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ZERO-SUM PROBLEMS AND THEIR CONNECTIONS TO COVERS OF $\mathbb{Z}$

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ABSTRACT. In this talk we present a new powerful approach to zero-sum problems on abelian groups, and disclose their connections to covers of the integers by residue classes. The recent solution to the Kemnitz conjecture by C. Reiher will also be commented and simplified.

1. SOME CLASSICAL RESULTS ON ZERO-SUMS

In 1961 P. Erdős, A. Ginzburg and A. Ziv [Bull. Research Council. Israel] established the following celebrated theorem which initiated the study of zero-sums.

The EGZ Theorem. *For any* $c_1, \cdots, c_{2n-1} \in \mathbb{Z}$*, there is an* $I \subseteq [1, 2n-1] = \{1, \ldots, 2n-1\}$ *with* $|I| = n$ *such that* $\sum_{s\in I} c_s \equiv 0 \pmod{n}$*. In other words, given* $2n-1$ *(not necessarily distinct) elements of* $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$*, we can select* n *of them with the sum vanishing.*

The EGZ theorem can be easily reduced to the case where n is a prime (and hence $\mathbb{Z}_n$ is a field), and then deduced from the well-known Cauchy-Davenport theorem or the Chevalley-Warning theorem.

For a finite abelian group G (written additively), the *Davenport constant* $D(G)$ is defined as the smallest positive integer k such that for any $c_1, \ldots, c_k \in G$ there exists a nonempty $I \subseteq [1,k]$ with $\sum_{s\in I} c_s = 0$. Clearly $D(G) \leqslant |G|$, for, if $c_1, \ldots, c_{|G|} \in G$ then the following partial sums

$$s_0 = 0,\ s_1 = c_1,\ s_2 = c_1 + c_2,\ \ldots,\ s_{|G|} = c_1 + c_2 + \cdots + c_{|G|}$$

cannot be distinct.

Since $\sum_{s\in I} 1 \not\equiv 0 \pmod{n}$ for any $\emptyset \neq I \subseteq [1, n-1]$, we have $D(\mathbb{Z}_n) = n$.

For an abelian p-group $G \cong \mathbb{Z}_{p^{h_1}} \oplus \cdots \oplus \mathbb{Z}_{p^{h_l}}$, we set

$$L(G) = 1 + \sum_{t=1}^{l} (p^{h_t} - 1).$$

Obviously $D(G) \geqslant L(G)$, because the sequence consisting of $p^{h_1} - 1$ copies of $\langle 1, 0, \ldots, 0\rangle$,..., $p^{h_l} - 1$ copies of $\langle 0, \ldots, 0, 1\rangle$, does not have a nonempty zero-sum subsequence. In 1969 Olson [J. Number Theory] used the knowledge of group rings to show that $D(G) = L(G)$.

Olson's Theorem. *Let p be a prime and let G be an additive abelian p-group. Then, for any $c_1, \cdots, c_{L(G)} \in G$, there exists a nonempty $I \subseteq [1, L(G)]$ with $\sum_{s\in I} c_s = 0$.*

What is the smallest integer $k = s(\mathbb{Z}_n^2)$ such that every sequence of k elements in $\mathbb{Z}_n^2 = \mathbb{Z}_n \oplus \mathbb{Z}_n$ contains a zero-sum subsequence of length n? In 1983 Kemnitz [Ars Combin.] conjectured that $s(\mathbb{Z}_n^2) = 4n - 3$. That $s(\mathbb{Z}_n^2) \geqslant 4n - 3$ is easy, because the sequence consisting of $n - 1$ copies of

$$\langle 0, 0\rangle, \langle 0, 1\rangle, \langle 1, 0\rangle, \langle 1, 1\rangle \in \mathbb{Z}_n$$

has no zero-sum subsequence of length n.

In 1993 Alon and Dubiner showed that $s(\mathbb{Z}_n^2) \leqslant 6n-5$. In 2000 Rónyai [Combinatorica] was able to prove that $s(\mathbb{Z}_p^2) \leqslant 4p-2$ for every prime p, in 2001 W. D. Gao [J. Combin. Theory Ser. A] used Olson's result to deduce that $s(\mathbb{Z}_q^2) \leqslant 4q-2$ for any prime power q. All these results were obtained by various ingenious **algebraic methods**.

