The Turán number of $2P_7$

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Abstract

The Turán number of a graph H, denoted by $ex(n, H)$, is the maximum number of edges in any graph on n vertices which does not contain H as a subgraph. Let P_k denote the path on k vertices and let mP_k denote m disjoint copies of P_k. Bushaw and Kettle [Turán numbers of multiple paths and equibipartite forests, Combin. Probab. Comput. 20(2011) 837–853] determined the exact value of $ex(n, kP_\ell)$ for large values of n. Yuan and Zhang [The Turán number of disjoint copies of paths, Discrete Math. 340(2)(2017) 132–139] completely determined the value of $ex(n, kP_3)$ for all n, and also determined $ex(n, F_m)$, where F_m is the disjoint union of m paths containing at most one odd path. They also determined the exact value of $ex(n, P_3 \cup P_{2\ell+1})$ for $n \geq 2\ell + 4$. Recently, Bielak and Kieliszek [The Turán number of the graph $2P_5$, Discuss. Math. Graph Theory 36(2016) 683–694], Yuan and Zhang [Turán numbers for disjoint paths, arXiv: 1611.00981v1] independently determined the exact value of $ex(n, 2P_5)$. In this paper, we show that $ex(n, 2P_7) = \max\{[n, 14, 7], 5n - 14\}$ for all $n \geq 14$, where $[n, 14, 7] = (5n + 91 + r(r - 6))/2$, $n - 13 \equiv r \pmod 6$ and $0 \leq r < 6$.

Keywords: Turán number; extremal graphs; $2P_7$
AMS subject classification 2010: 05C35.

1 Introduction

Throughout this paper, we only consider simple graphs. For a graph G we use $V(G)$, $|G|$, $E(G)$, $e(G)$ to denote the vertex set, number of vertices, edge set, number of edges, respectively. For $S_1, S_2 \subseteq V(G)$ and $S_1 \cap S_2 = \emptyset$, denote by $e(S_1, S_2)$ the number of edges between S_1 and S_2. Let G and H be two disjoint graphs. By $G \cup H$ denote the disjoint

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union of graphs G and H and by kG denote the k disjoint copies of G. Denote by $G + H$ the graph obtained from $G \cup H$ by joining all vertices of G to all vertices of H. Let $\overline{G}$ be the complement of the graph G. Denote by P_n, C_n and K_n the path, cycle and complete graph on n vertices, respectively. For $S \subseteq V(G)$, let $G[S]$ denote the subgraph of G induced by S and let $|S|$ denote the cardinality of S. For a graph G and its subgraph H, by $G - H$ we mean a graph obtained from G by deleting all vertices of H with all incident edges. If H consists of a single vertex x, then we simple write $G - x$. For $v \in V(G)$, let $N_G(v)$ denote the set of vertices in G which are adjacent to v. We define $d_G(v) = |N_G(v)|$.

A graph is *H-free* if it does not contain H as a subgraph. The *Turán number* of a graph H, denoted by $ex(n, H)$, is the maximum number of edges in any H-free graph on n vertices, i.e.,

$$ex(n, H) = \max\{e(G) : H \nsubseteq G \text{ and } |G| = n\}.$$

Let $EX(n, H)$ denote the family of all H-free graphs on n vertices with $ex(n, H)$ edges. A graph in $EX(n, H)$ is called an *extremal graph* for H. Moreover, we denote by $ex_{con}(n, H)$ the maximum number of edges in any connected H-free graph on n vertices. The problem of determining Turán number for assorted graphs traces its history back to 1907, when Mantel (see, *e.g.*, [3]) proved $ex(n, C_3) = \lfloor n^2/4 \rfloor$. In 1941, Turán [13, 14] proved that the extremal graph for K_r is the complete $(r-1)$-partite graph, which is as balanced as possible (any two part sizes differ at most 1). The balanced complete $(r-1)$-partite graph on n vertices is called as the Turán graph denoted by $T_{r-1}(n)$. For sparse graphs, Erdős and Gallai [5] in 1959 proved the following well known result.

