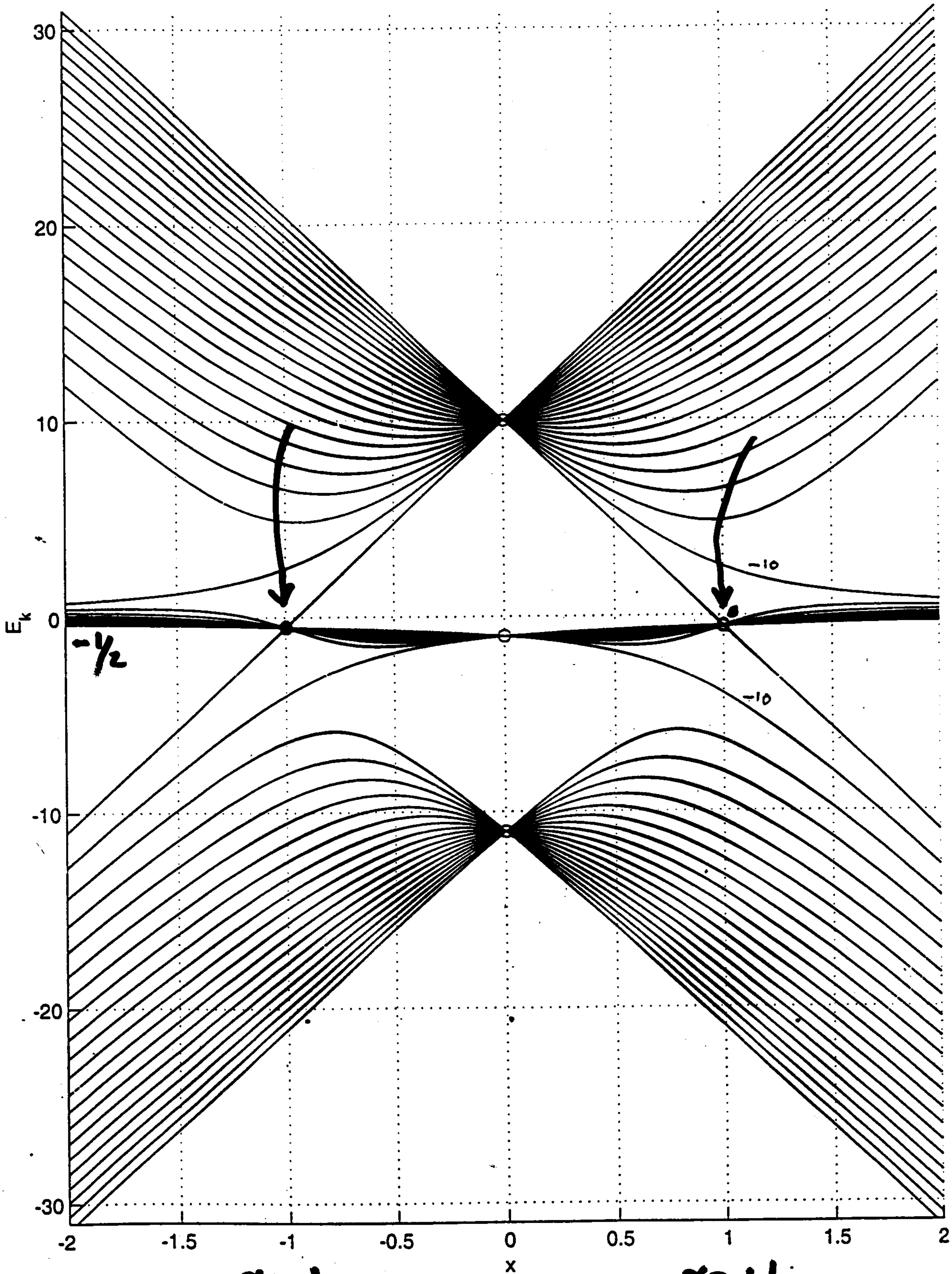
$\{M_{\mu\nu}, P_\mu, K_\mu, D \quad (\mu,\nu = 0,1,2,3)\}$ — 15 生成元


