# EDGE-TRANSITIVE HOMOGENEOUS FACTORISATIONS OF COMPLETE UNIFORM HYPERGRAPHS

HU YE CHEN AND ZAI PING LU

Abstract. For a finite set $V$ and a positive integer $k$ with $k \leq n := |V|$, letting $V^{\{k\}}$ be the set of all $k$-subsets of $V$, the pair $\mathcal{K}_n^k := (V, V^{\{k\}})$ is called the complete $k$-hypergraph on $V$, while each $k$-subset of $V$ is called an edge. A factorisation of the complete $k$-hypergraph $\mathcal{K}_n^k$ of index $s \geq 2$, simply a $(k,s)$-factorisation of order $n$, is a partition $\{E_1, E_2, \ldots, E_s\}$ of the edges into $s$ disjoint subsets such that each $k$-hypergraph $(V, E_i)$, called a factor, is a spanning subhypergraph of $\mathcal{K}_n^k$. Such a factorisation is homogeneous if there exist two transitive subgroups $G$ and $M$ of the symmetric group of degree $n$ such that $G$ induces a transitive action on the set $\{E_1, E_2, \ldots, E_s\}$ and $M$ lies in the kernel of this action.

In this paper, we give a classification of homogeneous factorisations of $\mathcal{K}_n^k$ which admit a group acting transitively on the edges of $\mathcal{K}_n^k$. It is shown that, for $6 \leq 2k \leq n$ and $s \geq 2$, there exists an edge-transitive homogeneous $(k,s)$-factorisation of order $n$ if and only if $(n,k,s)$ is one of $(32,3,5)$, $(32,3,31)$, $(33,4,5)$, $(2^d, 3, \frac{(2^d-1)(2^{d-1}-1)}{3})$ and $(q+1, 3, 2)$, where $d \geq 3$ and $q$ is a prime power with $q \equiv 1\,(\mathsf{mod}\ 4)$.

Keywords: uniform hypergraph, self-complementary hypergraph, edge-transitive, homogeneous factorisation, homogeneous permutation group.

## 1. introduction

Let $V$ be a finite (nonempty) set. For a positive integer $k \leq |V|$, we use $V^{\{k\}}$ to denote the set of all $k$-subsets of $V$. In this paper, a *$k$-uniform hypergraph* (or *$k$-hypergraph*) with vertex set $V$ and edge set $E$ is a pair $(V, E)$, where $E$ is a subset of $V^{\{k\}}$. Note that a 2-hypergraph is a graph. For a set $V$ of size $n$ and a positive integer $k \leq n$, we set $\mathcal{K}_n^k = (V, V^{\{k\}})$, which is called the *complete $k$-hypergraph* (on $V$). Two $k$-hypergraphs $\mathcal{H}_1 = (V_1, E_1)$ and $\mathcal{H}_2 = (V_2, E_2)$ are said to be *isomorphic* if there is a bijection $\phi$ between $V_1$ and $V_2$ such that $\phi$ induces a bijection between $E_1$ and $E_2$, while this bijection $\phi$ is called an isomorphism between $\mathcal{H}_1$ and $\mathcal{H}_2$.

Let $\mathcal{H} = (V, E)$ be a $k$-hypergraph. An isomorphism from $\mathcal{H}$ onto itself is called an *automorphism* of $\mathcal{H}$. Let $\mathsf{Aut}\mathcal{H}$ be the set of all automorphisms of $\mathcal{H}$. Then $\mathsf{Aut}\mathcal{H}$ is a subgroup of the symmetric group $\mathrm{Sym}(V)$. Note that $\mathrm{Sym}(V)$ acts transitively on $V^{\{k\}}$. Thus $\mathsf{Aut}\mathcal{H} = \mathrm{Sym}(V)$ if and only if either $\mathcal{H} = \mathcal{K}_n^k$ or $E = \emptyset$. For a subgroup $G \leq \mathsf{Aut}\mathcal{H}$, the hypergraph $\mathcal{H}$ is said to be $G$-vertex-transitive or $G$-edge-transitive if $G$ acts transitively on $V$ or $E$, respectively. The *complement* $\mathcal{H}^c$ of $\mathcal{H}$ is the $k$-hypergraph $(V, V^{\{k\}} \setminus E)$. Note that $\mathsf{Aut}\mathcal{H} = \mathsf{Aut}\mathcal{H}^c$. If there is an isomorphism $\tau : \mathcal{H} \to \mathcal{H}^c$, then $\mathcal{H}$ is said to be *self-complementary*, while the isomorphism $\tau$ is called an *antimorphism* of $\mathcal{H}$.

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Self-complementary uniform hypergraphs have been extensively studied, see [14, 16, 18, 19, 20, 25] and the references therein for self-complementary graphs, and see [7, 8, 9, 23, 24] for self-complementary uniform hypergraphs. In particular, Peisert [21] gave a complete classification for symmetric (i.e., vertex-transitive and edge-transitive) self-complementary graphs.

Let $k \geq 1$ and $s \geq 2$ be integers. A *factorisation* of $\mathcal{K}_n^k$ of index $s$ is a partition $\{E_1, E_2, \ldots, E_s\}$ of $V^{\{k\}}$ into $s$ disjoint subsets such that each $k$-hypergraph $(V, E_i)$ is a spanning subhypergraph, that is, for every $i \in \{1, 2, \ldots, s\}$, each $v \in V$ is contained in some $e \in E_i$. For convenience, we sometimes call such a factorisation a $(k, s)$-*factorisation* (on $V$) of order $n$, and call the resulting $k$-hypergraphs $(V, E_i)$ its factors. Two $(k, s)$-factorisations $\mathcal{F} = \{E_1, E_2, \ldots, E_s\}$ on $V$ and $\mathcal{E}$ on $U$ are said to be *isomorphic*, denoted by $\mathcal{F} \cong \mathcal{E}$, if there is a bijection $\phi : V \to U$ such that $\phi$ induces a bijection $V^{\{k\}} \to U^{\{k\}}$ and $\mathcal{E} = \{E_i^{\phi} \mid 1 \leq i \leq s\}$, while this bijection $\phi$ is called an *isomorphism* from $\mathcal{F}$ to $\mathcal{E}$.

Let $\mathcal{F} = \{E_1, E_2, \ldots, E_s\}$ be a $(k, s)$-factorisation on $V$. An isomorphism from $\mathcal{F}$ to itself is called an *automorphism* of $\mathcal{F}$. Let $\mathsf{Aut}\mathcal{F}$ be the set of all automorphisms of $\mathcal{F}$. Then it is easily shown that $\mathsf{Aut}\mathcal{F}$ is just the subgroup of $\mathrm{Sym}(V)$ which preserves the partition $\mathcal{F}$. For each $1 \leq i \leq s$, set $\mathcal{H}_i = (V, E_i)$, and let $\mathsf{Aut}(\mathcal{F}, E_i)$ be the subgroup of $\mathsf{Aut}\mathcal{F}$ fixing $E_i$ set-wise. Then $\mathsf{Aut}(\mathcal{F}, E_i) \leq \mathsf{Aut}\mathcal{H}_i$. The factorisation $\mathcal{F}$ is said to be *factor-transitive* if $\mathsf{Aut}\mathcal{F}$ acts transitively on the partition $\mathcal{F}$, and *vertex-transitive* (resp. *edge-transitive*) if further every factor $\mathcal{H}_i$ is $\mathsf{Aut}(\mathcal{F}, E_i)$-vertex-transitive (resp. $\mathsf{Aut}(\mathcal{F}, E_i)$-edge-transitive). (Note that, for $k = 2$, the edge-transitivity of factorisations considered in [15] is slightly more restricted than that given here.) The factorisation $\mathcal{F} = \{E_1, E_2, \ldots, E_s\}$ is said to be *homogeneous* if $\cap_{i=1}^{s} \mathsf{Aut}(\mathcal{F}, E_i)$, the kernel of $\mathsf{Aut}(\mathcal{F})$ acting on $\{E_1, E_2, \ldots, E_s\}$, is a transitive subgroup of $\mathrm{Sym}(V)$. Note that a vertex-transitive $(k, 2)$-factorisation if exists must be homogeneous.