The following lemma plays an indispensable role in the study of the Kemnitz conjecture.

Lemma 1.1 (Alon and Dubiner, 1993). *Let* q *be a prime power, and let* $c_1, \cdots, c_{3q}$ *be elements of* $\mathbb{Z}_q^2$ *with* $c_1 + \cdots + c_{3q} = 0$. *Then there is an* $I \subseteq [1, 3q]$ *with* $|I| = q$ *such that* $\sum_{s\in I} c_s = 0$.

In 2003 Zhi-Wei Sun found a new approach to zero-sum problems via polynomials rather than Olson's group ring method. Thus, he can prove Olson's theorem in an elementary way and obtain some stronger results. The key point in his new approach is the following fresh observation: *If* p *is a prime,* $n \in \mathbb{N}$ *and* $a \in \mathbb{Z}$, *then*

$$\binom{a-1}{p^n-1} \equiv \begin{cases} 1 \pmod p & \text{if } p^n \mid a, \\ 0 \pmod p & \text{otherwise.} \end{cases}$$

Lemma 1.2 (Z. W. Sun, 2003). *Let* $q > 1$ *be a power of a prime* p, *and let* $a_s, b_s \in \mathbb{Z}$ *for* $s = 1, \cdots, 4q-2$. *Set*

$$\mathcal{I} = \Big\{ I \subseteq [1, 4q-2] \colon \sum_{s\in I} a_s \equiv \sum_{s\in I} b_s \equiv 0 \pmod q \Big\}.$$

Then we have

$$|\{I \in \mathcal{I} \colon |I| = q\}| \equiv |\{I \in \mathcal{I} \colon |I| = 3q\}| + 2 \pmod p.$$

Applying Lemma 1.2 with $a_{4q-2} = b_{4q-2} = 0$, we obtain

Corollary 1.1. *Let* q *be a power of a prime* p*, and let* $a_s, b_s \in \mathbb{Z}$ *for* $s = 1, \cdots, 4q-3$*. Set*

$$\mathcal{I} = \Big\{ I \subseteq [1, 4q-3] \colon \sum_{s\in I} a_s \equiv \sum_{s \in I} b_s \equiv 0 \pmod q \Big\}.$$

Then

$$\begin{aligned} &|\{I \in \mathcal{I} \colon |I| = q\}| + |\{I \in \mathcal{I} \colon |I| = q-1\}| \\ \equiv &|\{I \in \mathcal{I} \colon |I| = 3q\}| + |\{I \in \mathcal{I} \colon |I| = 3q-1\}| + 2 \pmod p. \end{aligned}$$

Although we have essentially exhausted the power of algebraic methods to attack the Kemnitz conjecture, the conjecture remained open until October in 2003.

Quite recently, using a **combinatorial argument**, C. Reiher got the following sophisticated result.

Lemma 1.3 (C. Reiher, 2003)**.** *Let* p *be a prime and let* $a_s, b_s \in \mathbb{Z}$ *for* $s = 1, \cdots, 4p-3$*. Set*

$$\mathcal{I} = \Big\{ I \subseteq [1, 4p-3] \colon \sum_{s\in I} a_s \equiv \sum_{s \in I} b_s \equiv 0 \pmod p \Big\}.$$

Then, either $\{I \in \mathcal{I} \colon |I| = p\} \neq \emptyset$ *or*

$$|\{I \in \mathcal{I} \colon |I| = p-1\}| \equiv |\{I \in \mathcal{I} \colon |I| = 3p-1\}| \pmod p.$$

We remark that the prime power version of this lemma also holds.

`Sketch of the Proof`. For $J \subseteq [1, 4p-3]$ and $n = 1, 2, \cdots$ let

$$(n, J) := \left|\left\{I \subseteq J \colon |I| = n \ \& \ \sum_{i\in I} a_i \equiv \sum_{i\in I} b_i \equiv 0 \ (\mathrm{mod}\ p)\right\}\right|.$$

It is easy to show that if $|J| \in \{3p-1, 3p-2\}$ then $(2p, J) \equiv (p, J) - 1 \ (\mathrm{mod}\ p)$.