σ Pump Beam
Chopper
Beam
Combiner
Magnet
Cell
λ/4 Plate
Polarizing
Beamsplitter
Cube
Photodiode
Oven
Photodiode
π Probe Beam


ln ln I(t) / I(∞)
0
-5
-10
-15
0
10
20
30
40
50
60
70
Time, ms

~~Monica~~ ~~Jerry 1-800-800-2070 Ext 224~~ HAPPER

Eigenvalues E_k of $x(K+1/2)S_z + \mathbf{K}\cdot\mathbf{S}$; Spin quantum numbers are S = 1 and K = 10



E_T

22 Top

m = 11, 10, 9 ... -9, -10

E_D

21 Degenerate

m = 10, 9, ... -8, -9, -10

20 Bottom

m = 9, 8, 7, ... -9, -10

E_B

$x=-1$ $x=+1$

(Happer's calculation) $H=\vec{K}\cdot\vec{S}+xK S_3$

Student Version of MATLA

(Ⅲ). Physical Applications of Yangian

Example 1: Puzzle degeneracy in $^{87}Rb_2$

Experiment: 220°C, In vapor of ^{87}Rb there are 10^{-6} Rb_2 molecular, i.e. with total spin 1. It turs out that under applied magnetic field ~ 1500 Gauss there appears degeneracy.

The model is

$$H = \vec{K}\cdot\vec{S} + x(K+\tfrac{1}{2})S_3 \quad , \quad \vec{K}^2 = K(K+1)$$

where $\vec{K}$ — spin of the core of atom
$\vec{S}^2 = S(S+1)$, $S=1$.

For $x = \pm 1$ (corresponding to $B_0 = 1500$ Gauss along the z-axis)

$$E = \tfrac{1}{2}$$

Question: $\vec{B} \to -\vec{B} \Rightarrow |\vec{B}\rangle \xrightarrow{?} |-\vec{B}\rangle$

Both $|\vec{B}\rangle$ and $|-\vec{B}\rangle$ do have the same energy ($\frac{1}{2}$). When $\vec{B} \to -\vec{B}$ which operator can be introduced to transit $|\vec{B}\rangle$ to $|-\vec{B}\rangle$?

The operator is Yangian:

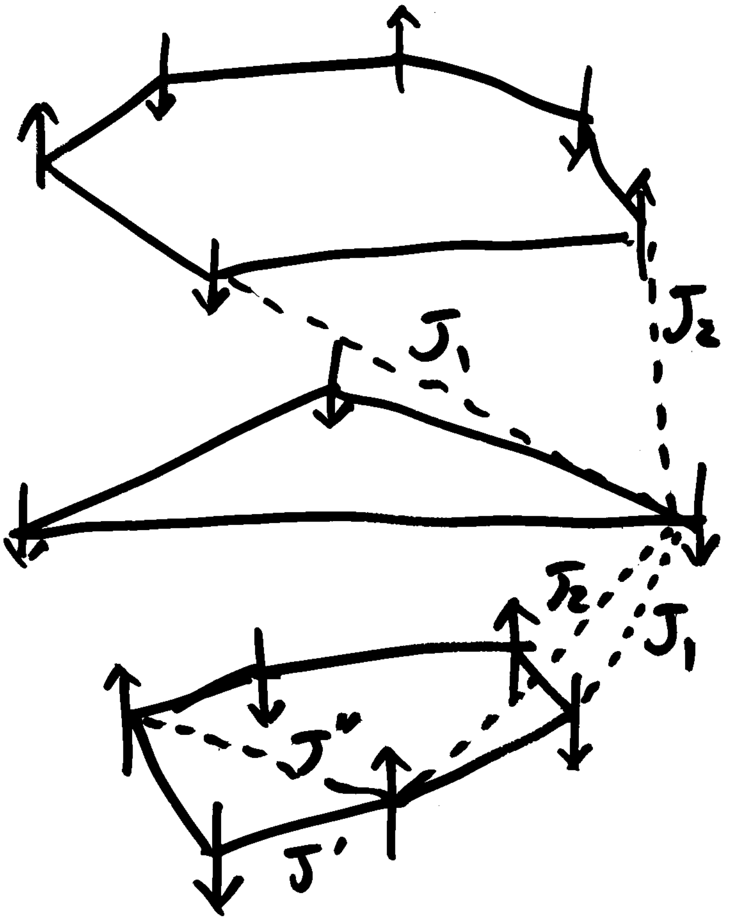
$$J_+ |\vec{B}\rangle = |-\vec{B}\rangle$$

$$\vec{J} = a\vec{S} + i\,\vec{S} \times \vec{K}$$

a is independent of M, only depends on K.

