Symplectic Graphs and Their Automorphisms

Joint work with Zhongming Tang

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1. Definition

$\mathbb{F}_q$

$\nu \geq 1$

$\mathbb{F}_q^{2\nu} = \{(a_1, a_2, \ldots, a_{2\nu})\colon a_i \in \mathbb{F}_q\}$,

$\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{F}_q^{2\nu}$

$[\alpha_1, \alpha_2, \ldots, \alpha_n]$ subspace spanned by $\alpha_1, \alpha_2, \ldots, \alpha_n$.

$$K = \begin{pmatrix} 0 & 1 & & & & & \\ -1 & 0 & & & & & \\ & & 0 & 1 & & & \\ & & -1 & 0 & & & \\ & & & & \ddots & & \\ & & & & & 0 & 1 \\ & & & & & -1 & 0 \end{pmatrix}_{2\nu \times 2\nu} .$$

$\mathrm{Sp}(2\nu, q)$ Symplectic graph over $\mathbb{F}_q$

$Vertex\ Set = \{1 - dim.\ subspaces\ of\ \mathbb{F}_q^{2\nu}\}$

$Adjacency : \forall\ \alpha, \beta \in \mathbb{F}_q^{2\nu}, \alpha \neq 0, \beta \neq 0.$

$[\alpha] \sim [\beta](\text{adjacent}) \Longleftrightarrow \alpha K^t \beta \neq 0.$

$q = 2,$
Rotman 1993
Rotman-Weichsel 1994
Godsil-Royle 2001

2. Strong Regularity

Theorem 1. *$Sp(2\nu, q)$ is strongly regular with parameters*

$$\left(\frac{q^{2\nu}-1}{q-1}, q^{2\nu-1}, q^{2\nu-2}(q-1), q^{2\nu-2}(q-1)\right)$$

and eigenvalues $q^{2\nu-1}, q^{\nu-1}, -q^{\nu-1}$.

Proof. $|\mathbb{F}_q^{2\nu}| = q^{2\nu} \Longrightarrow |V(Sp(2\nu, q))| = \frac{q^{2\nu}-1}{q-1}$.

Notation :

$V \subseteq \mathbb{F}_q^{2\nu}, V^{\perp} = \{\beta \in \mathbb{F}_q^{2\nu} : \beta K^t \alpha = 0, \forall\ \alpha \in V\}$

$[\alpha] \in V(Sp(2\nu, q)),\ dim[\alpha]^{\perp} = 2\nu - 1.$

$$deg[\alpha] = \#1 - dim\ subspaces[\beta] \quad s.t.\ \beta \notin [\alpha]^{\perp}$$
$$= \frac{q^{2\nu} - q^{2\nu-1}}{q-1} = q^{2\nu-1}.$$

$[\alpha], [\beta] \in V(Sp(2\nu, q)), [\alpha] \neq [\beta], [\alpha] \sim [\beta], or [\alpha] \nsim [\beta].$

$[\gamma] \sim [\alpha], [\beta] \Longleftrightarrow \gamma \notin [\alpha]^{\perp} \cup [\beta]^{\perp}$

$|[\alpha]^{\perp} \cup [\beta]^{\perp}| = |[\alpha]^{\perp}| + |[\beta]^{\perp}| - |[\alpha]^{\perp} \cap [\beta]^{\perp}|.$

$|[\alpha]^{\perp} \cap [\beta]^{\perp}| = |[\alpha, \beta]^{\perp}|.$

$p = q = \frac{|\mathbb{F}_q^{2\nu}| - |[\alpha]^{\perp} \cup [\beta]^{\perp}|}{q-1} = \frac{q^{2\nu} - 2q^{2\nu-1} + q^{2\nu-2}}{q-1} = q^{2\nu-2}(q-1).$

3. Chromatic Number

Definition. $V \subseteq \mathbb{F}_q^{2\nu}$ *totally isotropic, if* $V \subseteq V^{\perp}$.

Facts

(1) Totally isotropic subspaces are of $dim \leq \nu$.

