ON ZERO-SUM SUBSEQUENCES OF PRESCRIBED LENGTH

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Abstract. Let G be an additive finite abelian group with exponent $\exp(G) = m$. For any positive integer k, let $\mathsf{s}_{km}(G)$ be the smallest positive integer t such that every sequence S in G of length at least t has a zero-sum subsequence of length km. Let $\mathsf{D}(G)$ be the Davenport constant of G. In this paper we prove that if G is a finite abelian p-group with $\mathsf{D}(G) \le 4m$ then $\mathsf{s}_{km}(G) = km + \mathsf{D}(G) - 1$ for every $k \ge \frac{\mathsf{D}(G)}{m}$, which confirms a conjecture by Gao et al. recently, where $p \ge 7$ is a prime.

1. Introduction

Let G be an additive finite abelian group with exponent $\exp(G) = m$. Let $S = g_1 \cdot \ldots \cdot g_k$ be a sequence over G. We call S a zero-sum sequence if $0 = \sum_{i=1}^{k} g_i$. The Davenport constant, denoted by $\mathsf{D}(G)$, is the minimal integer t such that every sequence S over G of length $|S| \ge t$ has a nonempty zero-sum subsequence. For any positive integer k, define $\mathsf{s}_{km}(G)$ as the smallest positive integer t such that every sequence S in G of length at least t has a zero-sum subsequence of length km. For $k = 1$, $\mathsf{s}_m(G)$ is called the Erdős-Ginzburg-Ziv constant, it is a classical invariant in combinatorial number theory and has received a lot of attention, for example, see [1, 2, 3, 5, 7, 6, 8, 9, 10, 13, 14, 16, 17, 19, 20, 24, 27, 29, 31, 32], for related paper, see [4, 23, 33].

It is easy to verify that $\mathsf{s}_{km}(G) \ge km + \mathsf{D}(G) - 1$ holds for every $k \ge 1$, see [11]. In 1996, Gao [12] proved that

$$\mathsf{s}_{km}(G) = km + \mathsf{D}(G) - 1,$$

provided that $km \ge |G|$. In [18], Gao and Thangadurai proved that if $km < \mathsf{D}(G)$ then $\mathsf{s}_{km}(G) > km + \mathsf{D}(G) - 1$, and they introduced the invariant $l(G)$ which is defined as the smallest integer t such that $\mathsf{s}_{km}(G) = km + \mathsf{D}(G) - 1$ holds for every $k \ge t$. From the above we know that

$$\frac{\mathsf{D}(G)}{m} \le l(G) \le \frac{|G|}{m}.$$

Recently, Gao, Han, Peng and Sun [15] made the following conjecture:

Conjecture 1.1. Let G be a finite abelian group with $\exp(G) = m$. Then,

$$l(G) = \lceil \frac{\mathsf{D}(G)}{m} \rceil.$$

For cyclic groups G, we clearly have $l(G) = 1$ by the Erdős-Ginzburg-Ziv theorem. For finite abelian groups G of rank two, $l(G) = 2$ (see [15]). Let p be a prime and q be a power of p, Conjecture 1.1 was verified for $\mathbb{Z}_q^d$ where $1 \le d \le 4$ and except some cases when p is rather small, see [18, 28]. For arbitrary d, Kubertin [28] gave an upper bound for $\mathsf{s}_{kq}(\mathbb{Z}_q^d)$. Later, Gao, Han, Peng and Sun [15] verified

Conjecture 1.1 for finite abelian groups of the form $G = C_{mn} \oplus H$ where H is an arbitrary finite group with $\exp(H) = m \geq 2$, and $n \geq 2m|H| + 2|H|$. Soon after that, He [22] improved Kubertin's upper bound for $\mathsf{s}_{kq}(\mathbb{Z}_q^d)$ and also gave a general upper bound for $l(G)$ when G is a finite abelian p-group, where p is large.

Note that Gao and Thangadurai [18] used combinatorial method, while Kubertin [28] and He [22] employed the algebraic method of Rónyai [30]. In this paper, with a different method from previous authors, we verify Conjecture 1.1 for some finite abelian p-groups, the following is our main result.

Theorem 1.2. *Let $G = C_{p^{n_1}} \oplus \cdots \oplus C_{p^{n_r}}$ be an abelian p-group with $\exp(G) = p^{n_r} = m$. Then we have*

$$l(G) = \lceil \frac{\mathsf{D}(G)}{m} \rceil,$$

provided that

(1) *If $m < \mathsf{D}(G) \leq 2m$, where $p \geq 3$ is a prime.*
(2) *If $2m < \mathsf{D}(G) \leq 3m$, where $p \geq 7$ is a prime.*
(3) *If $3m < \mathsf{D}(G) \leq 4m$, where $p \geq 5$ is a prime.*

We also consider $\mathsf{s}_{km}(G)$ when $k < \lceil \frac{\mathsf{D}(G)}{m} \rceil$ and obtained the following result.

Theorem 1.3. *Let $G = C_{p^{n_1}} \oplus \cdots \oplus C_{p^{n_r}}$ be an abelian p-group with $\exp(G) = p^{n_r} = m$. Then we have*

(1) *If $2m < \mathsf{D}(G) \leq 3m$, $\mathsf{s}_{2m}(G) \leq 3m + \mathsf{D}(G) - 1$, for $p \geq 5$.*
(2) *If $3m < \mathsf{D}(G) \leq 4m$, $\mathsf{s}_{2m}(G) \leq 4m + \mathsf{D}(G) - 1$, for $p \geq 7$.*

2. Preliminaries

Let $\mathbb{N}$ denote the set of positive integers, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For a real number x, we denote by $\lfloor x \rfloor$ the largest integer that is less than or equal to x.

In the following sections, let G be an additive finite abelian p-group. By the Fundamental Theorem of finite abelian groups we have

$$G \cong C_{p^{n_1}} \oplus \cdots \oplus C_{p^{n_r}}$$

where $r = \mathsf{r}(G) \in \mathbb{N}_0$ is the rank of G, $n_1, \ldots, n_r \in \mathbb{N}$ are integers with $1 < n_1 \leq \ldots \leq n_r$, moreover, $n_1, \ldots, n_r$ are uniquely determined by G, and $p^{n_r} = \exp(G)$ is the *exponent* of G. For simplicity, let m denote the exponent of G.

Our notation follows the Chapter 5 of [21]. For $g_1, \ldots, g_l \in G$ (repetition allowed), we call $S = g_1 \cdot \ldots \cdot g_l$ a *sequence* over G. We write sequences S in the form

$$S = \prod_{g \in G} g^{\mathsf{v}_g(S)} \text{ with } \mathsf{v}_g(S) \in \mathbb{N}_0 \text{ for all } g \in G.$$

We call $\mathsf{v}_g(S)$ the *multiplicity* of g in S. We call $T = \prod_{g \in G} g^{\mathsf{v}_g(T)}$ a subsequence of S if $\mathsf{v}_g(T) \leq \mathsf{v}_g(S)$ for all $g \in G$, and denote by $T|S$.

For $S = g_1 \cdot \ldots \cdot g_l = \prod_{g \in G} g^{\mathsf{v}_g(S)}$, we call

- $|S| = l = \sum_{g \in G} \mathsf{v}_g(S) \in \mathbb{N}_0$ the *length* of S.
- $\sigma(S) = \sum\limits_{i=1}^{l} g_i = \sum_{g \in G} \mathsf{v}_g(S) g \in G$ the *sum* of S.
- S is a *zero-sum sequence* if $\sigma(S) = 0$.
- S is a *short zero-sum sequence* if it is a zero-sum sequence of length $|S| \in [1, \exp(G)]$

Using these concepts, we can define

- $\mathsf{D}(G)$ the smallest integer $l \in \mathbb{N}$ such that every sequence S of length $|S| \geq l$ has a non-empty zero-sum subsequence. The invariant $\mathsf{D}(G)$ is called the *Davenport constant* of G.
- $\eta(G)$ the smallest integer $l \in \mathbb{N}$ such that every sequence S of length $|S| \geq l$ has a short zero-sum subsequence.

Let $S = g_1 \cdot \ldots \cdot g_l$ be a sequence over G of length $|S| = l \in \mathbb{N}_0$ and let $g \in G$. For every $k \in \mathbb{N}_0$ let

$$N_g^k(S) = |\{I \subset [1,l] \mid \sum_{i\in I} g_i = g, |I| = k\}|$$

denote the number of subsequences T of S having sum $\sigma(T) = g$ and length $|T| = k$ (counted with the multiplicity of their appearance in S).

For convenience, let $N^k(S)$ denote $N_0^k(S)$.

Lemma 2.1. ([26]) *Let G be a finite abelian p-group, and $G = C_{p^{n_1}} \oplus \cdots \oplus C_{p^{n_r}}$, then*

$$\mathsf{D}(G) = \sum_{i=1}^{r}(p^{n_i} - 1) + 1.$$

Moreover, if S is a sequence over G with $|S| = l \geq D(G)$, then

$$1 - N^1(S) + N^2(S) + \cdots + (-1)^l N^l(S) \equiv 0 \pmod p.$$

Lemma 2.2. *Let G be a finite abelian p-group, and $G = C_{p^{n_1}} \oplus \cdots \oplus C_{p^{n_r}}$. Let S be a sequence over G of length $|S| = l \geq \mathsf{D}(G) + p^{n_r} - 1$ and $t = \lfloor \frac{|S|}{p^{n_r}} \rfloor$, then*

(1) $1 - N^{p^{n_r}}(S) + \cdots + (-1)^{tp^{n_r}} N^{tp^{n_r}}(S) \equiv 0 \pmod p$*;*
(2) *It is impossible to have*

$$N^{p^{n_r}}(S) \equiv \cdots \equiv N^{vp^{n_r}}(S) \equiv 0 \pmod p$$

where $v = \lfloor \frac{\mathsf{D}(G)+p^{n_r}-1}{p^{n_r}} \rfloor$.