Vertex-transitive factorisations of complete uniform hypergraphs are natural generalizations of vertex-transitive self-complementary uniform hypergraphs. In fact, each factor of a vertex-transitive $(k, 2)$-factorisation is a vertex-transitive self-complementary k-hypergraph. Conversely, every vertex-transitive self-complementary k-hypergraph together with its complement gives a vertex-transitive $(k, 2)$-factorisation.

As generalizations of vertex-transitive self-complementary graphs, homogeneous factorisations of complete graphs (complete 2-hypergraphs) were introduced in [17] (and for graphs in general in [5]). The reader is referred to [5, 6, 11, 17] for the theory of homogeneous factorisations of graphs. In [15], Li, Lim and Praeger classified the homogeneous factorisations of complete graphs with all factors admitting a common edge-transitive group. This motivates us to consider in this paper the problem of classifying edge-transitive homogeneous factorisations of complete $k$-hypergraphs, where $k \geq 3$.

After collecting some preliminary results on permutation groups in Section 2, a global analysing is given in Section 3 for edge-transitive homogeneous factorisations. In Section 4, some examples of edge-transitive homogeneous factorisations are constructed. Finally, our main result is presented in Section 5.

## 2. Preliminaries

In this section, we assume that $V$ is a finite nonempty set.

Let $G$ be a permutation group on $V$, that is, $G$ is a subgroup of the symmetric group $\mathrm{Sym}(V)$. For a subset $B \subseteq V$, denote by $G_B$ and $G_{(B)}$ the subgroups of $G$ fixing $B$ set-wise and point-wise, respectively. Then $G_{(B)}$ is normal in $G_B$ and is the kernel of $G_B$ acting on $B$. If $B$ is a singleton $\{v\}$ then $G_B = G_{(B)} = \{g \in G \mid v^g = v\}$. Write $G_v = \{g \in G \mid v^g = v\}$, and call it the *stabilizer* of $v$ in $G$. For $v \in V$, the *orbit* of $G$ containing $v$ is the subset $v^G := \{v^g \mid g \in G\}$. Note that $|v^G|$ equals to the index of $G_v$ in $G$, that is, $|v^G| = |G : G_v| = \frac{|G|}{|G_v|}$. If $G$ has only one orbit then $G$ is said to be *transitive*. The permutation group $G$ is *semiregular* if $G_v = 1$ for all $v \in V$, and *regular* if further $G$ is transitive on $V$.

Let $G$ be a transitive permutation group on $V$. A *block* of $G$ is a nonempty subset $B \subseteq V$ such that for every $g \in G$, either $B^g = B$ or $B^g \cap B = \emptyset$. A block is *trivial* if $|B| = 1$ or $B = V$, and *nontrivial* otherwise. Then $G$ is *primitive* if it has only trivial blocks. A partition $\mathcal{B}$ of $V$ is *$G$-invariant* if $B^g \in \mathcal{B}$ for $\forall B \in \mathcal{B}$ and $\forall g \in G$. Clearly, if $B$ is a block then $\{B^g \mid g \in G\}$ is a $G$-invariant partition. Conversely, for a $G$-invariant partition $\mathcal{B}$, every part $B \in \mathcal{B}$ is a block of $G$, and $\mathcal{B} = \{B^g \mid g \in G\}$. For a block $B$ and $v \in B$, we have $G_v \leq G_B$. This simple fact leads to a bijection between certain subgroups of $G$ and blocks of $G$, refer to [4, Theorem 1.5A, p.13].

**Lemma 2.1.** *Let $G$ be a transitive permutation group on $V$. Then $H \mapsto v^H$ defines a bijection between the subgroups containing $G_v$ and the blocks containing $v$. In particular, $G$ is primitive if and only if for $v \in V$, the stabilizer $G_v$ is a maximal subgroup of $G$.*

Let $G$ be a transitive permutation group on $V$, and let $\mathcal{B}$ be a $G$-invariant partition of $V$. Then $G$ induces a transitive permutation group $G^{\mathcal{B}}$ on $\mathcal{B}$ with kernel $G_{(\mathcal{B})} = \cap_{B\in\mathcal{B}} G_B$, and $G^{\mathcal{B}} \cong G/G_{(\mathcal{B})}$. An extreme case is that $G_{(\mathcal{B})}$ acts transitively on each part of $\mathcal{B}$. By [4, Theorem 1.6A, p.18], the following lemma holds.

**Lemma 2.2.** *Let $G$ be a transitive permutation group on $V$, and $M$ a normal subgroup of $G$. Then all $M$-orbits on $V$ form a $G$-invariant partition $\mathcal{B}$, and $|\mathcal{B}|$ is a divisor of $|G : M|$. In particular, all $M$-orbits have the same length, and if $G$ is primitive and $M \neq 1$ then $M$ is transitive.*

A $G$-invariant partition $\mathcal{B}'$ is a *refinement* of some $G$-invariant partition $\mathcal{B}$ if every part of $\mathcal{B}$ is the union of some parts of $\mathcal{B}'$. By Lemma 2.1, the following lemma is easily shown.

**Lemma 2.3.** *Let $G$ be a transitive permutation group on $V$, and let $\mathcal{B}$ and $\mathcal{B}'$ be $G$-invariant partitions. Then $\mathcal{B}'$ is a refinement of $\mathcal{B}$ if and only if $B = \cup_{g\in G_B}(B')^g$ for some $B \in \mathcal{B}$ and $B' \in \mathcal{B}'$.*

Let $k$ be an integer with $1 \leq k \leq |V|$, and let $V^{(k)}$ be the set of all ordered $k$-subsets of $V$. A permutation group $G$ is *$k$-transitive* or *$k$-homogeneous* on $V$ if $G$ acts transitively on $V^{(k)}$ or $V^{\{k\}}$, respectively. A permutation group $G$ on $V$ is *sharply $k$-transitive* if it is regular on $V^{(k)}$.

Clearly, a $k$-transitive permutation group is $k$-homogeneous, and a $k$-homogeneous permutation group is also $(|V| - k)$-homogeneous. It is easy to see that for $k \geq 2$, a $k$-homogeneous permutation group is primitive. For $k \geq 2$, all (finite) $k$-transitive permutation groups are known up to permutation isomorphism, see [2, 7.3 and 7.4] for example. (Recall that two permutation groups $G \leq \mathrm{Sym}(V)$ and $H \leq \mathrm{Sym}(U)$ are *permutation isomorphic* if there is a bijection $\lambda : V \to U$ and a group isomorphism $\phi : G \to H$ satisfying $\lambda(v)^\phi = \lambda(v^\phi)$ for all $v \in V$.) For $4 \leq 2k \leq |V|$, Kantor [12] determined all $k$-homogeneous but not $k$-transitive permutation groups, refer to [1, p.290]. These classification results will be used in the following sections.