The physical reason: when $\vec{B} = B_0 \vec{e}_z$ is applied it gives rise to Zeeman effect for spin $\vec{S}$. However, ther exist torque due to $\vec{K}$ that cancell the split of spectrum.

Example 2: V_{15} problem (3 spin structure, splitting)



$K_6[V_{15}As_6O_{42}(H_2O)]$
$8H_2O$

15 spin AF coupling

The resultant spin = 1/2

In different from the usual spin 1/2, now half-integer spin is gapped.

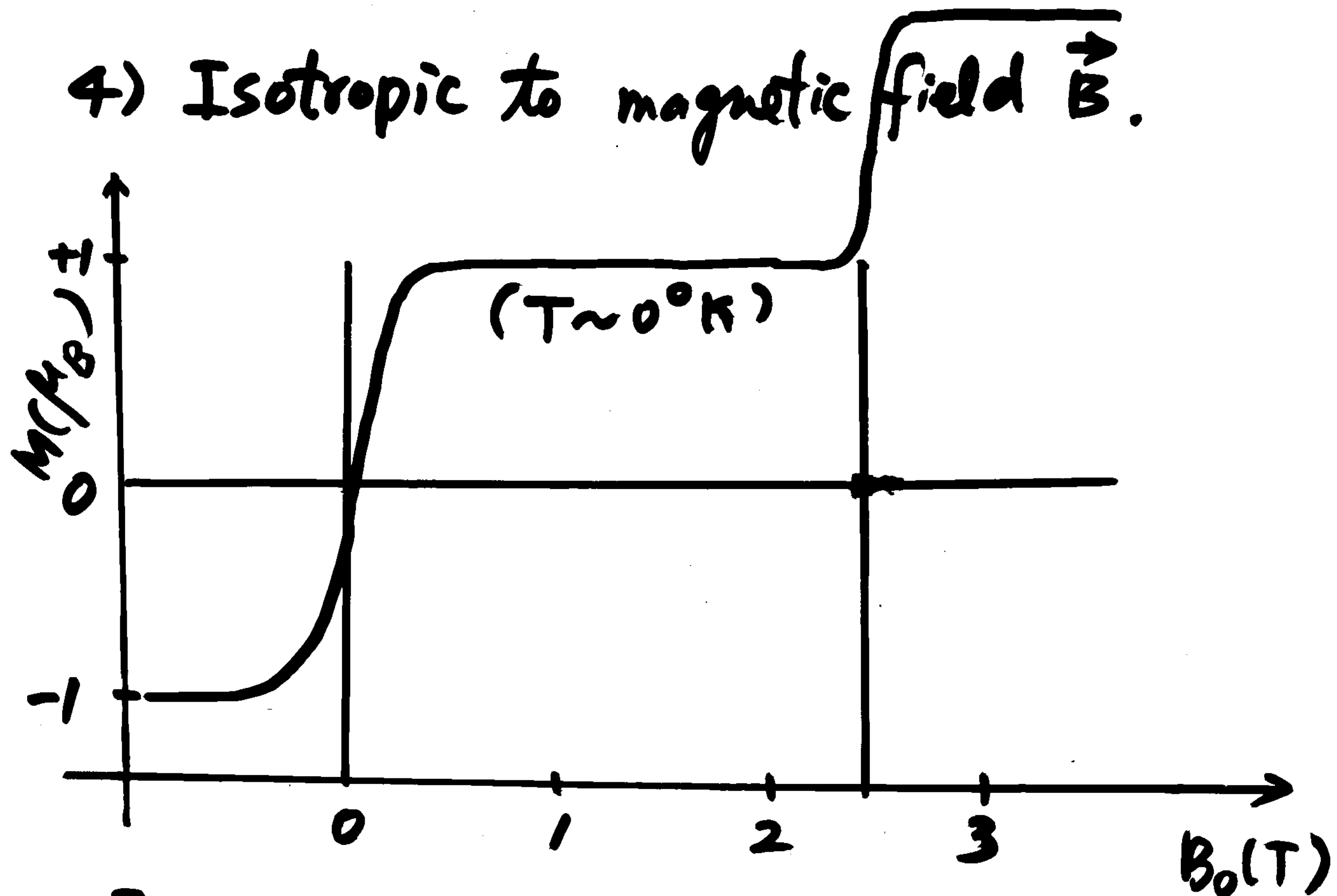
For V_{15} 1) Net spin = 1/2

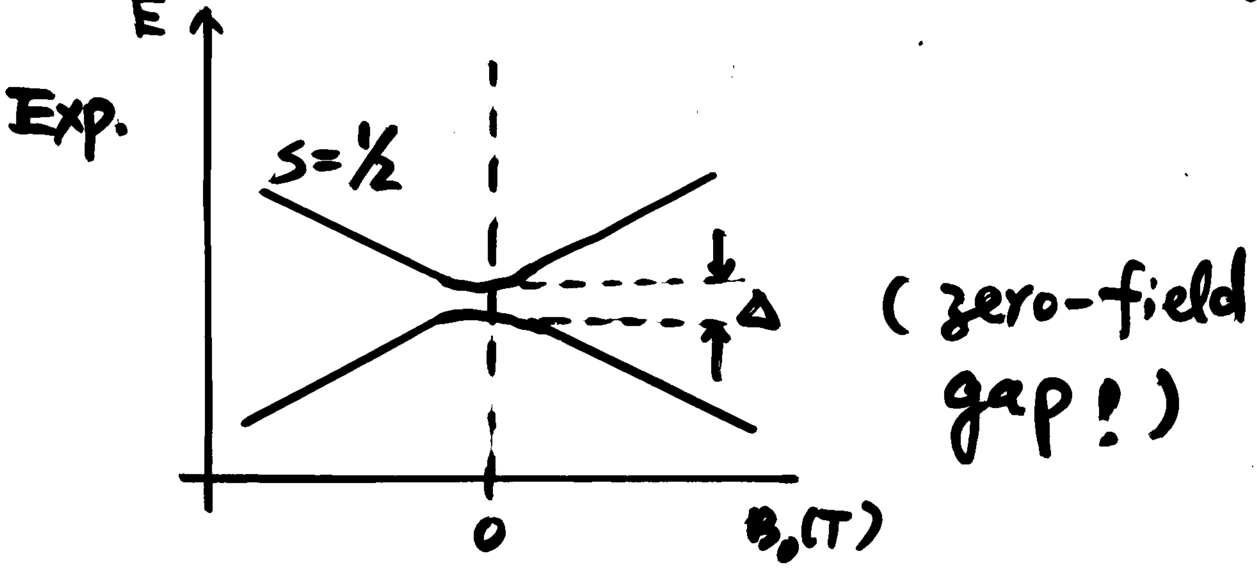
2) The interactions between spin-up and spin-down & dipolr coupling between moleculars can be neglected.

3) At low T°K   ground state : spin = ½

The first excitation state : spin = $3/2$

4) Isotropic to magnetic field $\vec{B}$.





Puzzle: Why there appears energy gap at zero field for spin-½ branch?

Difficulty: model should be isotropic and without applied field.