Proof. (1) was proved in [16] and (2) is an easy consequence, if $N^{p^{n_r}}(S) \equiv \cdots \equiv N^{vp^{n_r}}(S) \equiv 0 \pmod p$, then for any subsequence T of S of length $|T| = \mathsf{D}(G) + p^{n_r} - 1$, we also have $N^{p^{n_r}}(T) \equiv \cdots \equiv N^{vp^{n_r}}(T) \equiv 0 \pmod p$, but it contradicts (1). □

The following congruence will be used throughout this paper, it was first used by Lucas [25], we give a proof for the convenience of the reader.

Lemma 2.3. *Let a, b be positive integers with $a = a_np^n + \cdots + a_1p + a_0$ and $b = b_np^n + \cdots + b_1p + b_0$ be the p-adic expansions, where p is a prime, define $\binom{k}{0} = 1$ for $k \geq 0$. Then*

$$\binom{a}{b} \equiv \binom{a_n}{b_n}\binom{a_{n-1}}{b_{n-1}} \cdots \binom{a_0}{b_0} \pmod p.$$

Proof. We have

$$\begin{aligned}(1+x)^a &= (1+x)^{a_np^n+\cdots+a_1p+a_0} \\ &= (1+x^{p^n})^{a_n} \cdots (1+x^p)^{a_1}(1+x)^{a_0} \pmod p\end{aligned}$$

Since $0 \leq a_i \leq p-1$, comparing the coefficient of x^b, we get the desired result. □

3. Proof of the theorem 1.2 (1)

Throughout this section, assume that G satisfing the following condition:

$$m < \mathsf{D}(G) \le 2m,$$

and also assume that $p \ge 3$ is a prime.

Remark 3.1. We have $\lfloor \frac{\mathsf{D}(G)+m-1}{m} \rfloor = 2$ and $km + \mathsf{D}(G) - 1 = (k+1)m + t$, where $0 \le t \le m-1$.

Lemma 3.2. *We have*

$$\mathsf{s}_{2m}(G) = 2m + \mathsf{D}(G) - 1.$$

Proof. Let S be a sequence over G of length $2m + \mathsf{D}(G) - 1$, assume to the contrary that $N^{2m}(S) = 0$.

If $N^m(S) \neq 0$, then S contains a zero-sum subsequence J of length m, let $T = SJ^{-1}$, then $|T| = m + \mathsf{D}(G) - 1$ and $N^{2m}(T) = 0$ since $N^{2m}(S) = 0$. If $N^m(T) > 0$, then T contains a zero-sum subsequence U of length m, and JU is a zero-sum subsequence of S with $|JU| = 2m$, hence $N^{2m}(S) > 0$, a contradiction. Therefore we have $N^m(T) = N^{2m}(T) = 0$, which contradicts Lemma 2.2 as $|T| = m + \mathsf{D}(G) - 1$. So we have $N^m(S) = 0$.

Above all, we conclude that

$$N^m(S) = N^{2m}(S) = 0,$$

but since $2m + \mathsf{D}(G) - 1 \ge m + \mathsf{D}(G) - 1$, thus it contradicts Lemma 2.2. □

Lemma 3.3. *We have*

$$\mathsf{s}_{3m}(G) = 3m + \mathsf{D}(G) - 1.$$

Proof. Let S be a sequence over G of length $3m + \mathsf{D}(G) - 1$, assume to the contrary that $N^{3m}(S) = 0$.

If $N^m(S) \neq 0$, then S contains a zero-sum subsequence J of length m, let $T = SJ^{-1}$, then $|T| = 2m + \mathsf{D}(G) - 1$ and $N^{3m}(T) = 0$ since $N^{3m}(S) = 0$. By Lemma 3.2, we have $N^{2m}(T) > 0$, then $N^{3m}(S) > 0$, a contradiction. Therefore we can assume that $N^m(S) = 0$.

If $N^{2m}(S) \neq 0$, then by Lemma 2.2 we get:

$$1 + N^{2m}(S) + N^{4m}(S) \equiv 0 \pmod p. \tag{3.1}$$

We choose any subsequence $J_1 | S$ of length $2m + \mathsf{D}(G) - 1$, then

$$1 + N^{2m}(J_1) \equiv 0 \pmod p.$$

Now we consider

$$\sum_{|J_1| = 2m + \mathsf{D}(G) - 1} (1 + N^{2m}(J_1)) \equiv 0 \pmod p,$$

where the sum is extended over all $J_1 | S$ of length $2m + \mathsf{D}(G) - 1$. Analysing the number of times each subsequence is counted, one obtains

$$\binom{3m + \mathsf{D}(G) - 1}{2m + \mathsf{D}(G) - 1} + \binom{m + \mathsf{D}(G) - 1}{\mathsf{D}(G) - 1} N^{2m}(S) \equiv 0 \pmod p.$$

By Remark 3.1 and Lemma 2.3, that is

$$4 + 2N^{2m}(S) \equiv 0 \pmod p. \tag{3.2}$$

Similarly, choose any subsequence $J_2|S$ of length $m+\mathsf{D}(G)-1$, then

$$1+N^{2m}(J_2)\equiv 0 \pmod p.$$

Now we consider

$$\sum_{|J_2|=m+\mathsf{D}(G)-1}(1+N^{2m}(J_2))\equiv 0 \pmod p,$$

where the sum is extended over all $J_2|S$ of length $m+\mathsf{D}(G)-1$. Analysing the number of times each subsequence is counted, one obtains

$$\binom{3m+\mathsf{D}(G)-1}{m+\mathsf{D}(G)-1}+\binom{m+\mathsf{D}(G)-1}{\mathsf{D}(G)-m-1}N^{2m}(S)\equiv 0 \pmod p.$$

By Remark 3.1 and Lemma 2.3, that is

$$6+N^{2m}(S)\equiv 0 \pmod p. \tag{3.3}$$

Combining equations (3.2) and (3.3) one can deduces $8\equiv 0 \pmod p$, but this contradicts $p\geq 3$. Therefore $N^{2m}(S)=0$.

Above all, we conclude that

$$N^m(S)=N^{2m}(S)=0,$$

but since $3m+\mathsf{D}(G)-1\geq m+\mathsf{D}(G)-1$, thus it contradicts Lemma 2.2. □

Proof of Theorem 1.2 (1). Since $\lceil\frac{\mathsf{D}(G)}{m}\rceil=2$, we have to prove that $\mathsf{s}_{km}(G)=km+\mathsf{D}(G)-1$ holds for every $k\geq 2$.

We prove by induction on k, for $k=2,3$, it holds by Lemma 3.2 and 3.3. Suppose that $k\geq 3$ and the result holds for all positive integers n with $2\leq n\leq k$. Now we need to prove $\mathsf{s}_{(k+1)m}(G)=(k+1)m+\mathsf{D}(G)-1$.

Let S be a sequence over G of length $(k+1)m+\mathsf{D}(G)-1$, since

$$|S|\geq (k-1)m+\mathsf{D}(G)-1=\mathsf{s}_{(k-1)m}(G),$$

then S contains a zero-sum subsequence T of length $(k-1)m$. Let $J=ST^{-1}$, and

$$|J|=2m+\mathsf{D}(G)-1=\mathsf{s}_{2m}(G),$$

therefore J contains a zero-sum subsequence W of length $2m$, then TW is a zero-sum subsequence of length $(k+1)m$, this completes the proof. □

4. Proof of the theorem 1.2 (2)

Throughout this section, assume that G satisfing the following condition:

$$2m<\mathsf{D}(G)\leq 3m,$$

and also assume that $p\geq 7$ is a prime.

Remark 4.1. We have $\lfloor\frac{\mathsf{D}(G)+m-1}{m}\rfloor=3$ and $km+\mathsf{D}(G)-1=(k+2)m+t$, where $0\leq t\leq m-1$.

Lemma 4.2. *Let S be a sequence over G. If $N^{5m}(S)>0$, then $N^{3m}(S)>0$. Moreover, we get*

$$\mathsf{s}_{3m}(G)=3m+\mathsf{D}(G)-1.$$

Proof. Let S' be a zero-sum subsequence of S with $|S'| = 5m$. Assume to the contrary that $N^{3m}(S) = 0$, then we have $N^{3m}(S') = N^{2m}(S') = 0$.

If $N^m(S') \neq 0$, then S' contains a zero-sum subsequence J of length m, let $T = S'J^{-1}$, then $|T| = 4m \geq m + \mathsf{D}(G) - 1$ and $N^{3m}(T) = N^{2m}(T) = 0$. Lemma 2.2 implies $N^m(T) > 0$, then T contains a zero-sum subsequence T_1 of length m. Hence $|T_1 J| = 2m$ and $T_1 J$ is a zero-sum subsequence of S', thus $N^{2m}(S') > 0$, a contradiction.