Theorem 1.1 ([5]) *Let G be a P_k-free graph on n vertices and $n \geq k \geq 2$. Then $e(G) \leq (k-2)n/2$ with equality if and only if $n = (k-1)t$ and $G = tK_{k-1}$.*

For convenience, we first introduce the following symbols.

Definition 1.2 Let $n \geq m \geq \ell \geq 3$ be given three positive integers. If n can be written as $n = (m-1) + t(\ell - 1) + r$, where $t \geq 0$ and $0 \leq r < \ell - 1$, then we denote

$$[n, m, \ell] = \binom{m-1}{2} + t\binom{\ell-1}{2} + \binom{r}{2}.$$

Moreover, if $n \leq m - 1$, then we denote

$$[n, m, \ell] = \binom{n}{2}.$$

Definition 1.3 Let $s = \sum_{i=1}^{m} \lfloor k_i/2 \rfloor$ and k_i be positive integers. If $n \geq s$, then we denote

$$[n, s] = \binom{s-1}{2} + (s-1)(n-s+1).$$

Later, for all integers n and k, Faudree and Schelp [7] characterized all extremal graphs for P_k.

Theorem 1.4 ([7]) *Let G be a graph on $n = t(k-1)+r$ ($0 \leq t$ and $0 \leq r < k-1$) vertices. If G is P_k-free, then $e(G) \leq [n, k, k]$. Moreover, the equality holds if and only if*

- $G = (tK_{k-1}) \cup K_r$ *or*
- $G = ((t-s-1)K_{k-1}) \cup (K_{(k-2)/2} + \overline{K_{k/2+s(k-1)+r}})$, *where k is even, $t > 0$, $r = k/2$ or $(k-2)/2$ and $0 \leq s < t$.*

Corollary 1.5 *For a positive integer $n \equiv r \ (mod\ k)$, $ex(n, P_{k+1}) = (n(k-1) + r(r-k))/2$.*

We see that $ex(n, P_k)$ has been determined for all integers $n \geq k$ and all extremal graphs has also been characterized. For connected graphs, Kopylov [8] and Balister, Győri, Lehel, and Schelp [1] determined $ex_{con}(n, P_k)$ and characterized all extremal graphs for all integers $n \geq k$. Recently, Lan, Shi and Song [9] studied the Turán number of paths in planar graphs.

Theorem 1.6 ([1, 8]) *Let G be a connected P_k-free graph on n vertices and $n \geq k \geq 4$. Then*

$$e(G) \leq \max\left\{\binom{k-2}{2} + (n-k+2), [n, \lfloor k/2 \rfloor] + c\right\},$$

where $k \equiv c \ (mod\ 2)$. Further, the equality holds if and only if $G = (K_{k-3} \cup \overline{K_{n-k+2}}) + K_1$ or $G = (K_{1+c} \cup \overline{K_{n-\lfloor (k+1)/2 \rfloor}}) + K_{\lfloor k/2 \rfloor - 1}$.

In 1962, Erdős [6] first studied the Turán number of kK_3. And later, Moon [11] and Simonovits [12] studied the case of kK_r. In 2011, Bushaw and Kettle [4] determined $ex(n, kP_\ell)$ for n sufficiently large.

Theorem 1.7 ([4]) *For integers $k \geq 2$, $\ell \geq 4$ and $n \geq 2\ell + 2k\ell(\lceil \ell/2 \rceil + 1)\binom{\ell}{\lfloor \ell/2 \rfloor}$,*

$$ex(n, kP_\ell) = \left[n, k\left\lfloor \frac{\ell}{2} \right\rfloor\right] + c,$$

where $\ell \equiv c \ (mod\ 2)$.