$\because$ There are spin $= 3/2$ and spin $= 1/2$ states.

Introducing 3 spins whose Hilbert space is

$$\left(\tfrac{1}{2} \otimes \tfrac{1}{2}\right) \otimes \tfrac{1}{2} = (1 \oplus 0) \otimes \tfrac{1}{2} = \tfrac{3}{2} \oplus \tfrac{1}{2} \oplus \tfrac{1}{2}'$$

There are 6j-coefficient description. However, we shall introduce the Yangian as collective quantum number. For general theory, we discuss

$$\left(\tfrac{1}{2} \otimes \tfrac{1}{2}\right) \otimes l = (1 \oplus 0) \otimes l = l+1, \boxed{\begin{matrix} l \\ l \end{matrix}}, l-1$$

The highest weight $l+1$

lowerest " $l-1$

multi-plicity 2: l

$$\vec{J} = \sum_{i=1}^{3} u_i \vec{S}_i + i h \sum_{i<j}^{3} \vec{S}_i \times \vec{S}_j \qquad \begin{matrix} S_1 : \frac{1}{2} \\ S_2 : \frac{1}{2} \end{matrix} \quad S_3 : \vec{l}$$

Denoting by $\Phi^1_{l,m}$ and $\Phi^2_{l,m}$ the eigenstates for two l-weights we have