Therefore, $N^m(S') = N^{2m}(S') = N^{3m}(S') = 0$, since $5m \geq m + \mathsf{D}(G) - 1$, this contradicts Lemma 2.2.

Next, we are going to investigate $\mathsf{s}_{3m}(G)$.

Let S be a sequence over G of length $3m + \mathsf{D}(G) - 1$, assume to the contrary that $N^{3m}(S) = 0$, then $N^{5m}(S) = 0$.

If $N^m(S) \neq 0$, then S contains a zero-sum subsequence J of length m, let $T = SJ^{-1}$, then $|T| = 2m + \mathsf{D}(G) - 1$ and $N^{2m}(T) = N^{3m}(T) = N^{4m}(T) = 0$. Then by Lemma 2.2 we get:

$$1 - N^m(T) \equiv 0 \pmod{p}. \tag{4.1}$$

We choose any subsequence $J_1 | T$ of length $m + \mathsf{D}(G) - 1$, then

$$1 - N^m(J_1) \equiv 0 \pmod{p}.$$

Now we consider

$$\sum_{|J_1| = m + \mathsf{D}(G) - 1} (1 - N^m(J_1)) \equiv 0 \pmod{p},$$

where the sum is extended over all $J_1 | T$ of length $m + \mathsf{D}(G) - 1$. Analysing the number of times each subsequence is counted, one obtains

$$\binom{2m + \mathsf{D}(G) - 1}{m + \mathsf{D}(G) - 1} - \binom{m + \mathsf{D}(G) - 1}{\mathsf{D}(G) - 1} N^m(S) \equiv 0 \pmod{p}.$$

By Remark 4.1 and Lemma 2.3, that is

$$4 - 3N^m(T) \equiv 0 \pmod{p}. \tag{4.2}$$

Combining equations (4.1) and (4.2) one can deduces $1 \equiv 0 \pmod{p}$, this is a contradiction. Therefore we can assume that $N^m(S) = 0$.

Then by Lemma 2.2 we get:

$$1 + N^{2m}(S) + N^{4m}(S) \equiv 0 \pmod{p}. \tag{4.3}$$

We choose any subsequence $J_1 | S$ of length $2m + \mathsf{D}(G) - 1$, then

$$1 + N^{2m}(J_1) + N^{4m}(J_1) \equiv 0 \pmod{p}.$$

Now we consider

$$\sum_{|J_1| = 2m + \mathsf{D}(G) - 1} (1 + N^{2m}(J_1) + N^{4m}(J_1)) \equiv 0 \pmod{p},$$

where the sum is extended over all $J_1 | S$ of length $2m + \mathsf{D}(G) - 1$. Analysing the number of times each subsequence is counted, one obtains

$$\binom{3m + \mathsf{D}(G) - 1}{2m + \mathsf{D}(G) - 1} + \binom{m + \mathsf{D}(G) - 1}{\mathsf{D}(G) - 1} N^{2m}(S) + \binom{\mathsf{D}(G) - m - 1}{\mathsf{D}(G) - 2m - 1} N^{4m}(S) \equiv 0 \pmod{p}.$$

By Remark 4.1 and Lemma 2.3, that is

$$5+3N^{2m}(S)+N^{4m}(S)\equiv 0 \pmod p. \tag{4.4}$$

Similarly, we choose any subsequence $J_2|S$ of length $m+\mathsf{D}(G)-1$, then one obtains

$$\binom{3m+\mathsf{D}(G)-1}{m+\mathsf{D}(G)-1}+\binom{m+\mathsf{D}(G)-1}{\mathsf{D}(G)-m-1}N^{2m}(S)\equiv 0 \pmod p.$$

By Remark 4.1 and Lemma 2.3, that is

$$10+3N^{2m}(S)\equiv 0 \pmod p. \tag{4.5}$$

Combining equations (4.3)-(4.5) one can deduces $4\equiv 0 \pmod p$, this contradicts $p\geq 3$. □

Lemma 4.3. *Let S be a sequence over G. If $N^{6m}(S)>0$, then $N^{4m}(S)>0$. Moreover, we get*

$$\mathsf{s}_{4m}(G)=4m+\mathsf{D}(G)-1.$$

Proof. Let S' be a zero-sum subsequence of S with $|S'|=6m$. Assume to the contrary that $N^{4m}(S)=0$, then we have $N^{4m}(S')=N^{2m}(S')=0$.

If $N^m(S')\neq 0$, then S' contains a zero-sum subsequence J of length m, let $T=S'J^{-1}$, then $|T|=5m\geq m+\mathsf{D}(G)-1$ and $N^{4m}(T)=N^{3m}(T)=N^{2m}(T)=N^m(T)=0$, but this contradicts Lemma 2.2. Thus we may assume that $N^m(S')=0$.

Since $N^m(S')=N^{5m}(S')=N^{2m}(S')=N^{4m}(S')=0$ and $N^{6m}(S')=1$, Lemma 2.2 implies that

$$1-N^{3m}(S')+N^{6m}(S')\equiv 0 \pmod p. \tag{4.6}$$

Similar to above, we choose a subsequence $J_1|S'$ of length $5m$, one obtains

$$\binom{6m}{5m}-\binom{4m}{3m}N^{3m}(S')\equiv 0 \pmod p.$$

By Remark 4.1 and Lemma 2.3, that is

$$6-4N^{3m}(S')\equiv 0 \pmod p. \tag{4.7}$$

Combining equations (4.6)-(4.7) one can deduces that $2\equiv 0 \pmod p$, but this contradicts $p\geq 5$. So we have $N^{4m}(S')>0$.

Next, we are going to investigate $\mathsf{s}_{4m}(G)$.

Let S be a sequence over G of length $4m+\mathsf{D}(G)-1$, assume to the contrary that $N^{4m}(S)=0$, then $N^{6m}(S)=0$.

If $N^m(S)\neq 0$, then S contains a zero-sum subsequence J of length m, let $T=SJ^{-1}$, then $|T|=3m+\mathsf{D}(G)-1$, by Lemma 4.2 we know that $N^{3m}(T)>0$, thus $N^{4m}(S)>0$, a contradiction. Therefore we can assume that $N^m(S)=0$.

If $N^{2m}(S)\neq 0$, then S contains a zero-sum subsequence J of length $2m$, let $T=SJ^{-1}$, then $|T|=2m+\mathsf{D}(G)-1$, and $N^m(T)=N^{2m}(T)=N^{4m}(T)=0$, then by Lemma 2.2 we get:

$$1-N^{3m}(T)\equiv 0 \pmod p. \tag{4.8}$$

Similar to above, we choose any subsequence $J_1|T$ of length $m+\mathsf{D}(G)-1$, one obtains

$$\binom{2m+\mathsf{D}(G)-1}{m+\mathsf{D}(G)-1}-\binom{\mathsf{D}(G)-m-1}{\mathsf{D}(G)-2m-1}N^{3m}(T)\equiv 0 \pmod p.$$

By Remark 4.1 and Lemma 2.3, that is

$$4 - N^{3m}(T) \equiv 0 \pmod{p}. \tag{4.9}$$

Combining equations (4.8) and (4.9) one can deduces $3 \equiv 0 \pmod{p}$, this contradicts $p \geq 5$. Therefore we have $N^{2m}(S) = 0$.

By Lemma 2.2 we get

$$1 - N^{3m}(S) - N^{5m}(S) \equiv 0 \pmod{p}. \tag{4.10}$$

Similar to above, we choose a subsequence $J_1|S$ of length $3m + \mathsf{D}(G) - 1$, then one obtains

$$\begin{aligned}&\binom{4m+\mathsf{D}(G)-1}{3m+\mathsf{D}(G)-1} - \binom{m+\mathsf{D}(G)-1}{\mathsf{D}(G)-1}N^{3m}(S)\\ &- \binom{\mathsf{D}(G)-m-1}{\mathsf{D}(G)-2m-1}N^{5m}(S) \equiv 0 \pmod{p}.\end{aligned}$$

By Remark 4.1 and Lemma 2.3, that is

$$6 - 3N^{3m}(S) - N^{5m}(S) \equiv 0 \pmod{p}. \tag{4.11}$$

Similarly, we choose a subsequence $J_2|S$ of length $2m + \mathsf{D}(G) - 1$, then one obtains

$$\binom{4m+\mathsf{D}(G)-1}{2m+\mathsf{D}(G)-1} - \binom{m+\mathsf{D}(G)-1}{\mathsf{D}(G)-m-1}N^{3m}(S) \equiv 0 \pmod{p}.$$

By Remark 4.1 and Lemma 2.3, that is

$$15 - 3N^{3m}(S) \equiv 0 \pmod{p}. \tag{4.12}$$

Combining equations (4.10)-(4.12) one can deduces that $15 \equiv 0 \pmod{p}$, but this contradicts $p \geq 7$. □

Lemma 4.4. *Let S be a sequence over G. If $N^{7m}(S) > 0$, then $N^{5m}(S) > 0$. Moreover we get*

$$\mathsf{s}_{5m}(G) = 5m + \mathsf{D}(G) - 1.$$

Proof. Let S' be a zero-sum subsequence of S with $|S'| = 7m$. Assume to the contrary that $N^{5m}(S) = 0$, then we have $N^{5m}(S') = N^{2m}(S') = 0$.