Furthermore, their proof shows that their construction is optimal for $n = \Omega(k\ell^{3/2}2^\ell)$. Moreover, Bushaw and Kettle conjectured that their construction is optimal for $n = \Omega(k\ell)$. Recently, Lidický et al. [10] extended Bushaw and Kettle's result and determined $ex(n, F_m)$ for n sufficiently large, where $F_m = \bigcup_{i=1}^m P_{k_i}$ and $k_1 \geq k_2 \geq \ldots \geq k_m$.

Theorem 1.8 ([10]) *Let $F_m = \bigcup_{i=1}^m P_{k_i}$ and $k_1 \geq k_2 \geq \ldots \geq k_m$. If at least one k_i is not 3, then for n sufficiently large,*

$$ex(n, F_m) = \left[n, \sum_{i=1}^m \left\lfloor \frac{k_i}{2} \right\rfloor\right] + c,$$

where $c = 1$ if all k_i are odd, and $c = 0$ otherwise. Moreover, the extremal graph is unique.

However, they did not consider $ex(n, F_m)$ for smaller n. Recently, Yuan and Zhang [15, 16] completely determined the value of $ex(n, kP_3)$ and characterized all the extremal graphs for all n. Furthermore, they proved the following result in which F_m contains at most one odd path and proposed Conjecture 1.10.

Theorem 1.9 ([15]) *Let $k_1 \geq k_2 \geq \ldots \geq k_m \geq 3$, $n \geq \sum_{i=1}^{m} k_i$ and $F_m = \bigcup_{i=1}^{m} P_{k_i}$. If there is at most one odd in $\{k_1, k_2, \ldots, k_m\}$, then*

$$ex(n, F_m) = \max\left\{[n, k_1, k_1], [n, k_1 + k_2, k_2], \ldots, \left[n, \sum_{i=1}^{m} k_i, k_m\right], \left[n, \sum_{i=1}^{m} \left\lfloor \frac{k_i}{2} \right\rfloor\right]\right\}.$$

Moreover, if $k_1, k_2, \ldots, k_m$ are even, then the extremal graphs are characterized.

Conjecture 1.10 ([15]) *Let $k_1 \geq k_2 \geq \ldots \geq k_m \geq 3$, $k_1 > 3$ and $F_m = \bigcup_{i=1}^{m} P_{k_i}$. Then*

$$ex(n, F_m) = \max\left\{[n, k_1, k_1], [n, k_1 + k_2, k_2], \ldots, \left[n, \sum_{i=1}^{m} k_i, k_m\right], \left[n, \sum_{i=1}^{m} \left\lfloor \frac{k_i}{2} \right\rfloor\right] + c\right\},$$

where $c = 1$ if all of $k_1, k_2, \ldots, k_m$ are odd, and $c = 0$ otherwise. Moreover, the extremal graphs are

$$EX(n, P_{k_1}), \ldots, K_{\sum_{i=1}^{m} k_i - 1} \cup H \text{ for } H \in EX(n - \textstyle\sum_{i=1}^{m} k_i + 1, P_{k_m}), \text{ and}$$

$$K_{\sum_{i=1}^{m} \lfloor k_i/2 \rfloor - 1} + (K_{1+c} \cup \overline{K_{n - \sum_{i=1}^{m} \lfloor k_i/2 \rfloor - c}}).$$

When there are at least two odd integers in $\{k_1, k_2, \ldots, k_m\}$, Yuan and Zhang also determined $ex(n, P_3 \cup P_{2\ell+1})$ for $n \geq 2\ell + 4$ and characterized all extremal graphs. Bielak and Kieliszek [2] and Yuan and Zhang [15] independently determined $ex(n, 2P_5)$ and characterized all extremal graphs. In this paper, we prove the following result, which partially confirms Conjecture 1.10.

