On L-Borderenergetic Graphs with Maximum Degree at Most 4^*

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Abstract

If a graph G of order n has the same Laplacian energy as the complete graph K_n does, i.e., if $\mathcal{LE}(G) = 2(n-1)$, then G is said to be L-borderenergetic. In this paper, we first prove that there are no 2-connected L-borderenergetic graphs of order $n \geq 5$ with maximum degree $\Delta = 3$, which improves the result in [B. Deng, X. Li, J. Wang, Further results on L-Borderenergetic Graphs, MATCH Commun. Math. Comput. Chem., 77(2017)607–616]. Then by surveying the L-borderenergetic graphs with maximum degree $\Delta = 4$, we present two asymptotically tight bounds on their sizes.

1 Introduction

Let G be a simple graph of order n and size m and $\{d_1, d_2, \cdots, d_n\}$ be its degree sequence. Denote the maximum degree and average degree of G by Δ and $\overline{d}(= 2m/n)$, respectively. Let $Z_g(G) = \sum_{i=1}^{n} d_i^2$, called the first Zagreb index of G. Denote the complete graph of order n by K_n. The adjacency matrix of G is denoted by $A(G)$, whose eigenvalues are $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$, which consist of the spectrum of G. If $D(G)$ is the diagonal matrix of

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the vertex degrees of G, $L(G) = D(G) - A(G)$ is defined to be the Laplacian matrix of G. The Laplacian spectrum of G is composed of its eigenvalues $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_{n-1} \geq \mu_n = 0$. For details on spectral graph theory, see [3].

The energy [9] and the Laplacian energy [14] of a graph G, denoted by $\mathcal{E}(G)$ and $\mathcal{LE}(G)$, respectively, are defined as

$$\mathcal{E}(G) = \sum_{i=1}^{n} |\lambda_i|,$$

and

$$\mathcal{LE}(G) = \sum_{i=1}^{n} |\mu_i - \overline{d}|.$$

For more information on graph energy and its applications in chemistry, we can refer to [8, 10, 11, 17].

Recently, the concept of *borderenergetic* graphs [7] was proposed, namely graphs of order n satisfying $\mathcal{E}(G) = 2(n-1)$. The corresponding results on borderenergetic graphs can be seen in [4, 15, 21, 22, 24]. Similarly, some related topics on energy of graphs have been studied; see [1, 12, 13, 16, 18–20].

For the Laplacian energy of graphs, a similar concept as borderenergetic graphs, called L-borderenergetic graphs, was proposed by F. Tura [26]. That is, a graph G of order n is L-borderenergetic if $\mathcal{LE}(G) = \mathcal{LE}(K_n)$. Note that $\mathcal{LE}(K_n) = 2(n-1)$. More results on L-borderenergetic graphs, we can refer to [5, 6, 23, 26–28].

In [6], a main result is presented as follow. Let $t(G)$ be the number of vertices of degree 3 in G.

Theorem 1. *If G is a 2-connected graph with maximum degree $\Delta = 3$ and $t(G) \geq 7$, then G is not L-borderenergetic.*

In this paper, we obtain a better result, i.e. Theorem 2, which improves Theorem 1.

Theorem 2. *If G is a 2-connected graph of order $n \geq 5$ with maximum degree $\Delta = 3$, then G is not L-borderenergetic.*

When $n = 4$, it is easy to check that graph $K_4 - e$, i.e., the graph obtained by deleting an edge from K_4, is L-borderenergetic. Note that $K_4 - e$ is a 2-connected graph with maximum degree $\Delta = 3$.

On the other hand, we will focus on the L-borderenergetic graphs with maximum degree $\Delta = 4$. In chemical graph theory [2, 25], it is well known that, as carbon atoms are 4-valent, a chemical graph is the graph has no vertex of degree greater than 4. Using the Koolen-Moulton and the McClelland types of inequalities on the Laplacian energy, we present two asymptotically tight bounds on their sizes of the L-borderenergetic graphs with maximum degree $\Delta = 4$. These two types of inequalities below are given by Gutman and Zhou [14].

The Koolen-Moulton type of inequality on the Laplacian energy:

$$\mathcal{LE}(G) \leq \frac{2m}{n} + \sqrt{(n-1)[2M - (\frac{2m}{n})^2]}. \tag{1}$$

The McClelland type of inequality on the Laplacian energy:

$$\mathcal{LE}(G) \leq \sqrt{2Mn}, \tag{2}$$

where $M = m + \frac{1}{2}\sum_{i=1}^{n}(d_i - \frac{2m}{n})^2$.

2 Proof of Theorem 2

Proof. From the L-borderenergetic graphs with $4 \leq n \leq 9$ depicted in [5], we know that when $5 \leq n \leq 9$, there are no 2-connected L-borderenergetic graphs with maximum degree $\Delta = 3$. So the following discussion is under the condition $n \geq 10$.