If $N^m(S') \neq 0$, then S' contains a zero-sum subsequence J of length m, let $T = S'J^{-1}$, then $|T| = 6m \geq m + \mathsf{D}(G) - 1$ and $N^{5m}(T) = N^{4m}(T) = N^{2m}(T) = N^m(T) = 0$, $N^{6m}(T) = 1$. By Lemma 2.2,

$$1 - N^{3m}(T) + N^{6m}(T) \equiv 0 \pmod{p}. \tag{4.13}$$

Similar to above, we choose a subsequence $J_1|T$ of length $5m$, then one obtains

$$\binom{6m}{5m} - \binom{3m}{2m}N^{3m}(T) \equiv 0 \pmod{p}.$$

By Remark 4.1 and Lemma 2.3, that is

$$6 - 3N^{3m}(T) \equiv 0 \pmod{p}. \tag{4.14}$$

Similarly, we choose a subsequence $J_2|T$ of length $4m$, then one obtains

$$\binom{6m}{4m} - \binom{3m}{m}N^{3m}(T) \equiv 0 \pmod{p}.$$

By Remark 4.1 and Lemma 2.3, that is

$$15 - 3N^{3m}(T) \equiv 0 \pmod{p}. \tag{4.15}$$

Combining equations (4.13)-(4.15) one can deduces that $9 \equiv 0 \pmod{p}$, but this contradicts $p \geq 5$. Thus we may assume that $N^m(S') = N^{6m}(S') = 0$.

We choose a subsequence T from S' of length $7m-1$. Then $N^m(T) = N^{2m}(T) = N^{5m}(T) = N^{6m}(T) = 0$ and one obtains

$$1 - N^{3m}(T) + N^{4m}(T) \equiv 0 \pmod{p}. \tag{4.16}$$

Similarly, we choose a subsequence $J_1|T$ of length $6m-1$, then one obtains

$$\binom{7m-1}{6m-1} - \binom{4m-1}{3m-1}N^{3m}(T) + \binom{3m-1}{2m-1}N^{4m}(T) \equiv 0 \pmod{p}.$$

By Remark 4.1 and Lemma 2.3, that is

$$6 - 3N^{3m}(T) + 2N^{4m}(T) \equiv 0 \pmod{p}. \tag{4.17}$$

Similarly, we choose a subsequence $J_2|T$ of length $5m-1$, then one obtains

$$\binom{7m-1}{5m-1} - \binom{4m-1}{2m-1}N^{3m}(T) + \binom{3m-1}{m-1}N^{4m}(T) \equiv 0 \pmod{p}.$$

By Remark 4.1 and Lemma 2.3, that is

$$15 - 3N^{3m}(T) + N^{4m}(T) \equiv 0 \pmod{p}. \tag{4.18}$$

Combining equations (4.16)-(4.18) one can deduces that $6 \equiv 0 \pmod{p}$, but this contradicts $p \geq 5$. Therefore $N^{5m}(S') > 0$.

Next, we are going to investigate $\mathsf{s}_{5m}(G)$.

Let S be a sequence over G of length $5m + \mathsf{D}(G) - 1$, assume to the contrary that $N^{5m}(S) = 0$.

If $N^m(S) \neq 0$, then S contains a zero-sum subsequence J of length m, let $T = SJ^{-1}$, then $|T| = 4m + \mathsf{D}(G) - 1$. By Lemma 4.3, we have $N^{4m}(T) > 0$, then $N^{5m}(S) > 0$, a contradiction. Therefore we can assume that $N^m(S) = 0$.

If $N^{2m}(S) \neq 0$, then S contains a zero-sum subsequence J of length $2m$, let $T = SJ^{-1}$, then $|T| = 3m + \mathsf{D}(G) - 1$. By Lemma 4.2 we have $N^{3m}(T) > 0$, then $N^{5m}(S) > 0$, a contradiction. Therefore we can assume that $N^{2m}(S) = 0$.

If $N^{3m}(S) \neq 0$, then S contains a zero-sum subsequence J of length $3m$, let $T = SJ^{-1}$, then $|T| = 2m + \mathsf{D}(G) - 1$ and $N^m(T) = N^{2m}(T) = N^{4m}(T) = 0$. By Lemma 2.2, we get

$$1 - N^{3m}(T) \equiv 0 \pmod{p}. \tag{4.19}$$

Similarly, we choose any subsequence $J_1|T$ of length $m + \mathsf{D}(G) - 1$, one obtains

$$\binom{2m + \mathsf{D}(G) - 1}{m + \mathsf{D}(G) - 1} - \binom{\mathsf{D}(G) - m - 1}{\mathsf{D}(G) - 2m - 1}N^{3m}(T) \equiv 0 \pmod{p}.$$

By Remark 4.1 and Lemma 2.3, that is

$$4 - N^{3m}(T) \equiv 0 \pmod{p}. \tag{4.20}$$

Combining equations (4.19) and (4.20) one can deduces $3 \equiv 0 \pmod{p}$, but this contradicts $p \geq 5$.

Above all, we conclude that

$$N^m(S) = N^{2m}(S) = N^{3m}(S) = 0,$$

But since $5m + \mathsf{D}(G) - 1 \geq m + \mathsf{D}(G) - 1$, thus it contradicts Lemma 2.2. $\square$

Proof of Theorem 1.2 (2). Since $\lceil \frac{\mathsf{D}(G)}{m} \rceil = 3$, we have to prove that $\mathsf{s}_{km}(G) = km + \mathsf{D}(G) - 1$ holds for every $k \geq 3$.

We prove by induction on k, for $k = 3, 4, 5$, it holds by Lemma 4.2-4.4. Suppose that $k \geq 4$ and the result holds for all positive integers n with $2 \leq n \leq k$. Now we need to prove $\mathsf{s}_{(k+1)m}(G) = (k+1)m + \mathsf{D}(G) - 1$.

Let S be a sequence over G of length $(k+1)m + \mathsf{D}(G) - 1$, since

$$|S| \geq (k-2)m + \mathsf{D}(G) - 1 = \mathsf{s}_{(k-2)m}(G),$$

then S contains a zero-sum subsequence T of length $(k-2)m$. Let $J = ST^{-1}$, and

$$|J| = 3m + \mathsf{D}(G) - 1 = \mathsf{s}_{3m}(G),$$

therefore J contains a zero-sum subsequence W of length $3m$, then TW is a zero-sum subsequence of length $(k+1)m$, this completes the proof. □

5. Proof of the theorem 1.2 (3)

Throughout this section, assume that G satisfing the following condition:

$$3m < \mathsf{D}(G) \leq 4m,$$

and also assume that $p \geq 5$ is a prime.

Remark 5.1. We have $\lfloor \frac{\mathsf{D}(G)+m-1}{m} \rfloor = 4$ and $km + \mathsf{D}(G) - 1 = (k+3)m + t$, where $0 \leq t \leq m-1$.

Lemma 5.2. *Let S be a sequence over G. If $N^{6m}(S) > 0$ or $N^{7m}(S) > 0$, then $N^{4m}(S) > 0$. Moreover, we get*

$$\mathsf{s}_{4m}(G) = 4m + \mathsf{D}(G) - 1.$$

Proof. **(i)** Let S' be a zero-sum subsequence of S with $|S'| = 6m$. Assume to the contrary that $N^{4m}(S) = 0$, then we have $N^{4m}(S') = N^{2m}(S') = 0$.

If $N^m(S') \neq 0$, then S' contains a zero-sum subsequence J of length m, let $T = S'J^{-1}$, then $|T| = 5m \geq m + \mathsf{D}(G) - 1$ and $N^{4m}(T) = N^m(T) = 0$. Lemma 2.2 implies $N^{2m}(T) = N^{3m}(T) > 0$, then T contains a zero-sum subsequence T_1 of length $3m$. Hence $|T_1J| = 4m$ and T_1J is a zero-sum subsequence of S', a contradiction.

Therefore we may assume $N^m(S') = N^{5m}(S') = 0$. We choose a subsequence $T|S'$ of length $|T| = 6m-1$ therefore $N^m(T) = N^{2m}(T) = N^{4m}(T) = N^{5m}(T) = 0$, Lemma 2.2 implies that

$$1 - N^{3m}(T) \equiv 0 \pmod{p}. \tag{5.1}$$

Similar to above, we choose a subsequence J from T of length $5m-1$, one obtains

$$\binom{6m-1}{5m-1} - \binom{3m-1}{2m-1} N^{3m}(T) \equiv 0 \pmod{p}.$$

By Remark 5.1 and Lemma 2.3, that is

$$5 - 2N^{3m}(T) \equiv 0 \pmod{p}. \tag{5.2}$$

Combining equations (5.1) and (5.2) one can deduces $3 \equiv 0 \pmod{p}$, this contradicts $p \geq 5$.

(ii) Let S' be a zero-sum subsequence of S with $|S'| = 7m$. Assume to the contrary that $N^{4m}(S) = 0$, then we have $N^{4m}(S') = N^{3m}(S') = 0$.

If $N^m(S') \neq 0$, then S' contains a zero-sum subsequence J of length m, let $T = S'J^{-1}$, then T is a zero-sum subsequence of length $6m$, therefore by (i) one can get that $N^{4m}(T) > 0$, moreover $N^{4m}(S') > 0$.