Theorem 1.11 *For $n \geq 14$,*

$$ex(n, 2P_7) = \max\{[n, 14, 7], 5n - 14\}.$$

Moreover, the extremal graphs are $K_{13} \cup H$ for $H \in EX(n - 13, P_7)$ when $n \leq 21$ and $K_5 + (K_2 \cup \overline{K_{n-7}})$ when $n \geq 22$.

2 Proof of Theorem 1.11

We first present some useful lemmas. In the following, we say that u *hits* v or v *hits* u if two vertices u and v are adjacent. Otherwise, we say that u *misses* v or v *misses* u if u and v are not adjacent. We say a vertex set A hits (misses) a vertex set B, it means that each vertex of A is adjacent (non-adjacent) to each vertex of B.

Lemma 2.1 (Observation 2 of [15]) *Let $k_1 \geq k_2 \geq 3$ be two positive integers. If $n_1 \geq k_1$, then $[n_1, k_1 + k_2, k_2] + [n_2, k_2, k_2] \leq [n_1 + n_2, k_1 + k_2, k_2]$.*

Lemma 2.2 (Observation 5 of [15]) *Let $k_1 \geq k_2 \geq 3$ be two positive integers. If $n_1 \geq k_1 + k_2$, then $[n_1, \lfloor k_1/2 \rfloor + \lfloor k_2/2 \rfloor] + [n_2, k_2, k_2] < [n_1 + n_2, \lfloor k_1/2 \rfloor + \lfloor k_2/2 \rfloor]$.*

Lemma 2.3 *Let G be a connected $2P_7$-free graph on $n \geq 14$ vertices. Then*

$$e(G) \leq \max\{[n, 14, 7], 5n - 14\}.$$

with equality only when $n \geq 22$ and $G = K_5 + (\overline{K_{n-7}} \cup K_2)$.

Proof. Let $G \neq K_5 + (\overline{K_{n-7}} \cup K_2)$ be any connected $2P_7$-free graph on n vertices with $e(G) \geq \max\{[n, 14, 7], 5n - 14\}$ edges. Note that $\max\{[n, 14, 7], 5n - 14\} = [n, 14, 7]$ when $n \leq 21$ and $\max\{[n, 14, 7], 5n-14\} = 5n-14$ when $n \geq 22$. Since $\max\{[n, 14, 7], 5n-14\} \geq ex_{con}(n, P_{13})$, by Theorem 1.6, G contains P_{13} as a subgraph. Let $P_{13} = x_1x_2 \ldots x_{13}$ be a subgraph of G. Then

$(*)$ each vertex of $G - P_{13}$ cannot hit two adjacent vertices in P_{13}.

Notice that each vertex in $G-P_{13}$ misses $\{x_1, x_6, x_8, x_{13}\}$ and can not hit both x_p and x_{p+8} for $p \in \{2, 3, 4\}$. Moreover, if y is an isolated vertex in $G-P_{13}$, then by $(*)$, $|N_G(y) \cap V(P_{13})| \leq 5$; if y is not an isolated vertex in $G - P_{13}$, then $N_G(y) \cap V(P_{13}) \subseteq \{x_3, x_4, x_7, x_{10}, x_{11}\}$ and so $|N_G(y) \cap V(P_{13})| \leq 3$ by $(*)$; if $P_k = y_1y_2 \ldots y_k \subseteq G - P_{13}$ and $k \geq 3$ such that y_1 hits P_{13}, then y_1 can only hit x_7. Now we will prove the following useful Facts.

Fact 1. $e(G[V(P_{13})]) \leq 74$.

Since G is connected and $n \geq 14$, at least one vertex of $V(G)\backslash V(P_{13})$ hits P_{13}, say x_i. Then either $i \geq 6$ or $i \leq 8$. Without loss of generality, we may assume that $i \geq 6$. For $1 \leq j \leq i-2$, if both $x_{13}x_j \in E(G)$ and $x_{i+1}x_{j+1} \in E(G)$, then G contains $2P_7$ as a subgraph, a contradiction. Thus $e(G[V(P_{13})]) \leq 74$.