Now assume that $\{I \in \mathcal{I} \colon |I| = p\} = \emptyset$, i.e., $(p, J) = 0$ for any $J \subseteq [1, 4p-3]$. Let N denote the number of partitions $[1, 4p-3] = I_1 \cup I_2 \cup I_3$ satisfying

$$|I_1| = p-1, \quad |I_2| = p-2, \quad |I_3| = 2p$$

and furthermore

$$\sum_{i\in I_1} a_i \equiv \sum_{i\in I_1} b_i \equiv 0 \ (\mathrm{mod}\ p), \quad \sum_{i\in I_3} a_i \equiv \sum_{i\in I_3} b_i \equiv 0 \ (\mathrm{mod}\ p)$$

(and hence $\sum_{i\in[1,4p-3]\setminus I_2} a_i \equiv \sum_{i\in[1,4p-3]\setminus I_2} b_i \equiv 0 \ (\mathrm{mod}\ p)$). We count N in two ways. Observe that

$$N = \sum_{I_1} (2p, [1, 4p-3] \setminus I_1) \equiv \sum_{I_1} (-1) = -(p-1, [1, 4p-3]) \ (\mathrm{mod}\ p).$$

On the other hand,

$$N = \sum_{I_2} (2p, [1, 4p-3] \setminus I_2) \equiv \sum_{[1,4p-3]\setminus I_2} (-1) = -(3p-1, [1, 4p-3]) \ (\mathrm{mod}\ p).$$

So $(p-1, [1, 4p-3]) \equiv (3p-1, [1, 4p-3]) \ (\mathrm{mod}\ p)$. $\square$

Combining Reiher's lemma, the Alon-Dubiner lemma and Corollary 1.1, we immediately get a proof of the Kemnitz conjecture which is shorter than Reiher's original proof.

Theorem 1.1 (C. Reiher, 2003)**.** *The Kemnitz conjecture is true, that is,* $s(\mathbb{Z}_n^2) = 4n - 3$.

What does Reiher's solution teach us? **A mixed use of algebraic methods and combinatorial methods might be more powerful!**

What about $s(\mathbb{Z}_n^3)$? C. Elsholtz [Combinatorica, in press] proved that $s(\mathbb{Z}_n^3) \geqslant 9n - 8$ for any odd $n > 1$. On the basis of this result, many zero-sum experts conjectured that $s(\mathbb{Z}_n^3) = 9n - 8$ for every odd integer $n = 3, 5, \ldots$. This seems very difficult.

2. Connections to covers of $\mathbb{Z}$

For $a \in \mathbb{Z}$ and $n \in \mathbb{Z}^+ = \{1, 2, 3, \cdots\}$ we call

$$a(n) = a + n\mathbb{Z} = \{a + nx : \ x \in \mathbb{Z}\}$$

a residue class with modulus n.

For a finite system

$$A = \{a_s(n_s)\}_{s=1}^k = \{a_1(n_1), \ldots, a_k(n_k)\}$$

of residue classes, its *covering function* is defined by

$$w_A(x) = |\{1 \leqslant s \leqslant k\colon x \in a_s(n_s)\}|,$$

which is periodic modulo $N_A = [n_1, \cdots, n_k]$. *The arithmetic mean of the covering function in a period equals the sum* $\sum_{s=1}^k 1/n_s$. In fact, letting $[x \equiv a_s (\mathrm{mod}\ n_s)]$ be the characteristic function of the residue class $a_s(n_s)$

we then have

$$\begin{aligned}\sum_{x=0}^{N_A-1} w_A(x) &= \sum_{x=0}^{N_A-1}\sum_{s=1}^{k}[x \equiv a_s \ (\mathrm{mod}\ n_s)] \\ &= \sum_{s=1}^{k}\sum_{x=0}^{N_A-1}[x \equiv a_s \ (\mathrm{mod}\ n_s)] = \sum_{s=1}^{k}\frac{N_A}{n_s}.\end{aligned}$$

Let m be a positive integer. If $w_A(x) \geqslant m$ for all $x \in \mathbb{Z}$, then we call system A an *m-cover* of $\mathbb{Z}$; if $w_A(x) = m$ for all $x \in \mathbb{Z}$, then we call system A an *exact m-cover* of $\mathbb{Z}$. By the above, $\sum_{s=1}^{k} 1/n_s \geqslant m$ for any m-cover A, and $\sum_{s=1}^{k} 1/n_s = m$ if A is an exact m-cover of $\mathbb{Z}$.