Therefore we may assume $N^m(S') = N^{6m}(S') = 0$. If $N^{2m}(S') \neq 0$, then S' contains a zero-sum subsequence J of length $2m$, let $T = S'J^{-1}$, then T is a zero-sum subsequence of length $5m$, and $N^{4m}(T) = N^m(T) = 0$. Lemma 2.2 implies $N^{2m}(T) = N^{3m}(T) > 0$, then T contains a zero-sum subsequence T_1 of length $3m$. Hence $|T_1J| = 4m$ and T_1J is a zero-sum subsequence of S'. Moreover we can suppose $N^{2m}(S') = 0$.

Above all, we conclude that

$$N^m(S') = N^{2m}(S') = N^{3m}(S') = N^{4m}(S') = N^{5m}(S') = N^{6m}(S') = 0,$$

this contradicts Lemma 2.2 since $|S'| = 7m \geq m + \mathsf{D}(G) - 1$.

Next, we are going to investigate $\mathsf{s}_{4m}(G)$.

Let S be a sequence over G of length $4m + \mathsf{D}(G) - 1$, assume that $N^{4m}(S) = 0$, then $N^{6m}(S) = N^{7m}(S) = 0$.

If $N^m(S) \neq 0$, then S contains a zero-sum subsequence J of length m, let $T = SJ^{-1}$, then $|T| = 3m + \mathsf{D}(G) - 1$ and $N^{3m}(T) = N^{4m}(T) = N^{5m}(T) = N^{6m}(T) = 0$. By Lemma 2.2 we get:

$$1 - N^m(T) + N^{2m}(T) \equiv 0 \pmod{p}. \tag{5.3}$$

Similar to above, we choose any subsequence $J_1 | T$ of length $2m + \mathsf{D}(G) - 1$, then one obtains

$$\binom{3m + \mathsf{D}(G) - 1}{2m + \mathsf{D}(G) - 1} - \binom{2m + \mathsf{D}(G) - 1}{m + \mathsf{D}(G) - 1} N^m(T) + \binom{m + \mathsf{D}(G) - 1}{\mathsf{D}(G) - 1} N^{2m}(T) \equiv 0 \pmod{p}.$$

By Remark 5.1 and Lemma 2.3, that is

$$6 - 5N^m(T) + 4N^{2m}(T) \equiv 0 \pmod{p}. \tag{5.4}$$

Moreover, we choose any subsequence $J_2 | T$ of length $m + \mathsf{D}(G) - 1$ and one obtains

$$\binom{3m + \mathsf{D}(G) - 1}{m + \mathsf{D}(G) - 1} - \binom{2m + \mathsf{D}(G) - 1}{\mathsf{D}(G) - 1} N^m(T) + \binom{m + \mathsf{D}(G) - 1}{\mathsf{D}(G) - 1 - m} N^{2m}(T) \equiv 0 \pmod{p}.$$

By Remark 5.1 and Lemma 2.3, that is

$$15 - 10N^m(T) + 6N^{2m}(T) \equiv 0 \pmod{p}. \tag{5.5}$$

Combining equations (5.3)-(5.5), one deduces that $1 \equiv 0 \pmod{p}$, this is a contradiction, therefore we may assume $N^m(S) = 0$.

If $N^{2m}(S) \neq 0$, then S contains a zero-sum subsequence J of length $2m$, let $T = SJ^{-1}$, then $|T| = 2m + \mathsf{D}(G) - 1$ and $N^m(T) = N^{2m}(T) = N^{4m}(T) = N^{5m}(T) = 0$. Lemma 2.2 implies

$$1 - N^{3m}(T) \equiv 0 \pmod{p}. \tag{5.6}$$

Similar to above, choosing any subsequence $J | T$ of length $m + \mathsf{D}(G) - 1$, we obtain

$$\binom{2m + \mathsf{D}(G) - 1}{m + \mathsf{D}(G) - 1} - \binom{\mathsf{D}(G) - 1 - m}{\mathsf{D}(G) - 1 - 2m} N^{3m}(T) \equiv 0 \pmod{p}$$

that is

$$5-2N^{3m}(T)\equiv 0 \pmod p. \tag{5.7}$$

Combining equations (5.6) and (5.7), one can deduces $3\equiv 0 \pmod p$, this contradicts to $p\ge 5$. Moreover we can suppose $N^{2m}(S)=0$.

If $N^{3m}(S)\neq 0$, then S contains a zero-sum subsequence J of length $3m$, let $T=SJ^{-1}$, then $|T|=2m+\mathsf{D}(G)-1$ and $N^m(T)=N^{2m}(T)=0$. If $N^{3m}(T)\neq 0$ or $N^{4m}(T)\neq 0$, then $N^{6m}(S)>0$ or $N^{7m}(S)>0$ and both case implies that $N^{4m}(S)>0$. Therefore we may also assume that $N^{3m}(S)=0$.

Above all, we conclude that

$$N^m(S)=N^{2m}(S)=N^{3m}(S)=N^{4m}(S)=0,$$

this contradicts to $|S|\ge m+\mathsf{D}(G)-1$ and Lemma 2.2. □

Lemma 5.3. *Let S be a sequence over G. If $N^{7m}(S)>0$ or $N^{8m}(S)>0$, then $N^{5m}(S)>0$. Moreover, we get*

$$\mathsf{s}_{5m}(G)=5m+\mathsf{D}(G)-1.$$

Proof. **(i)** Let S' be a zero-sum subsequence of S with $|S'|=7m$. Assume to the contrary that $N^{5m}(S)=0$, then we have $N^{5m}(S')=N^{2m}(S')=0$.

If $N^m(S')\neq 0$, then S' contains a zero-sum subsequence J of length m, let $T=S'J^{-1}$, then $|T|=6m$ and by Lemma 5.2, we can get $N^{4m}(T)>0$, hence $N^{5m}(S')>0$, therefore we may assume $N^m(S')=N^{6m}(S')=0$.

let $T=S'\setminus\{g\}$, where g is any element of S', so $|T|=7m-1$ and $N^m(T)=N^{2m}(T)=N^{5m}(T)=N^{6m}(T)=0$, Lemma 2.2 implies that

$$1-N^{3m}(T)+N^{4m}(T)\equiv 0 \pmod p. \tag{5.8}$$

Similar to above, we choose a subsequence J from T of length $6m-1$, one obtains

$$\binom{7m-1}{6m-1}-\binom{4m-1}{3m-1}N^{3m}(T)+\binom{3m-1}{2m-1}N^{4m}(T)\equiv 0 \pmod p$$

By Remark 5.1 and Lemma 2.3, that is

$$6-3N^{3m}(T)+2N^{4m}(T)\equiv 0 \pmod p. \tag{5.9}$$

Moreover, we choose a subsequence J' from T of length $5m-1$, one obtains

$$\binom{7m-1}{5m-1}-\binom{4m-1}{2m-1}N^{3m}(T)+\binom{3m-1}{m-1}N^{4m}(T)\equiv 0 \pmod p$$

By Remark 5.1 and Lemma 2.3, that is

$$15-3N^{3m}(T)+N^{4m}(T)\equiv 0 \pmod p. \tag{5.10}$$

Combining (5.8-5.10) one can deduces $3\equiv 0 \pmod p$, this contradicts $p\ge 5$. Therefore, $N^{5m}(S')>0$.

(ii) Let S' be a zero-sum subsequence of S with $|S'|=8m$. Assume to the contrary that $N^{5m}(S)=0$, then we have $N^{5m}(S')=N^{3m}(S')=0$.

If $N^m(S')\neq 0$, then $N^{7m}(S')\neq 0$ and by (i), $N^{5m}(S')>0$, therefore we may assume that $N^m(S')=N^{7m}(S')=0$.

let $T=S'\setminus\{g\}$, where g is any element of S', so $|T|=8m-1$ and $N^m(T)=N^{3m}(T)=N^{5m}(T)=N^{7m}(T)=0$, Lemma 2.2 implies that

$$1+N^{2m}(T)+N^{4m}(T)+N^{6m}(T)\equiv 0 \pmod p. \tag{5.11}$$

Similar to above, we choose a subsequence J_1 from T of length $7m-1$, one obtains

$$\binom{8m-1}{7m-1}+\binom{6m-1}{5m-1}N^{2m}(T)+\binom{4m-1}{3m-1}N^{4m}(T)$$
$$+\binom{2m-1}{m-1}N^{6m}(T)\equiv 0 \pmod p.$$

By Remark 5.1 and Lemma 2.3, that is

$$7+5N^{2m}(T)+3N^{4m}(T)+N^{6m}(T)\equiv 0 \pmod p. \tag{5.12}$$

Moreover, we choose a subsequence J_2 from T of length $6m-1$, one obtains

$$\binom{8m-1}{6m-1}+\binom{6m-1}{4m-1}N^{2m}(T)+\binom{4m-1}{2m-1}N^{4m}(T)\equiv 0 \pmod p$$

By Remark 5.1 and Lemma 2.3, that is

$$21+10N^{2m}(T)+3N^{4m}(T)+N^{6m}(T)\equiv 0 \pmod p. \tag{5.13}$$

Furthermore, we choose a subsequence J_3 from T of length $5m-1$, one obtains

$$\binom{8m-1}{5m-1}+\binom{6m-1}{3m-1}N^{2m}(T)+\binom{4m-1}{m-1}N^{4m}(T)\equiv 0 \pmod p$$

By Remark 5.1 and Lemma 2.3, that is

$$35+10N^{2m}(T)+N^{4m}(T)\equiv 0 \pmod p. \tag{5.14}$$

Combining equations (5.11)-(5.14), one can deduces that $96\equiv 0 \pmod p$, this contradicts $p\geq 5$, therefore $N^{5m}(S')\neq 0$.