现在 $S_1 = S_2 = \frac{1}{2}, S_3 = l$．计算 $\mathbf{J}^2$，并将 $\mathbf{J}^2$ 作用在归一化基上面 Φ_{S,S_z}，$\mathbf{S} = \mathbf{S}_1 + \mathbf{S}_2 + \mathbf{l}$（总“自旋”）

$$\Phi_{l+1,l+1}, (\text{最高权基}), \Phi_{l-1,l-1}(\text{最低权基})$$
$$\Phi^{(1)}_{l,i}, \Phi^{(2)}_{l,i}(\text{重权独立基})(-l \le i \le l)$$

重要的是两个重权独立基，计算出：

$$\begin{aligned}
\mathbf{J}^2\Phi^{(1)}_{l,i} &= \{\frac{3}{4}u_1^2 + \frac{3}{4}u_2^2 + l(l+1)u_3^2 + \frac{1}{2}u_1u_2 - u_2u_3 - u_1u_3 \\
&+ h^2[-2l(l+1) - \frac{1}{4}]\}\Phi^1_{l,i} - \sqrt{l(l+1)}(u_1 - u_2 + h)(u_3 + h)\Phi^2_{l,i}
\end{aligned}$$

$$\begin{aligned}
\mathbf{J}^2\Phi^{(2)}_{l,i} &= -\sqrt{l(l+1)}(u_1 - u_2 - h)(u_3 - h)\Phi^{(1)}_{l,i} \\
&+ \{\frac{3}{4}(u_1 - u_2)^2 + l(l+1)u_3^2 - \frac{3}{4}u_1^2 - \frac{3}{4}u_2^2\}\Phi^{(2)}_{l,i}
\end{aligned}$$

其特征根为(eigenvalues)

$$\begin{aligned}
\lambda^{(\pm)}_l &= \frac{3}{4}u_1^2 + \frac{3}{4}u_2^2 + l(l+1)u_3^2 - \frac{1}{2}(u_1u_2 + u_2u_3 + u_1u_3) \\
&+ h^2[-l(l+1) - \frac{1}{2}] \pm \frac{1}{2}\sqrt{p}
\end{aligned}$$

$$\begin{aligned}
p &= [2u_1u_2 - u_2u_3 - u_1u_3 + h^2(-2l(l+1) + \frac{1}{2})]^2 \\
&+ 4l(l+1)[(u_1 - u_2)^2 - h^2](u_3^2 - h^2)
\end{aligned}$$

而 $\mathbf{J}^2$ 的本征向量为(The eigenstates of $\vec{J}^2$ are)

$$\begin{aligned}
\alpha^{(\pm)}_{l,i} &= \frac{1}{|\alpha^{\pm}_{l,i}|}(\beta_\pm\Phi^{(1)}_{l,i} + \gamma_\pm\Phi^{(3)}_{l,i}) \\
\beta_\pm &= 2u_1u_2 - u_1u_3 + h^2[-2l(l+1) + \frac{1}{2}] + \sqrt{p} \\
\gamma_\pm &= -2\sqrt{l(l+1)}(u_1 - u_2 + h)(u_3 + h)
\end{aligned}$$

即 $\mathbf{J}^2$ 的本征态为 $\Phi^{(1)}_{l,i}$ 与 $\Phi^{(2)}_{l,i}$ 态的线性变换，当 u_1, u_2 与 u_3 任意时，不一定是幺正变换．现选

$$u_2 = u_1 + u_3$$

i.e. the eigenstates of $\vec{J}^2$ is linear combination of $\Phi^1_{l,m}$ and $\Phi^2_{l,m}$ that in general is not unitary for arbitrary u_1, u_2 and u_3.

When we take $\underline{u_2 = u_1 + u_3}$ and introducing

$$\cos\varphi = \{2u_1^2 - u_3^2 + h^2[-2l(l+1) + \tfrac{1}{2}]\}/\sqrt{P}$$

$$\sin\varphi = \{2\sqrt{l(l+1)}\,(u_3^2 - h^2)\}/\sqrt{P}$$

$\Rightarrow$

$$\begin{pmatrix} \alpha_{l,m}^{(+)} \\ \alpha_{l,m}^{(-)} \end{pmatrix} = \begin{pmatrix} \cos\varphi/2 & -\sin\varphi/2 \\ \sin\varphi/2 & \cos\varphi/2 \end{pmatrix} \begin{pmatrix} \Phi_{l,m}^{1} \\ \Phi_{l,m}^{2} \end{pmatrix}$$

When $P \neq 0$, $\lambda_l^{(+)} \neq \lambda_l^{(-)}$, i.e. discret eigenvalues.