The first nontrivial cover of $\mathbb{Z}$ with distinct moduli was the following one discovered by P. Erdős.

$$B = \{0(2),\ 0(3),\ 1(4),\ 5(6),\ 7(12)\}.$$

Note that $N_B = 12$ and B covers $0, 1, \ldots, 11$. Observe that the sum of reciprocals of the moduli in the cover B equals

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{6} + \frac{1}{12} = 1\frac{1}{3}.$$

In 1976 Š. Porubský asked whether every exact m-cover is a union of m exact 1-covers. Choi supplied the following exact 2-cover

$$\{1(2); 0(3); 2(6); 0,4,6,8(10); 1,2,4,7,10,13(15); 5,11,12,22,23,29(30)\},$$

which is not a union of two exact 1-covers. In 1991, using a graph-theoretic argument M. Z. Zhang [J. Sichuan Univ. (Nat. Sci. Ed.)] proved that for each $m = 2, 3, \cdots$ there are infinitely many exact m-covers of $\mathbb{Z}$ which cannot be a union of an n-cover and an $(m-n)$-cover with $0 < n < m$.

In 1989, by using the Riemann zeta function, M. Z. Zhang [J. Sichuan Univ. (Nat. Sci. Ed.)] showed the following surprising result: *If* $A = \{a_s(n_s)\}_{s=1}^k$ *is a cover of* $\mathbb{Z}$ *then* $\sum_{s\in I} 1/n_s \in \mathbb{Z}^+$ *for some* $I \subseteq [1,k]$.

By a mixed use of tools from number theory, combinatorics, linear algebra and analysis, Z. W. Sun [Acta Arith. 72(1995); Trans. Amer. Math. Soc. 348(1996)] obtained the following local-global result:

Let $m_1, \ldots, m_k$ *be integers relatively prime to the moduli* $n_1, \ldots, n_k$ *of* $A = \{a_s(n_s)\}_{s=1}^k$ *respectively. Then* A *forms an* m*-cover of* $\mathbb{Z}$ *if and only if it covers* $|S(A)|$ *consecutive integers at least* m *times, where*

$$S(A) = \left\{ \left\{ \sum_{s\in I} \frac{m_s}{n_s} \right\} : I \subseteq [1,k] \right\}$$

and $\{\alpha\}$ stands for the fractional part of a real number α.

This result is stronger than a conjecture of P. Erdős which says that system A is a cover of $\mathbb{Z}$ if it covers integers from 1 to 2^k. Moreover, this local-global result initiated the later investigation [Z. W. Sun, Acta Arith. 81(1997); Proc. Amer. Math. Soc. 127(1999); Combinatorica 23(2003)] of the structure of $S(A)$.

Covers of $\mathbb{Z}$ and zero-sum problems have been investigated separately for about 50 years. In last year Z. W. Sun was able to establish connections between covers of $\mathbb{Z}$ and some classical theorems on zero-sums such as $D(\mathbb{Z}_n) = n$, the EGZ theorem, the Alon-Dubiner lemma and Olson's theorem. Before his unified work no one else had realized that zero-sum problems are connected with covers of $\mathbb{Z}$.

Theorem 2.1 (Z. W. Sun, 2003). *Let* $A = \{a_s(n_s)\}_{s=1}^k$ *and let* q *be a* *prime power. If A forms a q-cover of $\mathbb{Z}$, then for any $m_1, \ldots, m_k \in \mathbb{Z}$ there exists a nonempty $I \subseteq [1,k]$ such that $\sum_{s\in I} m_s/n_s \in q\mathbb{Z}$.*

In the case $q = 1$, this yields an extension of both Zhang's result and the basic fact $D(\mathbb{Z}_n) = n$. (Note that $\{r(n)\}_{r=0}^{n-1}$ is a cover of $\mathbb{Z}$.)