(2) $\exists$ totally isotropic subspaces of $dim\, \nu$.

maximal totally isotropic subspaces

Lemma. $\exists$ *maximal totally isotropic subspaces* $V_i, i = 1, 2, \ldots, q^\nu + 1$ *of* $\mathbb{F}_q^{2\nu}$ *s.t.*

$$\mathbb{F}_q^{2\nu} = V_1 \cup V_2 \cup \cdots \cup V_{q^\nu+1},$$

$$V_i \cap V_j = \{0\}, \textit{ for all } i \neq j.$$

Proposition. $Sp(2\nu, q)$ is $(q^\nu + 1)$-partite.

Proof. Let $X_i = \{[\alpha] \colon \alpha \in V_i, \alpha \neq 0\}$. Then

$$VSp(2\nu, q) = X_1 \cup X_2 \cup \cdots \cup X_{q^\nu+1},$$

$X_i \cap X_j = \phi$, for all $i \neq j$.

Every edge has one end in some X_i and one end in some $X_j, i \neq j$.

Theorem 2. $\chi(Sp(2\nu, q)) = q^{\nu} + 1.$

Proof. Let $\chi(Sp(2\nu, q)) = n$.

By Proposition, $n \leq q^{\nu} + 1$.

$$V(Sp(2\nu, q)) = Y_1 \cup Y_2 \cup \cdots \cup Y_n, Y_i \cap Y_j = \phi.$$

No edge connects any two vertices in the same Y_i,

$$\sum_{i=1}^{n} |Y_i| = \frac{q^{2\nu} - 1}{q - 1} = \frac{q^{\nu} - 1}{q - 1}(q^{\nu} + 1).$$

Suppose $n < q^\nu + 1$.

$\Longrightarrow \exists\ i \quad s.t. |Y_i| > \frac{q^\nu - 1}{q-1}$.

Let $W_i = [\alpha \colon [\alpha] \in Y_i]$

$\Longrightarrow W_i$ totally isotropic,

$\Longrightarrow dim W_i \leq \nu$,

$\Longrightarrow |Y_i| \leq \frac{q^\nu - 1}{q-1}$. Contradiction

4. Automorphisms

Definition. $T \quad 2\nu \times 2\nu/\mathbb{F}_q$

T symplectic, if $TK^tT = K$

T generalized symplectic, if $TK^tT = kK, k \in \mathbb{F}_q^*$

$$Sp_{2\nu}(\mathbb{F}_q) = \{T \text{ symplectic}\},$$

$$GSp_{2\nu}(\mathbb{F}_q) = \{T \text{ generalized symplectic}\},$$

$$q = 2 \Longrightarrow GSp_{2\nu}(\mathbb{F}_q) = Sp_{2\nu}(\mathbb{F}_q).$$

Proposition. $T \in GL_{2\nu}(\mathbb{F}_q)$, Define

$$\sigma_T\colon V(Sp(2\nu, q)) \longmapsto V(Sp(2\nu, q))$$

$$[\alpha] \longmapsto [\alpha T]$$

(1) $T \in GSp_{2\nu}(\mathbb{F}_q) \Longleftrightarrow \sigma_T \in Aut(Sp(2\nu, q))$,

(2) $T \in Sp_{2\nu}(\mathbb{F}_q), \sigma_T = id. \Longrightarrow T = \pm I^{(2\nu)}$.

Proof. “ $\Rightarrow$ ” $T \in GSp_{2\nu}(\mathbb{F}_q)$, $[\alpha], [\beta] \in V(Sp(2\nu, q))$.

$(\alpha T)K^t(\beta T) = \alpha T K^t T^t \beta = k\alpha K^t \beta, k \in \mathbb{F}_q^*,$

$\alpha K^t \beta = 0 \Longleftrightarrow (\alpha T)K^t(\beta T) = 0.$

“ $\Leftarrow$ ”

$\sigma_T \in Aut(Sp(2\nu, q))\colon \alpha K^t \beta = 0 \Longleftrightarrow (\alpha T)K^t(\beta T) = 0.$

$\Longrightarrow \alpha K = k(\alpha T K^t T), k \in \mathbb{F}_q^*.$

$\Longrightarrow K = kTK^tT.$

Theorem 3. $Aut(Sp(2\nu, 2)) = Sp_{2\nu}(\mathbb{F}_2)$.