## 3. Global analysing

Let $\mathcal{F} = \{E_1, E_2, \ldots, E_s\}$ be an edge-transitive homogeneous $(k, s)$-factorisation on $V$ of order $n$ and index $s \geq 2$. Then $|E_i| = \frac{\binom{n}{k}}{s} < \binom{n}{k}$ for $1 \leq i \leq s$. For $v \in V$, set $E_i(v) = \{e \in E_i \mid v \in e\}$. Noting that $\cap_{i=1}^s \mathsf{Aut}(\mathcal{F}, E_i)$ is transitive on $V$, the size $|E_i(v)|$ is independent of the choice of $v$. We have $n|E_i(v)| = |E_i|k = k\frac{\binom{n}{k}}{s} < k\binom{n}{k}$. It follows that $2 \leq k \leq n-2$. For each $i \leq s$, set $E_i^{op} = \{V \setminus e \mid e \in E_i\}$. Let $\mathcal{F}^{op} = \{E_1^{op}, E_2^{op}, \ldots, E_s^{op}\}$. The following lemma is trivial, which allows us to assume that $2k \leq n$.

**Lemma 3.1.** $\mathsf{Aut}\mathcal{F} = \mathsf{Aut}\mathcal{F}^{op}$, $\mathsf{Aut}(\mathcal{F}, E_i) = \mathsf{Aut}(\mathcal{F}^{op}, E_i^{op})$, *and* $\mathcal{F}^{op}$ *is an edge-transitive homogeneous* $(n-k, s)$*-factorisation of order* $n$.

For the rest of this section, we assume that $4 \leq 2k \leq n$ and $M \leq G \leq \mathsf{Aut}\mathcal{F}$ such that

(a) $M$ is normal in $G$ and lies in the kernel of $G$ acting on $\{E_1, E_2, \ldots, E_s\}$; and
(b) $M$ is transitive but not $k$-homogeneous on $V$, and $G$ is $k$-homogeneous on $V$.

Note that such $G$ and $M$ always exist, for example, $G = \mathsf{Aut}\mathcal{F}$ and $M = \cap_{i=1}^s \mathsf{Aut}(\mathcal{F}, E_i)$.

**Claim 1.** $\mathcal{F} = \{E_1, E_2, \ldots, E_s\}$ is a $G$-invariant partition of $V^{\{k\}}$, and $MG_e \leq G_{E_i}$ for $e \in E_i \in \mathcal{F}$.

*Proof.* By the choice of $G$, we know that $G$ is transitive on $V^{\{k\}}$ and preserves the factorisation $\mathcal{F} = \{E_1, E_2, \ldots, E_s\}$. In particular, each $E_i \in \mathcal{F}$ is a block of $G$ acting on $V^{\{k\}}$, and so $G_e \leq G_{E_i}$ for $e \in E_i$. Since $M$ fixes each $E_i$ set-wise, the claim follows. □

**Claim 2.** All $M$-orbits on $V^{\{k\}}$ have the same length $|M : M_e|$ for any given $e \in V^{\{k\}}$, and the number of $M$-orbits on each $E_i$ is equal to $t := \frac{|G_{E_i}:M|}{|G_e:M_e|} = \frac{\binom{n}{k}}{s|M:M_e|}$.

*Proof.* Since $M$ is normal in $G$ and $G$ is transitive on $V^{\{k\}}$, all $M$-orbits on $V^{\{k\}}$ have the same length, see Lemma 2.2. Let $t$ be the number of $M$-orbits on $E_i$. Note that every $M$-orbit on $V^{\{k\}}$ has length $|M : M_e|$, where $e \in V^{\{k\}}$. Then $t = \frac{|E_i|}{|M:M_e|}$. Without loss of generality, we let $e \in E_i$. Then $|E_i| = |G_{E_i} : G_e|$, and so $t = \frac{|G_{E_i}:G_e|}{|M:M_e|} = \frac{|G_{E_i}|}{|M|}\frac{|M_e|}{|G_e|} = \frac{|G_{E_i}:M|}{|G_e:M_e|}$. On the other hand, we have $|E_i| = \frac{|V^{\{k\}}|}{s} = \frac{\binom{n}{k}}{s}$. Then the claim follows. □

For each $i \in \{1, 2, \ldots, s\}$, denote by $\mathcal{E}_i = \{E_i^j \mid 1 \leq j \leq t\}$ the set of $M$-orbits on $E_i$. Set $\mathcal{E} = \cup_{i=1}^s \mathcal{E}_i$. Then we have the next two claims.

**Claim 3.** $\mathcal{E}$ is a $G$-invariant refinement of $\mathcal{F}$ and an edge-transitive homogeneous $(k, st)$-factorisation of order $n$, and $\{\mathcal{E}_i \mid 1 \leq i \leq s\}$ is a $G$-invariant partition of $\mathcal{E}$; in particular, if $G$ induces a primitive permutation group $G^{\mathcal{E}}$ on $\mathcal{E}$ then $\mathcal{F} = \mathcal{E}$.

*Proof.* Note that $\mathcal{E}$ consists of all $M$-orbits on $V^{\{k\}}$, and $|\mathcal{E}| = st$. Since $M$ is normal in $G$, we know that $\mathcal{E}$ is a $G$-invariant partition of $V^{\{k\}}$, see Lemma 2.2. Then the first part of this claim follows. Considering the transitive action induced by $G$ on $\mathcal{E}$, each $\mathcal{E}_i$ is in fact an orbit of $G_{E_i}$, and $G_{\mathcal{E}_i} = G_{E_i}$. For $E \in \mathcal{E}_i$, recalling that $E_i$ is a block of $G$ acting on $V^{\{k\}}$, we have $G_E \le G_{E_i}$. Then $\mathcal{E}_i$ is a block of $G$ acting on $\mathcal{E}$, see Lemma 2.1. Thus $\{\mathcal{E}_i \mid 1 \le i \le s\}$ is a $G$-invariant partition of $\mathcal{E}$. If $G$ acts primitively on $\mathcal{E}$ then $\mathcal{E}_i$ has size 1, and so $\mathcal{E}_i = \{E_i\}$, yielding $\mathcal{F} = \mathcal{E}$. Then our claim holds. □

**Claim 4.** $G_E = MG_e$ for $e \in E \in \mathcal{E}$ and, if $E \subseteq E_i$ then $E_i = \cup_{g \in G_{E_i}} E^g$.

*Proof.* Let $E \in \mathcal{E}$. Then $E$ is a block of $G$ acting on $V^{\{k\}}$. Thus $G_e \le G_E$ for $e \in E$, and $G_E$ is transitive on $E$. Since $M$ is transitive on $E$, we have $G_E = MG_e$. Then this claim follows from Claim 3 and Lemma 2.3. □

Assume further that $k \ge 3$. Since $G$ is $k$-homogeneous, $G$ is $(k-1)$-transitive on $V$, refer to [4, Theorem 9.4B]. Then $G$ has a unique minimal normal subgroup (see [4, Theorem 7.2B] for example), which is either a finite nonabelian simple group or isomorphic to $\mathbb{Z}_p^d$ for some prime $p$ and integer $d \ge 1$. Clearly, this minimal normal subgroup is contained in $M$. Recalling that $M$ is not $k$-homogeneous on $V$, the next lemma follows from [12].