For this spin structure we have the set of quantum numbers

$$\{\vec{S}_1^2 = \tfrac{3}{4},\ \vec{S}_2^2 = \tfrac{3}{2},\ \vec{S}_3^2 = l(l+1),\ \vec{J}^2(u_1, u_3)\}$$

The conclusion is independent of $(\vec{S}_1 \otimes \vec{S}_2) \otimes \vec{S}_3$, $\vec{S}_1 \otimes (\vec{S}_2 \otimes \vec{S}_3)$ and $(\vec{S}_1 \otimes \vec{S}_3) \otimes \vec{S}_2$ (order of direct products)

With the Yangian description and the eigenstates of $\{\vec{l}^2, \vec{J}^2(u_1, u_3)\}$ we introduce the Hamiltonian

$$H = (a\vec{S}_2 + b\vec{l}) \cdot \vec{S}_1 = H_0 \qquad \vec{S}_1, \vec{S}_2 : \text{spin } 1/2.$$

that is isotropic to external magnetic field, i.e. $\vec{B} = 0$.

For $\vec{B}=0$ $\quad H_0 = -(a\vec{\ell} + b\vec{S}_2)\cdot\vec{S}_1$

$$H_0 \Phi_{\ell+1,m} = \left(-\frac{\ell}{2}a - \frac{1}{4}b\right)\Phi_{\ell+1,m}$$

$$H_0 \Phi_{\ell-1,m} = \left(\frac{\ell+1}{2}a - \frac{1}{4}b\right)\Phi_{\ell-1,m}$$

$$H_0 \begin{pmatrix} \Phi^1_{\ell,m} \\ \Phi^2_{\ell,m} \end{pmatrix} = \begin{pmatrix} \frac{1}{2}(a-b) & \frac{1}{2}\sqrt{\ell(\ell+1)}\,a \\ \frac{1}{2}\sqrt{\ell(\ell+1)}\,a & \frac{3}{4}b \end{pmatrix}\begin{pmatrix} \Phi^1_{\ell,m} \\ \Phi^2_{\ell,m} \end{pmatrix}$$

The eigenvalues

$$\lambda_{\ell+1} = -\frac{\ell}{2}a - \frac{1}{4}b, \quad \lambda_{\ell-1} = \frac{\ell+1}{2}a - \frac{1}{4}b$$

$$\lambda_\ell^{\pm} = \frac{1}{4}(a+b) \pm \frac{1}{2}\sqrt{\ell(\ell+1)a^2 + \left(\frac{a}{2}-b\right)^2}$$

$$\begin{pmatrix} \alpha^+_{\ell,m} \\ \alpha^-_{\ell,m} \end{pmatrix} = \begin{pmatrix} \cos\varphi/2 & -\sin\varphi/2 \\ \sin\varphi/2 & \cos\varphi/2 \end{pmatrix}\begin{pmatrix} \Phi^1_{\ell,m} \\ \Phi^2_{\ell,m} \end{pmatrix}$$

$$\cos\varphi = \left(\frac{a}{2}-b\right)/\omega, \quad \sin\varphi = \sqrt{\ell(\ell+1)}\,a/\omega$$

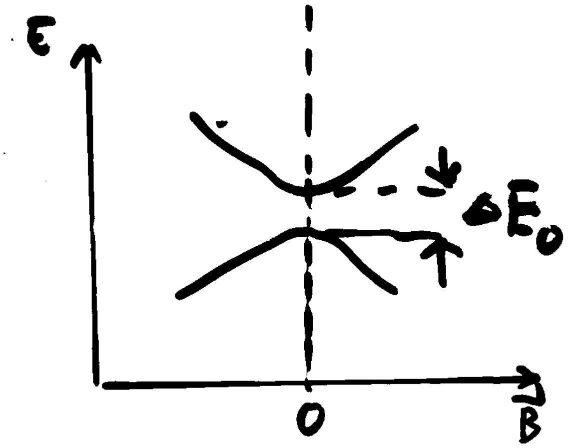
$$\omega^2 = \ell(\ell+1)a^2 + \left(\frac{a}{2}-b\right)^2$$