Fact 2. If there exists a $P_3 = y_1y_2y_3 \subseteq G - P_{13}$ such that y_1 hits P_{13}, then we have $e(G[V(P_{13})]) \leq 57$.

Clearly, y_1 must hit x_7 and so G contains a copy of P_7 with vertices $x_4, x_5, x_6, x_7, y_1, y_2, y_3$. Therefore, $\{x_1, x_2, x_3, x_5, x_6\}$ misses $\{x_{11}, x_{12}, x_{13}\}$. Symmetrically, $\{x_8, x_9, x_{11}, x_{12}, x_{13}\}$ misses $\{x_1, x_2, x_3\}$. So $e(G[V(P_{13})]) \leq 78 - (2 \cdot 15 - 9) = 57$. □

Fact 3. If there exists a non-isolated vertex in $G - P_{13}$, that hits one vertex of P_{13}, then we have $e(G[V(P_{13})]) \leq 68$.

Let y be a non-isolated vertex in $G - P_{13}$, that hits one vertex, say x_i of P_{13}. Recall that $x_i \in \{x_3, x_4, x_7, x_{10}, x_{11}\}$. If $x_i \in \{x_3, x_4\}$, then $\{x_1, x_2, \ldots, x_{i-1}\}$ misses $\{x_{i+1}, x_{i+2}, x_9, x_{12}, x_{13}\}$ and so $e(G[V(P_{13})]) \leq 68$. Symmetrically, if $x_i \in \{x_{10}, x_{11}\}$, then $e(G[V(P_{13})]) \leq 68$. Now assume that $x_i = x_7$. Then $\{x_1, x_2, x_{i-1}, x_{i-2}\}$ misses $\{x_{12}, x_{13}\}$ and symmetrically

$\{x_{i+1}, x_{i+2}, x_{12}, x_{13}\}$ misses $\{x_1, x_2\}$. So $e(G[V(P_{13})]) \leq 78 - (2 \cdot 8 - 4) = 66$. $\square$

Fact 4. If there exists a non-isolated vertex in $G - P_{13}$, that hits two vertices of P_{13}, then we have $e(G[V(P_{13})]) \leq 59$.

Let y be a non-isolated vertex in $G - P_{13}$, that hits two vertices, say x_i and x_j ($i < j$), of P_{13}. Recall that $\{x_i, x_j\} \subseteq \{x_3, x_4, x_7, x_{10}, x_{11}\}$ and $\{x_i, x_j\} \neq \{x_3, x_{11}\}$. If $x_i = x_3$, then by $(*)$, $x_j \in \{x_7, x_{10}\}$. Thus $\{x_1, x_2\}$ misses $\{x_4, x_5, x_6, x_8, x_9, x_{11}, x_{12}, x_{13}\}$ and $\{x_{j-2}, x_{j-1}\}$ misses $\{x_{12}, x_{13}\}$. So $e(G[V(P_{13})]) \leq 58$. Symmetrically, if $x_j = x_{11}$, then by $(*)$, $x_i \in \{x_4, x_7\}$ and so $e(G[V(P_{13})]) \leq 58$. Now we can assume that $x_i \neq x_3$ and $x_j \neq x_{11}$. If $x_i = x_4$, then $x_j \in \{x_7, x_{10}\}$. Thus $\{x_1, x_2, x_3\}$ misses $\{x_5, x_6, x_9, x_{12}, x_{13}\}$ and $\{x_{j-2}, x_{j-1}\}$ misses $\{x_{12}, x_{13}\}$. So $e(G[V(P_{13})]) \leq 59$. Symmetrically, if $x_j = x_{10}$, then $x_i \in \{x_4, x_7\}$ and so $e(G[V(P_{13})]) \leq 59$. $\square$

Fact 5. If there exists an isolated vertex in $G - P_{13}$, that hits five vertices of P_{13}, then $e(G[V(P_{13})]) \leq 50$.