Next, we are going to investigate $\mathsf{s}_{5m}(G)$.

Let S be a sequence over G of length $5m+\mathsf{D}(G)-1$, assume that $N^{5m}(S)=0$, then $N^{7m}(S)=N^{8m}(S)=0$.

If $N^{m}(S)\neq 0$, then S contains a zero-sum subsequence J of length m, let $T=SJ^{-1}$, then $|T|=4m+\mathsf{D}(G)-1$ and lemma 5.2 implies $N^{4m}(T)>0$, hence $N^{5m}(S)>0$. Therefore we may assume that $N^{m}(S)=0$.

If $N^{2m}(S)\neq 0$, then S contains a zero-sum subsequence J of length $2m$, let $T=SJ^{-1}$, then $|T|=3m+\mathsf{D}(G)-1$ and $N^{m}(T)=N^{3m}(T)=N^{5m}(T)=N^{6m}(T)=0$. By Lemma 2.2 we get:

$$1+N^{2m}(T)+N^{4m}(T)\equiv 0 \pmod p. \tag{5.15}$$

Similar to above, we choose a subsequence T_1 from T of length $2m+\mathsf{D}(G)-1$, one obtains

$$\binom{3m+\mathsf{D}(G)-1}{2m+\mathsf{D}(G)-1}+\binom{m+\mathsf{D}(G)-1}{\mathsf{D}(G)-1}N^{2m}(T)$$
$$+\binom{\mathsf{D}(G)-m-1}{\mathsf{D}(G)-2m-1}N^{4m}(T)\equiv 0 \pmod p.$$

By Remark 5.1 and Lemma 2.3, that is

$$6+4N^{2m}(T)+2N^{4m}(T)\equiv 0 \pmod p. \tag{5.16}$$

Moreover, we choose a subsequence T_2 from T of length $m+\mathsf{D}(G)-1$, one obtains

$$\begin{aligned}\binom{3m+\mathsf{D}(G)-1}{m+\mathsf{D}(G)-1} &+ \binom{m+\mathsf{D}(G)-1}{\mathsf{D}(G)-m-1}N^{2m}(T)\\ &+\binom{\mathsf{D}(G)-m-1}{\mathsf{D}(G)-3m-1}N^{4m}(T)\equiv 0 \pmod p.\end{aligned}$$

By Remark 5.1 and Lemma 2.3, that is

$$15+6N^{2m}(T)+N^{4m}(T)\equiv 0 \pmod p. \tag{5.17}$$

Combining equations (5.15)-(5.17), one deduces that $6\equiv 0 \pmod p$, this contradicts $p\geq 5$, therefore we can suppose that $N^{2m}(S)=0$.

If $N^{3m}(S)\neq 0$, then S contains a zero-sum subsequence J of length $3m$, let $T=SJ^{-1}$, then $|T|=2m+\mathsf{D}(G)-1$ and $N^{m}(T)=N^{2m}(T)=N^{4m}(T)=N^{5m}(T)=0$. By Lemma 2.2 we get:

$$1-N^{3m}(T)\equiv 0 \pmod p. \tag{5.18}$$

Similar to above, we choose a subsequence T_1 from T of length $m+\mathsf{D}(G)-1$, one obtains

$$\binom{2m+\mathsf{D}(G)-1}{m+\mathsf{D}(G)-1} - \binom{\mathsf{D}(G)-m-1}{\mathsf{D}(G)-2m-1}N^{3m}(T)\equiv 0 \pmod p.$$

By Remark 5.1 and Lemma 2.3, that is

$$5-2N^{3m}(T)\equiv 0 \pmod p. \tag{5.19}$$

Combining equations (5.18) and (5.19), one deduces that $3\equiv 0 \pmod p$, this contradicts $p\geq 5$, therefore we may also assume that $N^{3m}(S)=0$.

If $N^{4m}(S)\neq 0$, then S contains a zero-sum subsequence J of length $4m$, let $T=SJ^{-1}$, then $|T|=m+\mathsf{D}(G)-1$ and $N^{m}(T)=N^{2m}(T)=N^{3m}(T)=0$. But by Lemma 2.2, $N^{4m}(T)\neq 0$, thus there exist $U|T$ with $|U|=4m$, and JU is a zero-sum sequence of length $8m$, by (ii) it is obvious that $N^{5m}(JU)>0$, therefore $N^{5m}(S)>0$ and we may also assume that $N^{4m}(S)=0$.

Above all, we conclude that

$$N^{m}(S)=N^{2m}(S)=N^{3m}(S)=N^{4m}(S)=0,$$

this contradicts $|S|=5m+\mathsf{D}(G)-1\geq m+\mathsf{D}(G)-1$ and Lemma 2.2. □

Lemma 5.4. *Let S be a sequence over G. If $N^{8m}(S)>0$ or $N^{9m}(S)>0$, then $N^{6m}(S)>0$. Moreover, we get*

$$\mathsf{s}_{6m}(G)=6m+\mathsf{D}(G)-1.$$

Proof. **(i)** Let S' be a zero-sum subsequence of S with $|S'|=8m$. Assume to the contrary that $N^{6m}(S)=0$, then we have $N^{6m}(S')=N^{2m}(S')=0$.

If $N^{m}(S')\neq 0$, then S' contains a zero-sum subsequence J of length m, let $T=S'J^{-1}$, then $|T|=7m$ and by Lemma 5.3, we can get $N^{5m}(T)>0$, hence $N^{6m}(S')>0$, therefore we may assume that $N^{m}(S')=N^{7m}(S')=0$.

If $N^{3m}(S')\neq 0$, then S' contains a zero-sum subsequence J of length $3m$, let $T=S'J^{-1}$, then T is a zero-sum sequence of length $5m$, and $N^{m}(T)=N^{2m}(T)=N^{3m}(T)=N^{4m}(T)=0$, this contradicts to Lemma 2.2, therefore we may also assume that $N^{3m}(S')=N^{5m}(S')=0$.

Lemma 2.2 implies that

$$1 + N^{4m}(S') \equiv 0 \pmod p. \tag{5.20}$$

Similar to above, we choose a subsequence J from S' of length $7m$, one obtains

$$\binom{8m}{7m} + \binom{4m}{3m} N^{4m}(S') \equiv 0 \pmod p$$

By Remark 5.1 and Lemma 2.3, that is

$$8 + 4N^{4m}(S') \equiv 0 \pmod p. \tag{5.21}$$

Combining equations (5.20) and (5.21), one can deduces $4 \equiv 0 \pmod p$, this contradicts to $p \geq 5$. Therefore, $N^{6m}(S') > 0$.

(ii) Let S' be a zero-sum subsequence of S with $|S'| = 9m$. Assume to the contrary that $N^{6m}(S) = 0$, then we have $N^{6m}(S') = N^{3m}(S') = 0$.

If $N^m(S') \neq 0$, then S' contains a zero-sum subsequence J of length m, let $T = S'J^{-1}$, then $|T| = 8m$ and by (i), we can get $N^{6m}(T) > 0$, hence $N^{6m}(S') > 0$, therefore we may assume that $N^m(S') = N^{8m}(S') = 0$.

If $N^{2m}(S') \neq 0$, then S' contains a zero-sum subsequence J of length $2m$, let $T = S'J^{-1}$, then $|T| = 7m$ and by Lemma 5.2, we can get $N^{4m}(T) > 0$, hence $N^{6m}(S') > 0$, therefore we may also assume that $N^{2m}(S') = N^{7m}(S') = 0$.

If $N^{4m}(S') \neq 0$, then S' contains a zero-sum subsequence J of length $4m$, let $T = S'J^{-1}$, then $|T| = 5m$ and $N^m(T) = N^{2m}(T) = N^{3m}(T) = 0$, by Lemma 2.2, we can get $N^{4m}(T) > 0$, hence $N^{8m}(S') > 0$ and by (i) $N^{6m}(S') > 0$, and we may also assume that $N^{4m}(S') = 0$.

Above all, we conclude that

$$N^m(S') = N^{2m}(S') = N^{3m}(S') = N^{4m}(S') = 0,$$

this contradicts $|S'| = 9m \geq m + \mathsf{D}(G) - 1$ and Lemma 2.2.

Next, we are going to investigate $\mathsf{s}_{6m}(G)$.

Let S be a sequence over G of length $6m + \mathsf{D}(G) - 1$, assume that $N^{6m}(S) = 0$, then $N^{8m}(S) = N^{9m}(S) = 0$.

If $N^m(S) \neq 0$, then S contains a zero-sum subsequence J of length m, let $T = SJ^{-1}$, then $|T| = 5m + \mathsf{D}(G) - 1$ and Lemma 5.3 implies $N^{5m}(T) > 0$, hence $N^{6m}(S) > 0$. Therefore we may assume that $N^m(S) = 0$.

If $N^{2m}(S) \neq 0$, then S contains a zero-sum subsequence J of length $2m$, let $T = SJ^{-1}$, then $|T| = 4m + \mathsf{D}(G) - 1$ and Lemma 5.2 implies $N^{4m}(T) > 0$, hence $N^{6m}(S) > 0$. Therefore we may also assume that $N^{2m}(S) = 0$.