$\ell = \frac{1}{2} \Rightarrow 3$ spin-$\frac{1}{2}$ system.

$$H_0 = \begin{bmatrix} \frac{1}{2}(a-b) & \frac{\sqrt{3}}{4}a \\ \frac{\sqrt{3}}{4}a & \frac{3}{4}b \end{bmatrix} = \begin{bmatrix} \frac{1}{2}(a-b) & 0 \\ 0 & \frac{3}{4}b \end{bmatrix} + \begin{bmatrix} 0 & \Delta_0/2 \\ \Delta_0/2 & 0 \end{bmatrix}$$

$$(\Delta_0/2 = \frac{\sqrt{3}}{4}a)$$

Δ_0 : Energy gap



$$\Delta E_0 = \sqrt{\frac{3}{4}a^2 + (\frac{a}{2}-b)^2}$$

In this model the "zero spliting" comes from the "no-equality" of 3 spins and the commutivity between the Hamiltonian and squire of Yangian.

When there is applied $\vec{B} = B_0 \vec{e}_z$ the interaction

$$H_I = -\vec{B}\cdot\sum_{i=1}^{3}\vec{S}_i = -\vec{B}\cdot\vec{S} = -B_0 S_z$$

$\because [\vec{J}^2, S_z] = 0$ so a set

$$\{\vec{S}^2 \;\; S_z \;\; \vec{J}^2\}$$

form a complete quantum number set

$$H_I\, \Phi^1_{l,m} = m\, \Phi^1_{l,m}$$

$$H_I\, \Phi^2_{l,m} = m\, \Phi^1_{l,m}.$$

Example 3.

SYM ($N=4$) meets YB

Yangian symmetry

(iniated by Dolan-Nappi-Witten)

Yang-Mills field:

$$D_\mu = \partial_\mu + igA_\mu \qquad A_\mu = A_\mu^a I_a$$

$$[D_\mu, D_\nu] = F_{\mu\nu} = F_{\mu\nu}^a I_a$$

For $I_a \in SU(2)$ $\quad a=1,2,3$

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + ig(\vec{A}_\mu \times \vec{A}_\nu)^a$$

Super-YM:

Besides the isospin indices a

there are other internal degree N.

$N = 1, 2, 3, 4$

↑ usual YM.

Quantum correction

$$D \longrightarrow D + \delta\hat{D}$$

$$\delta\hat{D} = (\text{Hamiltonian})\ \hat{H}$$

$$[\hat{H},\ Y(SO(6))] = 0$$

i.e. $\hat{H}$ does have $Y(SO(6))$ symmetry

but $\hat{H}$ can be written in the form

$$\hat{H} = \sum_{j=1}^{N} \hat{H}_{j,j+1}$$

$$\hat{H}_{j,j+1} = -\psi(-\vec{J}_{j,j+1}) - \psi(\vec{J}_{j,j+1}+1)$$

$$+2\psi(1)$$

$$\psi(x) = \frac{d\log\Gamma(x)}{dx} \qquad \psi(1): \text{Euler const.}$$

$$J^2_{j,j+1} = (S_j + S_{j+1})^2$$

$S_j^2 = S(S+1) = 0$ i.e. $S = 0$ or -1

The $\hat{H}$ had been obtained by Kulish... through Yang-Baxer approach and describes a magnetic chain with $S=-1$ at each lattice. It is ∞ representation of $SU(2)$!

On the other hand, Lipatov gave the Hamiltonian early based on the non-peturbative QCD correction to the free parton model in high energy collision ($x \sim 10^{-5}$). He added the Feyman diagraphs of leading terms to guarantee the Unitarity.

The Lipatov model may be checked next year by exp.

Quantum solutions ? open
(colorless, massive)

monopole
Instatons ..

Classical solutions → Chern #s
(color, massless)

→ conserved
currents

Higgs

Y-M QT → Basic Theory
(F-P. 't Hooft
.... Asym. Freed. ..)

→ QCD
→ Models (weinberg. Lipatov
...)