Let y be an isolated vertex in $G - P_{13}$, that hits exactly five vertices, say $x_i, x_j, x_k, x_\ell, x_m$, $i < j < k < \ell < m$ of P_{13}. Recall that $\{x_i, x_j, x_k, x_\ell, x_m\} \subseteq V(P_{13}) \setminus \{x_1, x_6, x_8, x_{13}\}$ and y cannot hit both x_p and x_{p+8} for $p \in \{2, 3, 4\}$. Since y cannot hit two adjacent vertices in P_{13}, we have $x_k = x_7$, $\{x_i, x_j\} \subseteq \{x_2, x_3, x_4, x_5\}$ and $\{x_\ell, x_m\} \subseteq \{x_9, x_{10}, x_{11}, x_{12}\}$. Let $A = \{x_{i-1}, x_{j-1}, x_{k-1}, x_{\ell-1}, x_{m-1}, x_{13}\}$ and $B = \{x_1, x_{i+1}, x_{j+1}, x_{k+1}, x_{\ell+1}, x_{m+1}\}$. Then, A and B are independent sets and $|A \cap B| = 4$. Since $\{x_3, x_{11}\} \nsubseteq N_G(y)$, we have either $i = 2$ or $m = 12$. If $i = 2$ and $m = 12$, then $N_G(y) = \{x_2, x_5, x_7, x_9, x_{12}\}$, which implies that x_5 misses $\{x_{10}, x_{11}\}$. And symmetrically x_9 misses $\{x_3, x_4\}$. If $i = 2$ and $m \neq 12$, then $\ell = 9$ and $m = 11$, which implies that x_m misses $\{x_3, x_6\}$ and x_ℓ misses $\{x_q, x_{q+1}\} \subseteq \{x_1, \ldots, x_7\} \setminus N_G(y)$. If $i \neq 2$ and $m = 12$, then $i = 3$ and $j = 5$, which implies that x_i misses $\{x_8, x_{11}\}$ and x_j misses $\{x_q, x_{q+1}\} \subseteq \{x_7, \ldots, x_{13}\} \setminus N_G(y)$. For each of the above cases, we have $e(G[V(P_{13})]) \leq 78 - (\binom{|A|}{2} + \binom{|B|}{2} - \binom{|A \cap B|}{2}) - 4 = 50$. $\square$

Fact 6. If there exists an isolated vertex in $G - P_{13}$, that hits four vertices of P_{13}, then $e(G[V(P_{13})]) \leq 59$.

Let y be an isolated vertex in $G - P_{13}$, that hits exactly four vertices, say x_i, x_j, x_k, x_ℓ, $i < j < k < \ell$ of P_{13}. Recall that $\{x_i, x_j, x_k, x_\ell\} \subseteq V(P_{13}) \setminus \{x_1, x_6, x_8, x_{13}\}$ and y cannot hit both x_p and x_{p+8} for $p \in \{2, 3, 4\}$. Let $A = \{x_{i-1}, x_{j-1}, x_{k-1}, x_{\ell-1}, x_{13}\}$ and $B = \{x_1, x_{i+1}, x_{j+1}, x_{k+1}, x_{\ell+1}\}$. Then A and B are independent sets and $|A \cap B| \leq 3$. If $|A \cap B| \leq 2$, then $e(G[V(P_{13})]) \leq 78 - (\binom{|A|}{2} + \binom{|B|}{2} - 1) = 59$. Now we assume that $|A \cap B| = 3$. If $i = 2$ and $\ell = 12$, then $7 \in \{j, k\}$ which implies that x_3 misses x_{11} and x_p misses x_{p+9} for $p \in \{1, 4\}$. If $i = 2$, $\ell \neq 12$ and $7 \in \{j, k\}$, then x_{11} misses $\{x_3, x_6\}$. If $i = 2$, $\ell \neq 12$ and $7 \notin \{j, k\}$, then $N_G(y) = \{x_2, x_4, x_9, x_{11}\}$ which implies x_{11} misses $\{x_5, x_8\}$. If $\ell = 12$ and $i \neq 2$, then it is similar as the case of $i = 2$ and $\ell \neq 12$. If $i \neq 2$ and $\ell \neq 12$, then $N_G(y) = \{x_3, x_5, x_7, x_9\}$ which implies x_{11} misses $\{x_1, x_4\}$. For each of the above cases, $e(G[V(P_{13})]) \leq 78 - (\binom{|A|}{2} + \binom{|B|}{2} - \binom{|A \cap B|}{2}) - 2 = 59$. $\square$