**Lemma 3.2.** *Let $6 \le 2k \le n$, and let $G$ and $M$ be as above. If $G$ is not $k$-transitive on $V$ then, up to permutation isomorphism, one of the following occurs:*

(I) *$k = 3$, $n = 8$, and the pair $(G, M)$ is $(\mathrm{AGL}(1,8), \mathbb{Z}_2^3)$ or $(\mathrm{A\Gamma L}(1,8), \mathbb{Z}_2^3)$;*
(II) *$k = 3$, $n = 32$, and the pair $(G, M)$ is $(\mathrm{A\Gamma L}(1,32), \mathbb{Z}_2^5)$ or $(\mathrm{A\Gamma L}(1,32), \mathbb{Z}_2^5{:}\mathbb{Z}_{31})$;*
(III) *$k = 4$, $n = 32$, $M = \mathrm{PSL}(2,32)$, and $G = \mathrm{P\Gamma L}(2,32)$ is 4-homogeneous on $V$.*

We next determine the $k$-transitive candidates of $G$. In this case, $G$ is a 3-transitive permutation group of degree $n$. All 3-transitive finite permutation groups are explicitly known, refer to [2, 7.3 and 7.4]. Then we have the following lemma.

**Lemma 3.3.** *Assume that $G$ is $k$-transitive on $V$. Then $k = 3$ and, up to permutation isomorphism, one of the following occurs:*

(IV) *$G = \mathrm{AGL}(d,2)$ with $d \ge 3$, $M = \mathbb{Z}_2^d$ and $n = 2^d$;*
(V) *$G = \mathbb{Z}_2^4{:}\mathrm{A}_7 < \mathrm{AGL}(4,2)$, $M = \mathbb{Z}_2^4$ and $n = 16$;*
(VI) *$\mathrm{PSL}(2,q) \le M \le \mathrm{P\Sigma L}(2,q)$ and $\mathrm{PGL}(2,q) \le G \le \mathrm{P\Gamma L}(2,q)$ with $5 \le q = n - 1 \equiv 1 \,(\mathsf{mod}\ 4)$.*

*Proof.* Let $N$ be the minimal normal subgroup of $G$. Assume first that $N \cong \mathbb{Z}_p^d$ for some prime $p$ and integer $d \ge 1$. Then $6 \le 2k \le n = |V| = p^d$. By [2, 7.3], one of parts (IV) and (V) occurs.

Assume that $N$ is nonabelian simple. Checking Table 7.4 given in [2, 7.4], we know that either $k = 3$ and $N = \mathrm{PSL}(2,q)$ with odd $q$, or $N$ is $k$-transitive. Recall $N \le M$ and $M$ is not a $k$-homogeneous permutation group on $V$. We have $k = 3$ and $N = \mathrm{PSL}(2,q)$ with odd $q = n - 1$. Moveover, [12, Theorem 1] yields that $q \equiv 1 \,(\mathsf{mod}\ 4)$. Then part (VI) follows. □

Based on the above argument, we can formulate a method to construct up to isomorphism all possible edge-transitive homogeneous $(k, s)$-factorisations of order $n$, where $6 \le 2k \le n$ and $s \ge 2$.

**Construction 3.4.** Let $G$ be a permutation group on $V$ described as in one of (I)-(VI), and let $M$ be the minimal normal subgroup of $G$. Take a $k$-subset $e$ of $V$ and $v \in e$. Then $G = MG_v$. Take a subgroup $H$ of $G_v$ such that $G_e \le MH \ne G$. Let $E_1 = e^{MH}$, the $MH$-orbit containing $e$. Then $E_1$ consists of $|MH : (MG_e)|$ orbits of $M$ on $V^{\{k\}}$. It is easily shown that $\mathcal{F} := \{E_1^g \mid g \in G\}$ is an edge-transitive homogeneous $(k, s)$-factorisations on $V$, where $s = |G : (MH)|$. Write $G = \cup_{i=1}^{s} MHg_i$ with $g_i \in G_v$ and $g_1 = 1$, and set $E_i = E_1^{g_i}$ for $1 \le i \le s$. It is easily shown that $\mathcal{F} = \{E_i \mid 1 \le i \le s\}$.

## 4. Examples

In this section we construct some edge-transitive homogeneous $(k, s)$-factorisations of order $n$, where $s \ge 2$ and $6 \le 2k \le n$.

For a prime power $q$, denote by $\mathbb{F}_q$ the finite field of order $q$, and $\mathbb{F}_q^*$ the multiplicative group of $\mathbb{F}_q$. Then $\mathbb{F}_q^*$ is cyclic and of order $q-1$. For an integer $d \ge 1$, denote by $\mathbb{F}_q^d$ the $d$-dimensional vector space over $\mathbb{F}_q$. For each vector $\mathbf{u} \in \mathbb{F}_q^d$, denote by $\tau_{\mathbf{u}}$ the translation $\mathbb{F}_q^d \to \mathbb{F}_q^d$, $\mathbf{v} \mapsto \mathbf{v} + \mathbf{u}$. Set $T(d, q) = \{\tau_{\mathbf{u}} \mid \mathbf{u} \in \mathbb{F}_q^d\}$. Then $T(d, q)$ is normal in $\mathrm{A\Gamma L}(d, q)$, $\mathrm{AGL}(d, q) = T(d, q){:}\mathrm{GL}(d, q)$ and $\mathrm{A\Gamma L}(d, q) = T(d, q){:}\Gamma\mathrm{L}(d, q)$. Write $q = p^f$ for some prime $p$. Let $\sigma$ be the Frobenius automorphism of the filed $\mathbb{F}_q$, that is, $\sigma : \mathbb{F}_q \to \mathbb{F}_q$, $\xi \mapsto \xi^p$. Then $\Gamma\mathrm{L}(d, q) = \mathrm{GL}(d, q){:}\langle\sigma\rangle$ and $\mathrm{A\Gamma L}(d, q) = \mathrm{AGL}(d, q){:}\langle\sigma\rangle$, where $\sigma$ acts componentwise on the vectors in $\mathbb{F}_q^d$.

4.1. **Factorisations arising from the affine geometry** $\mathrm{AG}(d, 2)$**.** Let $d \ge 3$ be an integer. Note that each 3-subset of $\mathbb{F}_2^d$ is contained in a unique 2-dimensional affine subspace $\mathbf{v} + U$, where $\mathbf{v} \in \mathbb{F}_2^d$ and $U$ is a 2-dimensional subspace of $\mathbb{F}_2^d$. This allows us to give a partition of $(\mathbb{F}_2^d)^{\{3\}}$ whose parts are indexed by the 2-dimensional subspaces of $\mathbb{F}_2^d$.