If $N^{3m}(S) \neq 0$, then S contains a zero-sum subsequence J of length $3m$, let $T = SJ^{-1}$, and $|T| = 3m + \mathsf{D}(G) - 1$, and $N^m(T) = N^{2m}(T) = N^{3m}(T) = N^{5m}(T) = N^{6m}(T) = 0$, by Lemma 2.2, we can get $N^{4m}(T) > 0$ and

$$1 + N^{4m}(T) \equiv 0 \pmod p. \tag{5.22}$$

Similar to above, we choose a subsequence J from T of length $2m + \mathsf{D}(G) - 1$, one obtains

$$\binom{3m + \mathsf{D}(G) - 1}{2m + \mathsf{D}(G) - 1} + \binom{\mathsf{D}(G) - m - 1}{\mathsf{D}(G) - 2m - 1} N^{4m}(T) \equiv 0 \pmod p.$$

By Remark 5.1 and Lemma 2.3, that is

$$6 + 2N^{4m}(T) \equiv 0 \pmod p. \tag{5.23}$$

Combining equations (5.22) and (5.23), one deduces that $4 \equiv 0 \pmod p$, but this contradicts $p \geq 5$, therefore we may also assume that $N^{3m}(S) = 0$.

If $N^{4m}(S) \neq 0$, then S contains a zero-sum subsequence J of length $4m$, let $T = SJ^{-1}$, and $|T| = 2m + \mathsf{D}(G) - 1$, and $N^{m}(T) = N^{2m}(T) = N^{3m}(T) = 0$, by Lemma 2.2, we can get $N^{4m}(T) > 0$, this implies $N^{8m}(S) > 0$ and $N^{6m}(S) > 0$. Therefore we may also assume that $N^{4m}(S) = 0$.

Above all, we conclude that

$$N^{m}(S) = N^{2m}(S) = N^{3m}(S) = N^{4m}(S) = 0,$$

but it contradicts Lemma 2.2 since $|S| = 6m + \mathsf{D}(G) - 1 \geq m + \mathsf{D}(G) - 1$. □

Lemma 5.5. *Let S be a sequence over G. If $N^{9m}(S) > 0$ or $N^{10m}(S) > 0$, then $N^{7m}(S) > 0$. Moreover, we get*

$$\mathsf{s}_{7m}(G) = 7m + \mathsf{D}(G) - 1.$$

Proof. **(i)** Let S' be a zero-sum subsequence of S with $|S'| = 9m$. Assume to the contrary that $N^{7m}(S) = 0$, then we have $N^{7m}(S') = N^{2m}(S') = 0$.

If $N^{m}(S') \neq 0$, then S' contains a zero-sum subsequence J of length m, let $T = S'J^{-1}$, then T is a zero-sum sequence of length $8m$ and by Lemma 5.4, we can get $N^{6m}(T) > 0$, hence $N^{7m}(S') > 0$, therefore we may assume that $N^{m}(S') = N^{8m}(S') = 0$.

If $N^{3m}(S') \neq 0$, then S' contains a zero-sum subsequence J of length $3m$, let $T = S'J^{-1}$, then T is a zero-sum sequence of length $6m$, and by Lemma 5.2, we can get $N^{4m}(T) > 0$, hence $N^{7m}(S') > 0$, therefore we may also assume that $N^{3m}(S') = N^{6m}(S') = 0$.

If $N^{4m}(S') \neq 0$, then S' contains a zero-sum subsequence J of length $4m$, let $T = S'J^{-1}$, then T is a zero-sum sequence of length $5m$, and $N^{m}(T) = N^{2m}(T) = N^{3m}(T) = N^{4m}(T) = 0$, this contradicts Lemma 2.2 since $|T| = 5m \geq m + \mathsf{D}(G) - 1$, therefore we may also assume that $N^{4m}(S') = 0$.

Above all, we can conclude that

$$N^{m}(S') = N^{2m}(S') = N^{3m}(S') = N^{4m}(S') = 0,$$

but it contradicts Lemma 2.2 since $|S'| = 9m \geq m + \mathsf{D}(G) - 1$.

(ii) Let S' be a zero-sum subsequence of S with $|S'| = 10m$. Assume to the contrary that $N^{7m}(S) = 0$, then we have $N^{7m}(S') = N^{3m}(S') = 0$.

If $N^{m}(S') \neq 0$, then S' contains a zero-sum subsequence J of length m, let $T = S'J^{-1}$, then T is a zero-sum sequence of length $9m$ and by (i), we can get $N^{7m}(T) > 0$, hence $N^{7m}(S) > 0$, therefore we may assume that $N^{m}(S') = N^{9m}(S') = 0$.

If $N^{2m}(S') \neq 0$, then S' contains a zero-sum subsequence J of length $2m$, let $T = S'J^{-1}$, then T is a zero-sum sequence of length $8m$, and by Lemma 5.3, we can get $N^{5m}(T) > 0$, hence $N^{7m}(S') > 0$, therefore we may assume that $N^{2m}(S') = N^{8m}(S') = 0$.

If $N^{4m}(S') \neq 0$, then S' contains a zero-sum subsequence J of length $4m$, let $T = S'J^{-1}$, then T is a zero-sum sequence of length $6m$, and $N^{m}(T) = N^{2m}(T) = N^{3m}(T) = N^{4m}(T) = 0$, this contradicts Lemma 2.2 since $|T| = 6m \geq m + \mathsf{D}(G) - 1$ and we may assume that $N^{4m}(S') = 0$.

Above all, we conclude that

$$N^{m}(S') = N^{2m}(S') = N^{3m}(S') = N^{4m}(S') = 0,$$

but it contradicts Lemma 2.2 since $|S'| = 10m \geq m + \mathsf{D}(G) - 1$.

Next, we are going to investigate $\mathsf{s}_{7m}(G)$.

Let S be a sequence over G of length $7m+\mathsf{D}(G)-1$, assume that $N^{7m}(S)=0$, then $N^{9m}(S)=N^{10m}(S)=0$.

If $N^{m}(S)\neq 0$, then S contains a zero-sum subsequence J of length m, let $T=SJ^{-1}$, and $|T|=6m+\mathsf{D}(G)-1$, by Lemma 5.4, we can get $N^{6m}(T)>0$, hence $N^{7m}(S)>0$, therefore we may assume that $N^{m}(S)=0$.

If $N^{2m}(S)\neq 0$, then S contains a zero-sum subsequence J of length $2m$, let $T=SJ^{-1}$, and $|T|=5m+\mathsf{D}(G)-1$, by Lemma 5.3, we can get $N^{5m}(T)>0$, hence $N^{7m}(S)>0$, therefore we may also assume that $N^{2m}(S)=0$.

If $N^{3m}(S)\neq 0$, then S contains a zero-sum subsequence J of length $3m$, let $T=SJ^{-1}$, and $|T|=4m+\mathsf{D}(G)-1$, by Lemma 5.2, we can get $N^{4m}(T)>0$, hence $N^{7m}(S)>0$, therefore we may also assume that $N^{3m}(S)=0$.

If $N^{4m}(S)\neq 0$, then S contains a zero-sum subsequence J of length $4m$, let $T=SJ^{-1}$, and $|T|=3m+\mathsf{D}(G)-1$, and $N^{m}(T)=N^{2m}(T)=N^{3m}(T)=N^{5m}(T)=N^{6m}(T)=0$, by Lemma 2.2, we can get $N^{4m}(T)>0$ and

$$1+N^{4m}(T)\equiv 0 \pmod p. \tag{5.24}$$

Similar to above, we choose a subsequence J from T of length $2m+\mathsf{D}(G)-1$, one obtains

$$\binom{3m+\mathsf{D}(G)-1}{2m+\mathsf{D}(G)-1}+\binom{\mathsf{D}(G)-m-1}{\mathsf{D}(G)-2m-1}N^{4m}(T)\equiv 0 \pmod p.$$

By Remark 5.1 and Lemma 2.3, that is

$$6+2N^{4m}(T)\equiv 0 \pmod p. \tag{5.25}$$

Combining equations (5.24)-(5.25), one deduces that $4\equiv 0 \pmod p$, this contradicts to $p\geq 5$, therefore we may also assume that $N^{4m}(S)=0$.

Above all, we conclude that

$$N^{m}(S)=N^{2m}(S)=N^{3m}(S)=N^{4m}(S)=0,$$

but it contradicts Lemma 2.2 since $|S|=7m+\mathsf{D}(G)-1\geq m+\mathsf{D}(G)-1$. □

Proof of Theorem 1.2 (3). Since $\lceil\frac{\mathsf{D}(G)}{m}\rceil=4$, we have to prove that $\mathsf{s}_{km}(G)=km+\mathsf{D}(G)-1$ holds for every $k\geq 4$.

We prove by induction on k, for $k=4,5,6,7$, it holds by Lemma 5.2-5.5. Suppose that $k\geq 4$ and the result holds for all positive integers n with $4\leq n\leq k$. Now we need to prove $\mathsf{s}_{(k+1)m}(G)=(k+1)m+\mathsf{D}(G)-1$.