Fact 7. If there exists an isolated vertex in $G - P_{13}$, that hits three vertices of P_{13}, then $e(G[V(P_{13})]) \leq 67$.

Let y be an isolated vertex in $G - P_{13}$, that hits exactly three vertices, says x_i, x_j, x_k, $i < j < k$ of P_{13}. Recall that $\{x_i, x_j, x_k\} \subseteq V(P_{13}) \setminus \{x_1, x_6, x_8, x_{13}\}$ and y can not hit both x_p and x_{p+8} for $p \in \{2, 3, 4\}$. Let $A = \{x_{i-1}, x_{j-1}, x_{k-1}, x_{13}\}$ and $B = \{x_1, x_{i+1}, x_{j+1}, x_{k+1}\}$. Then both A and B are independent sets and $|A \cap B| \leq 2$. Hence, $e(G[V(P_{13})]) \leq 78 - (\binom{|A|}{2} + \binom{|B|}{2} - \binom{|A\cap B|}{2}) \leq 78 - (6 + 6 - 1) = 67$. $\square$

Let $P_k = y_1 y_2 \ldots y_k$, where $k \leq 6$, be the longest path in $G - P_{13}$ such that y_1 hits P_{13}. Let $H_1, H_2, \ldots, H_t$ be connected components of order at least 2 of $G - P_{13}$ and let H be a subgraph of G which consists of all isolated vertices of $G - P_{13}$. Note that $\sum_{i=1}^{t} |H_i| + |H| = n - 13$. Let $m(H_i)$ be the number of edges incident with the vertices of H_i and let H_1 be a component of $G - P_{13}$ which contains P_k as a subgraph. We first show the following claim.

Claim: For $1 \leq i \leq t$, $m(H_i) \leq 4|H_i|$.

Proof. We use induction on $|H_i|$. Recall that each vertex of H_i can hit at most three vertices of P_{13}. For $|H_i| = 2$, $m(H_i) = e(G[V(H_i)]) + e(V(H_i), V(P_{13})) \leq 7 \leq 4|H_i|$. If H_i has a pendant vertex x, then $d_G(x) \leq 4$. By induction hypothesis, we have $m(H_i) = m(H_i - x) + d_G(x) \leq 4(|H_i| - 1) + 4 \leq 4|H_i|$. Next if H_i has no pendant vertex, then each vertex of H_i must be an endpoint of a path of length at least two. This implies that each vertex of H_i can only hit x_7 of P_{13}. Thus, $m(H_i) = e(G[V(H_i)]) + e(V(H_i), V(P_{13})) \leq ex_{con}(|H_i|, P_7) + |H_i| \leq \frac{7}{2}|H_i|$ since H_i is P_7-free. $\square$

Let $\Delta(H) = \max\{d_G(v) | v \in V(H)\}$. Recall that $\Delta(H) \leq 5$. Now we would divide the proof into the following cases (in each case we assume, the previous cases do not hold).

Case 1. $\Delta(H) = 5$. Then by Fact 5 and the Claim,

$$e(G) \leq 50 + 5(n - 13) = 5n - 15 < \max\{[n, 14, 7], 5n - 14\},$$

a contradiction.