**Example 4.1.** For a 2-dimensional subspace $U$ of $\mathbb{F}_2^d$, let $E_U = \cup_{\mathbf{v} \in \mathbb{F}_2^d} (\mathbf{v} + U)^{\{3\}}$. Then $|E_U| = 2^d$, and $\{E_U \mid U \text{ a 2-dimensional subspace of } \mathbb{F}_2^d\}$ is a partition of $(\mathbb{F}_2^d)^{\{3\}}$. Clearly, the number of parts of this partition is equal to the number of 2-dimensional subspaces of $\mathbb{F}_2^d$, which is $\frac{(2^d-1)(2^{d-1}-1)}{3}$. Thus we have a $(3, \frac{(2^d-1)(2^{d-1}-1)}{3})$-factorisation of order $2^d$, namely,

$$\mathcal{F}_{(2^d;3,\frac{(2^d-1)(2^{d-1}-1)}{3})} = \{E_U \mid U \text{ a 2-dimensional subspace of } \mathbb{F}_2^d\}.$$

It is easily shown that, for each 2-dimensional subspace $U$ of $\mathbb{F}_2^d$, the set $E_U$ is an orbit of $T(d, 2)$ on $(\mathbb{F}_2^d)^{\{3\}}$; in fact, $T(d, 2)$ acts regularly on $E_U$. Since $T(d, 2)$ is normal in $\mathrm{AGL}(d, 2)$ and $\mathrm{AGL}(d, 2)$ is transitive on $(\mathbb{F}_2^d)^{\{3\}}$, we know that $\mathcal{F}_{(2^d;3,\frac{(2^d-1)(2^{d-1}-1)}{3})}$ is an edge-transitive homogeneous $(3, \frac{(2^d-1)(2^{d-1}-1)}{3})$-factorisation of order $2^d$. In particular,

$$\mathrm{AGL}(d, 2) \le \mathsf{Aut}\mathcal{F}_{(2^d;3,\frac{(2^d-1)(2^{d-1}-1)}{3})}.$$

By Lemmas 3.2 and 3.3, we conclude that

$$\mathsf{Aut}\mathcal{F}_{(2^d;3,\frac{(2^d-1)(2^{d-1}-1)}{3})} = \mathrm{AGL}(d, 2).$$

**Lemma 4.2.** *Let $\mathcal{E}$ be the set of $T(d,2)$-orbits on $(\mathbb{F}_2^d)^{\{3\}}$. Then* $\mathrm{AGL}(d,2)$ *induces a primitive permutation group on $\mathcal{E}$.*

*Proof.* Let $G = \mathrm{AGL}(d,2)$, $M = T(d,2)$ and $H = \mathrm{GL}(d,2)$. Then $G = M{:}H$, and $M$ lies in the kernel $G_{(\mathcal{E})}$ of $G$ acting on the $G$-invariant partition $\mathcal{E}$. Since $d \geq 3$, we know that $H \cong \mathrm{PSL}(d,q)$ is simple. It follows that $G_{(\mathcal{E})} = M$. Thus $G^{\mathcal{E}}$ is permutation isomorphic to $H^{\mathcal{E}}$, and $H^{\mathcal{E}} \cong HM/M \cong H$. Take $E \in \mathcal{E}$. Then there is a 2-dimensional subspace $U$ of $\mathbb{F}_2^d$ such that

$$E = \cup_{\mathbf{v}\in\mathbb{F}_2^d}(\mathbf{v}+U)^{\{3\}}.$$

Then $H_U \leq H_E$. It is well-known that $H = \mathrm{GL}(d,2)$ acts primitively on the set of 2-dimensional subspaces of $\mathbb{F}_2^d$. Then $H_U$ is a maximal subgroup of $H$ by Lemma 2.1. Since $M$ is intransitive on $(\mathbb{F}_2^d)^{\{3\}}$, we have $G \neq MG_e = G_E$, where $e \in E$. Noting that $G_E = G_E \cap (MH) = M(G_E \cap H) = MH_E$, we have $H_E \neq H$. Thus $H_E = H_U$ is maximal in $H$. Then $H^{\mathcal{E}}$ is primitive by Lemma 2.1, and hence $G^{\mathcal{E}}$ is primitive. □

Noting that $\mathbb{F}_{2^d}$ is a $d$-dimensional vector space over the field $\mathbb{F}_2$ of order 2, we may construct $\mathcal{F}_{(8;3,7)}$ and $\mathcal{F}_{(32;3,155)}$ alternatively as in the following two examples.

**Example 4.3.** Let $V = \mathbb{F}_8$, and set $\mathbb{F}_8^* = \langle\eta\rangle$. Then $V = \{0, \eta^i \mid 1 \leq i \leq 7\}$. It is easily shown that $\mathrm{AGL}(1,8)$ is regular on $V^{\{3\}}$. For $1 \leq i \leq 7$, take $e_i = \{0, \eta^{i-1}, \eta^i\} \in V^{\{3\}}$, and let $E_i = \{\{\xi, \eta^{i-1}+\xi, \eta^i+\xi\} \mid \xi \in \mathbb{F}_8\}$. Then $E_i = e_i^{T(1,8)}$, and $\{E_i \mid 1 \leq i \leq 7\}$ is a partition of $V^{\{3\}}$. Note that, identifying $\mathbb{F}_8$ with $\mathbb{F}_2^3$, the group $\mathrm{AGL}(1,8)$ is permutation isomorphic to a 3-homogeneous subgroup of $\mathrm{AGL}(3,2)$ with $T(1,8)$ corresponding to $T(3,2)$. It follows that $\{E_i \mid 1 \leq i \leq 7\}$ is isomorphic to $\mathcal{F}_{(8;3,7)}$.

**Example 4.4.** Let $V = \mathbb{F}_{32}$, and set $\mathbb{F}_{32}^* = \langle\eta\rangle$. Then $V = \{0, \eta^i \mid 1 \leq i \leq 31\}$. For $1 \leq i \leq 31$ and $1 \leq j \leq 5$, take $e_j^i = \{0, \eta^{(i-1)2^{j-1}}, \eta^{i2^{j-1}}\} \in V^{\{3\}}$, and let $E_j^i = \{\{\xi, \eta^{(i-1)2^{j-1}}+\xi, \eta^{i2^{j-1}}+\xi\} \mid \xi \in \mathbb{F}_{32}\}$. Set

$$\mathcal{F} = \{E_j^i \mid 1 \leq i \leq 31,\ 1 \leq j \leq 5\}.$$

It is easy to check that each $E_j^i$ is a $T(1,32)$-orbit containing $e_j^i$, $\mathsf{Aut}\mathcal{F} \geq \mathrm{A\Gamma L}(1,32)$ and $\mathcal{F}$ is an edge-transitive homogeneous $(3,155)$-factorisation of order 32. Note that, identifying $\mathbb{F}_{32}$ with $\mathbb{F}_2^5$, the group $\mathrm{A\Gamma L}(1,32)$ is permutation isomorphic to a 3-homogeneous subgroup of $\mathrm{AGL}(5,2)$ with $T(1,32)$ corresponding to $T(5,2)$. It follows that $\mathcal{F} \cong \mathcal{F}_{(32;3,155)}$.

In the following example, we construct two edge-transitive homogeneous factorisations of order 32 from $\mathcal{F}_{(32;3,155)}$.

**Example 4.5.** Let $V$ and $E_j^i$ be as Example 4.4.

(1) For $1 \leq j \leq 5$, let $E_j = \cup_{i=1}^{31} E_j^i$. Then each $E_j$ is one of the $\mathrm{AGL}(1,32)$-orbits on $V^{\{3\}}$, and $\mathrm{A\Gamma L}(1,32)$ is regular on $V^{\{3\}}$. Set

$$\mathcal{F}_{(32;3,5)} = \{E_j \mid 1 \leq j \leq 5\}.$$

Then $\mathcal{F}_{(32;3,5)}$ is an edge-transitive homogeneous $(3,5)$-factorisation of order 32.