Let S be a sequence over G of length $(k+1)m+\mathsf{D}(G)-1$, since

$$|S|\geq (k-3)m+\mathsf{D}(G)-1=\mathsf{s}_{(k-3)m}(G),$$

then S contains a zero-sum subsequence T of length $(k-3)m$. Let $J=ST^{-1}$, and

$$|J|=4m+\mathsf{D}(G)-1=\mathsf{s}_{4m}(G),$$

therefore J contains a zero-sum subsequence W of length $4m$, then TW is a zero-sum subsequence of length $(k+1)m$, this completes the proof. □

6. Proof of the theorem 1.3 (1)

Throughout this section, assume that G satisfing the following condition:

$$2m < \mathsf{D}(G) \leq 3m,$$

and also assume that $p \geq 5$ is a prime.

Remark 6.1. We have $\lfloor \frac{\mathsf{D}(G)+m-1}{m} \rfloor = 3$ and $km + \mathsf{D}(G) - 1 = (k+2)m + t$, where $0 \leq t \leq m-1$.

Lemma 6.2. *Let S be a sequence over G. If $N^{5m}(S) > 0$, then $N^{2m}(S) > 0$.*

Proof. This is an easy consequence of Lemma 4.2. □

Proof of Theorem 1.3 (1). Let S be a sequence over G of length $3m + \mathsf{D}(G) - 1$, and assume that $N^{2m}(S) = 0$, then $N^{5m}(S) = 0$ by Lemma 6.2.

If $N^m(S) \neq 0$, then S contains a zero-sum subsequence J of length m, let $T = SJ^{-1}$, then $|T| = 2m + \mathsf{D}(G) - 1 \geq m + \mathsf{D}(G) - 1$ and $N^m(T) = N^{2m}(T) = N^{4m}(T) = 0$, Lemma 2.2 implies that

$$1 - N^{3m}(T) \equiv 0 \pmod{p}. \tag{6.1}$$

We choose a subsequence J from T of length $m + \mathsf{D}(G) - 1 \geq m + \mathsf{D}(G) - 1$, then one obtains

$$\binom{2m + \mathsf{D}(G) - 1}{m + \mathsf{D}(G) - 1} - \binom{\mathsf{D}(G) - 1 - m}{\mathsf{D}(G) - 1 - 2m} N^{3m}(T) \equiv 0 \pmod{p}$$

By Remark 6.1 and Lemma 2.3, that is

$$4 - N^{3m}(T) \equiv 0 \pmod{p}. \tag{6.2}$$

Combining equations (6.1) and (6.2) one can deduces $3 \equiv 0 \pmod{p}$, this contradicts $p \geq 5$, therefore we may assume that $N^m(S) = 0$.

By Lemma 2.2 we get

$$1 - N^{3m}(S) + N^{4m}(S) \equiv 0 \pmod{p}. \tag{6.3}$$

Smilarly, we choose any subsequence $J_1 | S$ of length $2m + \mathsf{D}(G) - 1 \geq m + \mathsf{D}(G) - 1$, then one obtains

$$\begin{aligned} &\binom{3m + \mathsf{D}(G) - 1}{2m + \mathsf{D}(G) - 1} - \binom{\mathsf{D}(G) - 1}{m + \mathsf{D}(G) - 1 - m} N^{3m}(S) \\ &+ \binom{\mathsf{D}(G) - 1 - m}{\mathsf{D}(G) - 1 - 2m} N^{4m}(S) \equiv 0 \pmod{p}. \end{aligned}$$

By Remark 6.1 and Lemma 2.3, that is

$$5 - 2N^{3m}(S) + N^{4m}(S) \equiv 0 \pmod{p}. \tag{6.4}$$

Moreover, we choose any subsequence $J_2 | S$ of length $m + \mathsf{D}(G) - 1 \geq m + \mathsf{D}(G) - 1$, then one obtains

$$\binom{3m + \mathsf{D}(G) - 1}{m + \mathsf{D}(G) - 1} - \binom{\mathsf{D}(G) - 1}{\mathsf{D}(G) - 1 - 2m} N^{3m}(S) \equiv 0 \pmod{p}.$$

By Remark 6.1 and Lemma 2.3, that is

$$10 - N^{3m}(S) \equiv 0 \pmod{p}. \tag{6.5}$$

Combining equations (6.3)-(6.5), one deduces that $6 \equiv 0 \pmod{p}$, this contradicts $p \geq 5$ and completes the proof. □

7. Proof of the theorem 1.3 (2)

Throughout this section, assume that G satisfing the following condition:

$$3m < \mathsf{D}(G) \le 4m,$$

and also assume that $p \ge 7$ is a prime.

Remark 7.1. We have $\lfloor \frac{\mathsf{D}(G)+m-1}{m} \rfloor = 4$ and $km + \mathsf{D}(G) - 1 = (k+3)m + t$, where $0 \le t \le m-1$.

Lemma 7.2. *Let S be a sequence over G. If $N^{6m}(S) > 0$ or $N^{7m}(S) > 0$, then $N^{2m}(S) > 0$.*

Proof. This is an easy consequence of Lemma 5.2 and 5.3. □

Proof of Theorem 1.3 (2). Let S be a sequence over G of length $4m + \mathsf{D}(G) - 1$, and assume that $N^{2m}(S) = 0$, then $N^{6m}(S) = N^{7m}(S) = 0$ by Lemma 7.2.

If $N^m(S) \ne 0$, then S contains a zero-sum subsequence J of length m, let $T = SJ^{-1}$, then $|T| = 3m + \mathsf{D}(G) - 1 \ge m + \mathsf{D}(G) - 1$ and $N^m(T) = N^{2m}(T) = N^{5m}(T) = N^{6m}(T) = N^{7m}(T) = 0$.

Lemma 2.2 implies that

$$1 - N^{3m}(T) + N^{4m}(T) \equiv 0 \pmod p. \tag{7.1}$$

Similar to above, we choose any subsequence $J_1|S$ of length $2m + \mathsf{D}(G) - 1 \ge m + \mathsf{D}(G) - 1$, then one obtains

$$\binom{3m + \mathsf{D}(G) - 1}{2m + \mathsf{D}(G) - 1} - \binom{\mathsf{D}(G) - 1}{\mathsf{D}(G) - 1 - m} N^{3m}(T) + \binom{\mathsf{D}(G) - 1 - m}{\mathsf{D}(G) - 1 - 2m} N^{4m}(T) \equiv 0 \pmod p.$$

By Remark 7.1 and Lemma 2.3, that is

$$6 - 3N^{3m}(T) + 2N^{4m}(T) \equiv 0 \pmod p. \tag{7.2}$$

Moreover, we choose any subsequence $J_2|T$ of length $m + \mathsf{D}(G) - 1 \ge m + \mathsf{D}(G) - 1$, then one obtains

$$\binom{3m + \mathsf{D}(G) - 1}{m + \mathsf{D}(G) - 1} - \binom{\mathsf{D}(G) - 1}{\mathsf{D}(G) - 1 - 2m} N^{3m}(T) + \binom{\mathsf{D}(G) - 1 - m}{\mathsf{D}(G) - 1 - 3m} N^{4m}(T) \equiv 0 \pmod p.$$

By Remark 7.1 and Lemma 2.3, that is

$$15 - 3N^{3m}(T) + N^{4m}(T) \equiv 0 \pmod p. \tag{7.3}$$

Combining equations (7.1)-(7.3), one deduces that $2 \equiv 0 \pmod p$, this contradicts $p \ge 7$ and we can assume that $N^m(S) = 0$.

If $N^{3m}(S) \ne 0$, then S contains a zero-sum subsequence J of length $3m$, let $T = SJ^{-1}$, then $|T| = m + \mathsf{D}(G) - 1 \ge m + \mathsf{D}(G) - 1$ and $N^m(T) = N^{2m}(T) = N^{3m}(T) = N^{4m}(T) = 0$ which contradicts Lemma 2.2.

Now, by Lemma 2.2 we get

$$1 - N^{4m}(S) + N^{5m}(S) \equiv 0 \pmod p. \tag{7.4}$$

Similarly, We choose any subsequence $J_1|S$ of length $3m+\mathsf{D}(G)-1 \geq m+\mathsf{D}(G)-1$, then one obtains

$$\binom{4m+\mathsf{D}(G)-1}{3m+\mathsf{D}(G)-1} - \binom{\mathsf{D}(G)-1}{\mathsf{D}(G)-1-m} N^{4m}(S)$$
$$+ \binom{\mathsf{D}(G)-1-m}{\mathsf{D}(G)-1-2m} N^{5m}(S) \equiv 0 \pmod p.$$

By Remark 7.1 and Lemma 2.3, that is

$$7 - 3N^{4m}(S) + 2N^{5m}(S) \equiv 0 \pmod p. \tag{7.5}$$

Moreover, we choose any subsequence $J_2|S$ of length $2m+\mathsf{D}(G)-1 \geq m+\mathsf{D}(G)-1$, then one obtains

$$\binom{4m+\mathsf{D}(G)-1}{2m+\mathsf{D}(G)-1} - \binom{\mathsf{D}(G)-1}{\mathsf{D}(G)-1-2m} N^{4m}(S)$$
$$+ \binom{\mathsf{D}(G)-1-m}{\mathsf{D}(G)-1-3m} N^{4m}(S) \equiv 0 \pmod p.$$

By Remark 7.1 and Lemma 2.3, that is

$$21 - 3N^{4m}(S) + N^{5m}(S) \equiv 0 \pmod p. \tag{7.6}$$

Combining equations (7.4)-(7.6), one deduces that $5 \equiv 0 \pmod p$, this contradicts $p \geq 7$ and completes the proof. □

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