Conjecture 2.1 (Z. W. Sun, 2003). *We can replace the prime power q in Theorem 2.1 by any positive integer.*

Theorem 2.2 (Z. W. Sun, 2003). *Let $A = \{a_s(n_s)\}_{s=1}^k$ be a system of residue classes with $\{w_A(x) : x \in \mathbb{Z}\} \subseteq [p^n + p^h - 1, 2p^h]$, where p is a prime, $h, n \in \mathbb{N}$ and $h \geqslant n$. Then, for any $c_1, \ldots, c_k \in \mathbb{Z}_{p^n}$, there exists an $I \subseteq [1,k]$ such that $\sum_{s\in I} 1/n_s = p^h$ and $\sum_{s\in I} c_s = 0$.*

It is interesting to **view $\mathbf{1/n_s}$** in Theorem 2.2 **as a weight of $\mathbf{s}$** $\in [1,k]$. In the case $n_1 = \cdots = n_k = 1$ and $h = n$, Theorem 2.2 yields the famous EGZ theorem because $2p^n - 1$ copies of $0(1)$ form an exact $2p^n - 1$-cover of $\mathbb{Z}$.

Conjecture 2.2 (Z. W. Sun, 2003). *Let $A = \{a_s(n_s)\}_{s=1}^k$ cover each integer either exactly $2n-1$ times or exactly $2n$ times, where n is a positive integer. Then, for any $c_1, \ldots, c_k \in \mathbb{Z}_n$, there is an $I \subseteq [1,k]$ such that $\sum_{s\in I} 1/n_s = n$ and $\sum_{s\in I} c_s = 0$.*

Theorem 2.3 (Z. W. Sun, 2003). *Let $A = \{a_s(n_s)\}_{s=1}^k$ be an exact $3q$-cover of $\mathbb{Z}$ where q is a prime power. Then, for any $c_1, \ldots, c_k \in \mathbb{Z}_q^2$ with $c_1 + \cdots + c_k = 0$, there exists an $I \subseteq [1,k]$ such that $\sum_{s\in I} 1/n_s = q$ and $\sum_{s\in I} c_s = 0$.*

Since $3q$ copies of $0(1)$ form an exact $3q$-cover, Theorem 2.3 in the case $n_1 = \cdots = n_k = 1$ yields the Alon-Dubiner lemma.

Conjecture 2.3 (Z. W. Sun, 2003). *The prime power q in Theorem 2.3 can be replaced by any positive integer.*

Theorem 2.4 (Z. W. Sun, 2003). *Let G be an additive abelian p-group (where p is a prime), and let $A = \{a_s(n_s)\}_{s=1}^{k}$ be an $L(G)$-cover of $\mathbb{Z}$. Then, for any $m_1, \cdots, m_k \in \mathbb{Z}$ and $c_1, \cdots, c_k \in G$, there is a nonempty $I \subseteq [1, k]$ such that $\sum_{s\in I} m_s/n_s \in \mathbb{Z}$ and $\sum_{s\in I} c_s = 0$.*

In the case $n_1 = \cdots = n_k = 1$, this yields Olson's theorem.

It seems that we cannot have a similar extension of the (confirmed) Kemnitz conjecture.

We mention that Theorems 2.1–2.4 are special cases of a more general unified theorem obtained and announced by Z. W. Sun in 2003 (see Electron. Res. Announc. Amer. Math. Soc. 9(2003), 51–60, MR 2004i:11017). The full paper containing proofs of these theorems has been available from the public preprint server arXiv (`arXiv:math.NT/0305369`) since May, 2003. Here is a comment from a paper of V. Dimitrov: "**We note that a deep unification of zero-sum problems, subset sums and covers of $\mathbb{Z}$ (the three main areas of additive number theory, each initiated by P. Erdős) was very recently established by Zhi-Wei Sun**".