(2) For $1 \leq i \leq 31$, let $E^i = \cup_{j=1}^{5} E_j^i$. Set

$$\mathcal{F}_{(32;3,31)} = \{E^i \mid 1 \leq i \leq 31\}.$$

It is easy to see that $E^1$ is a $(T(1,32){:}\langle\sigma\rangle)$-orbit, where $\sigma$ is the Frobenius automorphism of $\mathbb{F}_{32}$. By Construction 3.4, we conclude that $\mathcal{F}_{(32;3,31)}$ is an edge-transitive homogeneous $(3,31)$-factorisation of order 32.

**Lemma 4.6.** $\mathsf{Aut}\mathcal{F}_{(32;3,5)} = \mathsf{Aut}\mathcal{F}_{(32;3,31)} = \mathrm{A\Gamma L}(1,32)$.

*Proof.* Let $s \in \{5,31\}$. Then $\mathsf{Aut}\mathcal{F}_{(32;3,s)} \geq \mathrm{A\Gamma L}(1,32)$. Suppose that $\mathsf{Aut}\mathcal{F}_{(32;3,s)} \neq \mathrm{A\Gamma L}(1,32)$. Then, by Lemmas 3.2 and 3.3, we conclude that $\mathsf{Aut}\mathcal{F}_{(32;3,s)}$ is permutation isomorphic to $\mathrm{AGL}(5,2)$. Thus $\mathcal{F}_{(32;3,s)}$ is isomorphic to an edge-transitive homogeneous $(3,s)$-factorisation $\mathcal{F}'$ (of order 32) arising from the action of $\mathrm{AGL}(5,2)$ on the vector space $\mathbb{F}_2^5$. Note that $\mathrm{AGL}(5,2)$ has a unique proper normal subgroup, which is $T(5,2)$. Let $\mathcal{E}$ be the set of $T(5,2)$-orbits on $(\mathbb{F}_2^5)^{\{3\}}$. Then $\mathcal{E} = \mathcal{F}_{(32;3,155)}$, see Example 4.1. By Claim 3 and Lemma 4.2, we get $\mathcal{F}' = \mathcal{E} = \mathcal{F}_{(32;3,155)}$. Thus $\mathcal{F}_{(32;3,s)} \cong \mathcal{F}_{(32;3,155)}$, yielding $s = 155$, a contradiction. This completes the proof. □

4.2. **Factorisations arising from the projective line** $\mathrm{PG}(1,q)$**.** Let $q = p^f$, where $p$ is a prime and $f$ is a positive integer. For a nonzero vector $(\alpha,\beta) \in \mathbb{F}_q^2$, denote by $[\alpha,\beta]$ the 1-dimensional subspace spanned by $(\alpha,\beta)$. Then the projective line $\mathrm{PG}(1,q)$ over the field $\mathbb{F}_q$ can be identified with $\mathbb{F}_q \cup \{\infty\}$ by

$$[\xi,1] \mapsto \xi,\ [1,0] \mapsto \infty,\ \xi \in \mathbb{F}_q.$$

The group $\mathrm{PGL}(2,q)$ then consists of all fractional linear mappings of the form

$$t_{\alpha\beta\gamma\delta} : \xi \mapsto \frac{\alpha\xi+\beta}{\gamma\xi+\delta},\ \alpha,\beta,\gamma,\delta \in \mathbb{F}_q \text{ with } \alpha\delta-\beta\gamma \neq 0$$

acting sharply 3-transitively on $\mathbb{F}_q \cup \{\infty\}$, where $\frac{\alpha\infty+\beta}{\gamma\infty+\delta} = \alpha\gamma^{-1}$ for $\gamma \neq 0$, $\frac{\alpha\infty+\beta}{\delta} = \infty$ for $\alpha \neq 0$ and $\frac{\zeta}{0} = \infty$ for $\zeta \in \mathbb{F}_q^*$. Note that $t_{\alpha\beta\gamma\delta} = t_{\alpha'\beta'\gamma'\delta'}$ if and only if the vector $(\alpha',\beta',\gamma',\delta')$ is a nonzero multiple of $(\alpha,\beta,\gamma,\delta)$. Further,

$$\mathrm{PSL}(2,q) = \{t_{\alpha\beta\gamma\delta} \mid \alpha,\beta,\gamma,\delta \in \mathbb{F}_q \text{ with } \alpha\delta-\beta\gamma \text{ a nonzero square in } \mathbb{F}_q\}.$$

The Frobenius automorphism of $\mathbb{F}_q$ induces a permutation on $\mathrm{PG}(1,q)$ by $\sigma : \xi \mapsto \xi^p$ with $\infty^p = \infty$. Then $t_{\alpha\beta\gamma\delta}^{\sigma} = t_{\alpha^p\beta^p\gamma^p\delta^p}$, $\mathrm{P\Gamma L}(2,q) = \mathrm{PGL}(2,q){:}\langle\sigma\rangle$ and $\mathrm{P\Sigma L}(2,q) = \mathrm{PSL}(2,q){:}\langle\sigma\rangle$. (See [1, p.192] and [4, p.242] for example.)

Let $e = \{0,1,\infty\}$. Noting that $\mathrm{PGL}(2,q)$ is sharply 3-transitive, we have $\mathrm{PGL}(2,q)_e \cong \mathrm{S}_3$. Since $|\mathrm{PGL}(2,q) : \mathrm{PSL}(2,q)| \leq 2$, we know that $|\mathrm{PSL}(2,q)_e|$ is divisible by 3. Let $g \in \mathrm{PGL}(2,q)_e$ such that $1^g = 1$ and $0^g = \infty$. Then $g = t_{0\beta\beta 0}$ for $0 \neq \beta \in \mathbb{F}_q$, and so $g \in \mathrm{PSL}(2,q)_e$ if and only if $-\beta^2$ is a square in $\mathbb{F}_q$, i.e., either $q$ is even or $q \equiv 1 \,(\mathsf{mod}\ 4)$. Thus $\mathrm{PGL}(2,q)_e = \mathrm{PSL}(2,q)_e$ if and only if either $q$ is even or $q \equiv 1 \,(\mathsf{mod}\ 4)$.