Case 2. $\Delta(H) = 4$ or $k \geq 3$ or there exists a non-isolated vertex in $G - P_{13}$ that hits two vertices of P_{13} ($k = 2$). Then by Facts 6, 2 and 4 and the Claim,

$$e(G) \leq 59 + 4(n - 13) = 4n + 7 < \max\{[n, 14, 7], 5n - 14\},$$

a contradiction.

Case 3. $\Delta(H) = 3$ ($k = 2$) or there exists a non-isolated vertex in $G - P_{13}$ that hits one vertex of P_{13} ($k = 2$). For $k = 2$, each component of $G - P_{13}$ is a star (with at least three vertices), or an edge, or an isolated vertex. For $1 \leq i \leq t$, $e(G[V(H_i)]) \leq |H_i| - 1$. $m_0 \leq \sum_{i=1}^{t}(2|H_i| - 1) + 3|H| = 3(n - 15) + 6 - \sum_{i=1}^{t} |H_i| - t \leq 3(n - 13)$. Then by Facts 7 and 3, we have

$$e(G) \leq 68 + 3(n - 13) = 3n + 29 < \max\{[n, 14, 7], 5n - 14\},$$

a contradiction.

Case 4. $\Delta(H) \leq 2$ and $k = 1$. Then by Fact 1,

$$e(G) \leq 74 + 2(n - 13) = 2n + 48 < \max\{[n, 14, 7], 5n - 14\},$$

a contradiction.

The proof is thus completed. $\square$

Proof of Theorem 1.11. Let G be any $2P_7$-free graph on n vertices with $e(G) \geq \max\{[n, 14, 7], 5n-14\}$. If G is connected, then by Lemma 2.3, $e(G) \leq \max\{[n, 14, 7], 5n-14\}$ when $n \geq 22$ and $e(G) < \max\{[n, 14, 7], 5n - 14\}$ when $n \leq 21$. Thus when G is connected, $e(G) \leq \max\{[n, 14, 7], 5n-14\}$ with equality holds if and only if $n \geq 22$ and $G = K_5 + (\overline{K_{n-7}} \cup K_2)$. Now we may assume that G is disconnected. By Lemma 1.4, G contains P_7 as a subgraph. Let C be a connected component with $n_1 \geq 7$ vertices which contains P_7 as a subgraph. Notice that C is $2P_7$-free and $G - C$ is P_7-free. If $n_1 \geq 22$, then by Lemma 2.3, $e(C) \leq 5n - 14$ and by Lemmas 1.4 and 2.2,

$$e(G) = e(C) + e(G - C) \leq 5n_1 - 14 + [n - n_1, 7, 7] < 5n - 14,$$

a contradiction. If $14 \leq n_1 \leq 21$, then by Lemma 2.3, $e(C) < [n_1, 14, 7]$ and by Lemmas 1.4 and 2.1,

$$e(G) = e(C) + e(G - C) < [n_1, 14, 7] + [n - n_1, 7, 7] \leq [n, 14, 7],$$

a contradiction. If $n_1 \leq 13$, then $e(G) \leq \binom{n_1}{2} + [n - n_1, 7, 7] \leq [n, 14, 7]$ with equality holds if and only if $C = K_{13}$ and $G - C \in EX(n - 13, P_7)$. But then when $n \geq 22$, $e(G) \geq \max\{[n, 14, 7], 5n - 14\} = 5n - 14 > [n, 14, 7]$, a contradiction. Thus when G is disconnected, $e(G) \leq \max\{[n, 14, 7], 5n - 14\}$ with equality holds if and only if $n \leq 21$, $G = K_{13} \cup H$ for $H \in EX(n - 13, P_7)$.

The proof is thus complete. $\square$

Acknowledgements: We wish to thank the two anonymous referees for their valuable suggestions and comments. This work was supported by the National Science Foundation and the Natural Science Foundation of Tianjin (No. 17JCQNJC00300).

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