Super YM

Integrable
Model with → ?
QCD

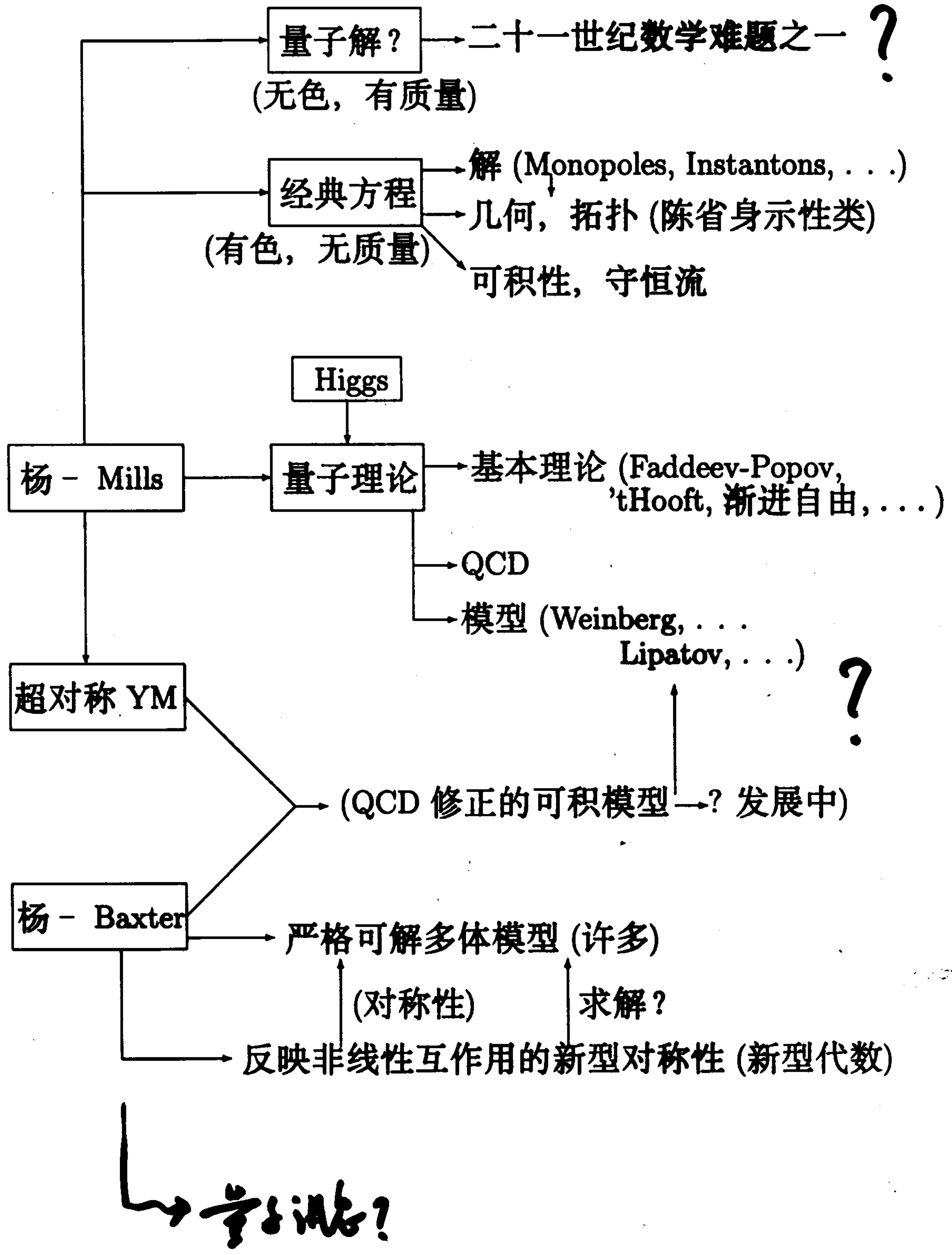
Y-B → Exactly solvable models

↑ symmetry ↓ solutions?

→ New symmetries for nonlinear
Models

→ Quantum Information

结语:


量子解？
二十一世纪数学难题之一
?
(无色，有质量)
经典方程
解 (Monopoles, Instantons, . . .)
几何，拓扑 (陈省身示性类)
(有色，无质量)
可积性，守恒流
Higgs
杨 − Mills
量子理论
基本理论 (Faddeev-Popov,
'tHooft, 渐进自由, . . .)
QCD
模型 (Weinberg, . . .
Lipatov, . . .)
?
超对称 YM
(QCD 修正的可积模型 → ? 发展中)
杨 − Baxter
严格可解多体模型 (许多)
(对称性)
求解？
反映非线性互作用的新型对称性 (新型代数)