**Example 4.7.** Let $V = \mathrm{PG}(1,q)$ with $q \equiv 1 \,(\mathsf{mod}\ 4)$. Then $\mathrm{PSL}(2,q)$ has exactly two orbits on $V^{\{3\}}$, and $\mathrm{PGL}(2,q) = \mathrm{PSL}(2,q) \cup \mathrm{PSL}(2,q)t_{\eta 001}$, where $\eta$ is a generator of the multiplicative group of $\mathbb{F}_q$. Set

$$E_1 = \{\{\frac{\beta}{\delta}, \frac{\alpha+\beta}{\gamma+\delta}, \frac{\alpha\eta+\beta}{\gamma\eta+\delta}\} \mid \alpha,\beta,\gamma,\delta \in \mathbb{F}_q,\ \alpha\delta-\beta\gamma = \eta^{2i-1},\ 1 \leq i \leq \frac{q-1}{2}\}$$

and

$$E_2 = \{\{\frac{\beta}{\delta}, \frac{\alpha+\beta}{\gamma+\delta}, \frac{\alpha\eta+\beta}{\gamma\eta+\delta}\} \mid \alpha,\beta,\gamma,\delta \in \mathbb{F}_q,\ \alpha\delta-\beta\gamma = \eta^{2i},\ 1 \leq i \leq \frac{q-1}{2}\}.$$

Then $E_1$ and $E_2$ are distinct $\mathrm{PSL}(2,q)$-orbits, and $E_1^{t_{\eta 001}} = E_2$. Moreover, since $\mathrm{PSL}(2,q)$ is normal in $\mathrm{P\Gamma L}(2,q)$, it is easily shown that $\{E_1, E_2\}$ is $\mathrm{P\Gamma L}(2,q)$-invariant. Thus $\mathcal{F}_{(q+1;3,2)} = \{E_1, E_2\}$ is an edge-transitive homogeneous $(3,2)$-factorisation of order $q+1$. Moreover, by Lemmas 3.2 and 3.3, we conclude that $\mathsf{Aut}\mathcal{F}_{(q+1;3,2)} = \mathrm{P\Gamma L}(2,q)$.

**Remark 4.8.** The factors of $\mathcal{F}_{(q+1;3,2)}$ constructed in Example 4.7 are complementary 3-hypergraphs admitting a 2-transitive group of automorphisms, which are essentially due to Taylor [26, Example 6.2]. Noting that $\mathsf{Aut}\mathcal{F}_{(q+1;3,2)}$ contains an element interchanging the parts of $\mathcal{F}_{(q+1;3,2)}$, those two 3-hypergraphs are self-complementary. Moreover, by [22, 27], a 3-hypergraph with 2-transitive automorphism group is self-complementary if and only if it is isomorphic to the factors of $\mathcal{F}_{(q+1;3,2)}$.

**Example 4.9.** Let $V = \mathrm{PG}(1,32)$. Then, by Lemma 3.2, $\mathrm{P\Gamma L}(2,32)$ is 4-homogeneous but not 4-transitive on $V$ (see also [1, 6.18, p.196]). Let $e = \{0,1,\eta,\eta^2\}$, where $\eta$ is a generator of the multiplicative group of $\mathbb{F}_{32}$. Then $\mathrm{P\Gamma L}(2,32)_e$ has order 4. Since $|\mathrm{P\Gamma L}(2,32) : \mathrm{PSL}(2,32)| = 5$, we have $\mathrm{P\Gamma L}(2,32)_e < \mathrm{PSL}(2,32)$. It follows that $\mathrm{PSL}(2,32)$ has 5 obits on $V^{\{4\}}$. Note that $\mathrm{P\Gamma L}(2,32) = \cup_{i=1}^{5}\mathrm{PSL}(2,32)\sigma^{i-1}$, where $\sigma$ is the Frobenius automorphism of the field $\mathbb{F}_{32}$. We may write those five orbits as follows:

$$E_i = \{\{\frac{\beta}{\delta}, \frac{\alpha+\beta}{\gamma+\delta}, \frac{\alpha\eta^{2^{i-1}}+\beta}{\gamma\eta^{2^{i-1}}+\delta}, \frac{\alpha\eta^{2^i}+\beta}{\gamma\eta^{2^i}+\delta}\} \mid \alpha,\beta,\gamma,\delta \in \mathbb{F}_{32}, \alpha\delta-\beta\gamma \neq 0\},\ 1 \leq i \leq 5.$$

Set

$$\mathcal{F}_{(33;4,5)} = \{E_i \mid 1 \leq 5 \leq i\}.$$

Then $\mathcal{F}_{(33;4,5)}$ is an edge-transitive homogeneous $(4,5)$-factorisation of order 33. By Lemmas 3.2 and 3.3, we conclude that $\mathsf{Aut}\mathcal{F}_{(33;4,5)} = \mathrm{P\Gamma L}(2,32)$.

## 5. The main result

Now we are ready to state and prove our main result.

**Theorem 5.1.** *Let $\mathcal{F}$ be an edge-transitive homogeneous $(k,s)$-factorisation of order $n$, where $s \geq 2$ and $6 \leq 2k \leq n$. Then $\mathcal{F} \cong \mathcal{F}_{(n;k,s)}$ with $n$, $k$, $s$ and $\mathsf{Aut}\mathcal{F}_{(n;k,s)}$ listed in Table 1 and defined in one of the examples in Section 4.*

| $n$ | $k$ | $s$ | Aut | Kernel | Condition | Reference |
|---|---|---|---|---|---|---|
| 32 | 3 | 5 | $\mathrm{A\Gamma L}(1,32)$ | $\mathrm{AGL}(1,32)$ | | Example 4.5 (1) |
| 32 | 3 | 31 | $\mathrm{A\Gamma L}(1,32)$ | $T(1,32)$ | | Example 4.5 (2) |
| 33 | 4 | 5 | $\mathrm{P\Gamma L}(2,32)$ | $\mathrm{PSL}(2,32)$ | | Example 4.9 |
| $2^d$ | 3 | $\frac{(2^d-1)(2^{d-1}-1)}{3}$ | $\mathrm{AGL}(d,2)$ | $T(d,2)$ | $d \geq 3$ | Example 4.1 |
| $q+1$ | 3 | 2 | $\mathrm{P\Gamma L}(2,q)$ | $\mathrm{P\Sigma L}(2,q)$ | $q \equiv 1 \pmod 4$ | Example 4.7 |

TABLE 1. Edge-transitive homogeneous factorisations

*Proof.* Assume that $\mathcal{F} = \{E_i \mid 1 \leq i \leq s\}$ is an edge-transitive homogeneous $(k,s)$-factorisation on $V$ of order $n$. Take $M \trianglelefteq G \leq \mathsf{Aut}\mathcal{F}$ such that $M$ fixes every $E_i$ set-wise, $G$ is transitive on $V^{\{k\}}$ and $M$ is transitive on $V$. Then, up to isomorphism of factorisations, we may let $G$ be one of the $k$-homogeneous permutation groups listed in Lemmas 3.2 and 3.3. Recall that $G$ has a unique minimal normal subgroup, which is transitive on $V$. We choose $M$ to be the minimal normal subgroup of $G$. Let $\mathcal{E}$ be the set of $M$-orbits on $V^{\{k\}}$. Then $\mathcal{E}$ is a refinement of $\mathcal{F}$. We next deal with all possible candidates of $G$ one by one.

Let $G$ be as in (I) of Lemma 3.2. Then $k = 3$, and we may choose $V = \mathbb{F}_8$ and $M = T(1, 8)$. Recall that every $E_i$ is the union of some $M$-orbits on $V^{\{3\}}$. Since $|V^{\{3\}}| = 56$ and $G$ contains a regular subgroup $\mathrm{AGL}(1, 8)$ (acting on $V^{\{3\}}$), the only possibility is that every $E_i$ has size 8 and is an $M$-orbit. Then, identifying $\mathbb{F}_8$ with $\mathbb{F}_2^3$, we have $\mathcal{F} \cong \mathcal{F}_{(8;3,7)}$, see Example 4.3. Thus line 4 of Table 1 occurs.