Proof. $\sigma\colon Sp_{2\nu}(\mathbb{F}_q) \longmapsto Aut(Sp(2\nu, 2))$,

$$T \longmapsto \sigma_T.$$

Remains to show : for any $\tau \in Aut(Sp(2\nu, 2))$,

$$\exists\ T \in Sp_{2\nu}(\mathbb{F}_q), s.t.\sigma_T = \tau.$$

$e_i = (0, \ldots, 0, 1, 0, \ldots, 0), \quad i = 1, 2, \ldots, 2\nu.$

$e_{2i-1} K^t e_{2i} = 1, \quad i = 1, 2, \ldots, \nu.$

$e_i K^t e_j = 0, \quad otherwise.$

$[e_{2i-1}] \sim [e_{2i}], [e_i] \nsim [e_j], \quad otherwise.$

$\tau[e_{2i-1}] \sim \tau[e_{2i}], \tau[e_i] \nsim \tau[e_j], \quad otherwise.$

Let $\tau[e_i] = [f_i], \quad i = 1, 2, \ldots, 2\nu.$

Then $f_{2i-1} K^t f_{2i} = 1, \quad i = 1, 2, \ldots, \nu.$

$f_i K^t f_j = o, \quad otherwise.$

By Witt's Theorem, $\exists\, T \in Sp_{2\nu}(\mathbb{F}_2)$,

$$\text{s.t. } f_i T = e_i, \quad i = 1, 2, \ldots, 2\nu.$$

Then $(\sigma_T \circ \tau)(e_i) = e_i, \quad i = 1, 2, \ldots, 2\nu.$

$(e_1 + e_2) K^t e_j = 0, \quad j = 3, 4, \ldots, 2\nu.$

$(e_1 + e_2) K^t e_1 = (e_1 + e_2) K^t e_2 = 0,$

$\Longrightarrow (\sigma_T \circ \tau)(e_1 + e_2) = e_1 + e_2,$ etc.

$\therefore \sigma_T \circ \tau = id, \quad \tau = \sigma_{T^{-1}}.$

Set $PSp_{2\nu}(\mathbb{F}_q) = Sp_{2\nu}(\mathbb{F}_q)/\{\pm I^{(2\nu)}\}$

Theorem4. $PSp_{2\nu}(\mathbb{F}_q) \subset Aut(Sp(2\nu, q))$,

$$Aut(Sp(2\nu, q)) = PSp_{2\nu}(\mathbb{F}_q) \cdot E,$$

where

$$E = \{\sigma \in Aut(Sp(2\nu, q)) \colon \sigma([e_i]) = [e_i], i = 1, 2, \ldots, 2\nu.\}$$

If $\nu = 1$,

$$E \cong S_{q-1}(\text{symmetric group on } q{-}1 \text{ elements})$$

If $\nu > 1$,

$$E \cong (\underbrace{\mathbb{F}_q^* \times \cdots \mathbb{F}_q^*}_{\nu}) \times_\varphi Aut(\mathbb{F}_q),$$

where φ is the natural homomorphism from $Aut(\mathbb{F}_q)$ to $Aut(\mathbb{F}_q^* \times \cdots \times \mathbb{F}_q^*)$ which maps any $\pi \in Aut(\mathbb{F}_q)$ to the following automorphism of $\mathbb{F}_q^* \times \cdots \times \mathbb{F}_q^*$.

$$\mathbb{F}_q^* \times \cdots \times \mathbb{F}_q^* \longmapsto (\mathbb{F}_q^* \times \cdots \times \mathbb{F}_q^*),$$

$$(k_1, \ldots, k_\nu) \longmapsto (\pi(k_1), \ldots, \pi(k_\nu)).$$