Let $G = \mathrm{A\Gamma L}(1, 32)$ be as in (II) of Lemma 3.2. Then $k = 3$, and we may choose $V = \mathbb{F}_{32}$ and $M = T(1, 32)$. If $\mathcal{F}$ is the set of $M$-orbits, that is, $\mathcal{F} = \mathcal{E}$, then $\mathcal{F} \cong \mathcal{F}_{(32;3,155)}$ by a similar argument as above (see also Example 4.4), and so line 4 of Table 1 occurs. Thus we assume that every $E \in \mathcal{F}$ consists of more than one $M$-orbits. Then $MG_e \leq G_E \neq MG_e$ for $e \in E$, see Claims 1-4. Since $32 \cdot 31 \cdot 5 = |\mathrm{A\Gamma L}(1, 32)| = |V^{\{3\}}|$, we know that $G$ is regular on $V^{\{3\}}$, and so $G_e = 1$. Checking the subgroups of $G = \mathrm{A\Gamma L}(1, 32)$, we conclude that either $G_E = \mathrm{AGL}(1, 32)$ or $G_E$ is conjugate to $T(1, 32){:}\langle\sigma\rangle$, where $\sigma$ is the Frobenius automorphism of $\mathbb{F}_{32}$. Thus $\mathcal{F}$ is isomorphic to one of $\mathcal{F}_{(32;3,5)}$ and $\mathcal{F}_{(32;3,31)}$, which are constructed in Example 4.5. By Lemma 4.6, one of the first two lines of Table 1 follows.

Let $G = \mathrm{P\Gamma L}(2, 32)$ be as in (III) of Lemma 3.2. Then $k = 4$, and we may choose $V = \mathrm{PG}(1, 32) = \mathbb{F}_{32} \cup \{\infty\}$ and $M = \mathrm{PSL}(2, 32)$. By the argument given in Example 4.9, we know that $M$ has 5-orbits on $V^{\{4\}}$. In particular, $G$ acts primitively on the set $\mathcal{E}$ of $M$-orbits. Then, by Claim 3, we have $\mathcal{F} = \mathcal{E}$. Then $\mathcal{F}$ is (isomorphic to) the factorisation $\mathcal{F}_{(33;4,5)}$ given in Example 4.9, and hence line 3 of Table 1 follows.

Let $G = \mathrm{AGL}(d, 2)$ be as in (IV) of Lemma 3.3. Then $k = 3$, $V = \mathbb{F}_2^d$, $M = T(d, 2)$ and $\mathcal{E} = \mathcal{F}_{(2^d;3,\frac{(2^d-1)(2^{d-1}-1)}{3})}$. By Lemma 4.2, $G^{\mathcal{E}}$ is primitive. Thus $\mathcal{F} = \mathcal{E}$ by Claim 3, and so line 4 of Table 1 follows.

Let $G = \mathbb{Z}_2^4{:}\mathrm{A}_7$ be as in (V) of Lemma 3.3. Then $\mathcal{E} = \mathcal{F}_{(16;3,35)}$. Arguing similarly as in the proof of Lemma 4.2, $G^{\mathcal{E}}$ is permutation isomorphic to a transitive subgroup $\mathrm{A}_7$ of $\mathrm{GL}(4, 2)$ acting on the 35 2-dimensional subspaces of $\mathbb{F}_2^4$. Checking the subgroups of $\mathrm{A}_7$ in the Atlas [3], we know that every subgroup of $\mathrm{A}_7$ with index 35 is maximal. It follows from Lemma 2.1 that $G^{\mathcal{E}}$ is primitive. Then $\mathcal{F} = \mathcal{E}$, and so line 4 of Table 1 occurs.

Finally, if $G$ is described as in (VI) of Lemma 3.3, then line 5 of Table 1 follows from the argument in Example 4.7. □

A $k$-hypergraph is said to be *symmetric* if it is both vertex-transitive and edge-transitive. Note that there is a bijection $(V, E) \mapsto (V, E^{op})$ between self-complementary $k$-hypergraphs and self-complementary $(n-k)$-hypergraphs of order $n$. The next result is a direct consequence of Theorem 5.1.

**Corollary 5.2.** *Let $k$ and $n$ be positive integers with $6 \leq 2k \leq n$. Then there exists a symmetric self-complementary $k$-hypergraph of order $n$ if and only if $k = 3$, $n - 1 \equiv 1 \pmod 4$ and $n - 1$ is a power of some odd prime.*

Let $\mathcal{H}$ be an edge-transitive self-complementary 3-hypergraph of order $n$. Then $n \geq 5$. Noting that the 5-cycle is a symmetric self-complementary 2-hypergraph, the next corollary follows.

**Corollary 5.3.** *There exists a symmetric self-complementary* 3*-hypergraph of order* $n \geq 5$ *if and only if either* $n = 5$*, or* $n-1 \equiv 1 \pmod 4$ *and* $n-1$ *is a prime power.*

We end this paper by the following remark on Theorem 5.1.

**Remark 5.4.** Let $n$, $k$ and $t$ be positive integers with $n > k \geq t$.

(1) A $k$-hypergraph $\mathcal{H} = (V, E)$ on $n$ vertices is *$t$-subset regular* if there is a constant $\lambda \geq 1$ such that each $t$-subset of $V$ is contained in exactly $\lambda$ edges. Let $\mathcal{F}$ be one of the factorisations in Theorem 5.1. Then the factors of $\mathcal{F}$ are $t$-subset regular $k$-hypergraphs with $t$ and $\lambda$ listed in Table 2. The reader is referred to [10, 13, 23, 24] for more examples and results on $t$-subset regular hypergraphs.

| $n$ | $k$ | $s$, $N$ | $t$ | $\lambda$ | Condition |
|---|---|---|---|---|---|
| 32 | 3 | 5 | 2 | 6 | |
| 32 | 3 | 31 | 1 | 15 | |
| 33 | 4 | 5 | 3 | 6 | |
| $2^d$ | 3 | $\frac{(2^d-1)(2^{d-1}-1)}{3}$ | 1 | 3 | $d \geq 3$ |
| $q+1$ | 3 | 2 | 2 | $\frac{q-1}{2}$ | $q \equiv 1 \pmod 4$ |

TABLE 2. The parameters $t$ and $\lambda$.

(2) Recall that a *large set* of $t$-$(n, k, \lambda)$ designs of size $N$, denoted by LS$[N](t, k, n)$, is a partition of the set of all $k$-subsets of an $n$-set into block sets of $N$ disjoint $t$-$(n, k, \lambda)$ designs, where $N\lambda = \binom{n-t}{k-t}$. Let $\mathcal{F}$ be one of the factorisations in Theorem 5.1. Note that a $t$-subset regular $k$-hypergraph is a $t$-$(n, k, \lambda)$ design, where $\lambda$ is number of edges containing a given $t$-subset. Then, using terminology from design theory, $\mathcal{F}$ is an LS$[N](t, k, n)$ in which all designs are flag-transitive and admit a common point-transitive group, where $N$, $t$, $k$ and $n$ are listed in Table 2.

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Huye Chen, Center for Combinatorics, LPMC-TJKLC, Nankai University, Tianjin 300071, China

*E-mail address*: `1120140003@mail.nankai.edu.cn`

Zaiping Lu, Center for Combinatorics, LPMC-TJKLC, Nankai University, Tianjin 300071, China

*E-mail address*: `lu@nankai.edu.cn`