For the case of $t(G) \geq 7$, the result follows by Theorem 1. Now we only need to discuss the case of $1 \leq t(G) \leq 6$. And we prove it by contradiction. Suppose G is L-borderenergetic. That is, $\mathcal{LE}(G) = \sum_{i=1}^{n} |\mu_i - \overline{d}| = 2(n-1)$. Then we have

$$(\sum_{i=1}^{n} |\mu_i - \overline{d}|)^2 = 4(n-1)^2. \tag{3}$$

From the left hand of above equation and the Cauchy-Schwarz inequality, we get

$$\begin{aligned}(\sum_{i=1}^{n} |\mu_i - \overline{d}|)^2 &\leq n\sum_{i=1}^{n}(\mu_i - \overline{d})^2 \\ &= n\sum_{i=1}^{n}(\mu_i^2 + \overline{d}^2 - 2\mu_i\overline{d}) \\ &= n(2m + \sum_{i=1}^{n} d_i^2 + n\overline{d}^2 - 4\overline{d}m) \end{aligned} \tag{4}$$

Since G has $t(G)$ vertices of degree 3 and $n - t(G)$ vertices of degree 2, we obtain

$$\overline{d} = \frac{3t(G) + 2(n - t(G))}{n}, \quad m = \frac{3t(G) + 2(n - t(G))}{2}.$$

When $t(G) = 1$, we get $\overline{d} = 2 + 1/n$, $m = n + 1/2$ and $\sum_{i=1}^{n} d_i^2 = 4n + 5$. Thus, by (3) and (4), we have

$$\begin{aligned} 4(n-1)^2 &= (\sum_{i=1}^{n} |\mu_i - \overline{d}|)^2 \\ &\le n[2n + 1 + 4n + 5 + n(2 + 1/n)^2 - 4(2 + 1/n)(n + 1/2)] \\ &= 2n^2 + 2n - 1, \end{aligned}$$

which is a contradiction as $n \ge 10$. With a similar way, we discuss the cases of $t = 2, 3, 4, 5, 6$.

When $t(G) = 2$, we get $\overline{d} = 2 + 2/n$, $m = n + 1$ and $\sum_{i=1}^{n} d_i^2 = 4n + 10$. By (3) and (4), we have $4(n-1)^2 \le 2(n^2 + 2n - 2)$.

When $t(G) = 3$, we get $\overline{d} = 2 + 3/n$, $m = n + 3/2$ and $\sum_{i=1}^{n} d_i^2 = 4n + 15$. By (3) and (4), we have $4(n-1)^2 \le 2n^2 + 6n - 9$.

When $t(G) = 4$, we get $\overline{d} = 2 + 4/n$, $m = n + 2$ and $\sum_{i=1}^{n} d_i^2 = 4n + 20$. By (3) and (4), we have $4(n-1)^2 \le 2(n^2 + 4n - 8)$.

When $t(G) = 5$, we get $\overline{d} = 2 + 5/n$, $m = n + 5/2$ and $\sum_{i=1}^{n} d_i^2 = 4n + 25$. By (3) and (4), we have $4(n-1)^2 \le 2n^2 + 10n - 25$.

When $t(G) = 6$, we get $\overline{d} = 2 + 6/n$, $m = n + 3$ and $\sum_{i=1}^{n} d_i^2 = 4n + 30$. By (3) and (4), we have $4(n-1)^2 \le 2(n^2 + 6n - 18)$.

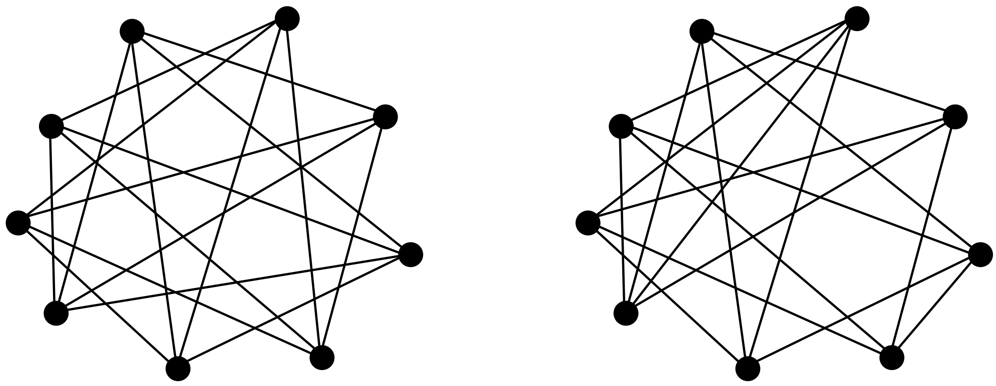
For above cases, it all makes contradictions as $n \ge 10$. Hence, we can see that G is not L-borderenergetic. $\square$

Indeed, when the maximum degree of a graph is 4, there exists 2-connected L-borderenergetic graphs. For example, G_1 and G_2 are two such graphs, see Figure 1. And their Laplacian spectra are given as follow.

$$LSp(G_1) = \{6, 6, 6, 5, 5, 3, 3, 2, 0\};$$

$$LSp(G_2) = \{6, 6, 6, 6, 3, 3, 3, 3, 0\}.$$

Moreover, we will survey the sizes of the L-borderenergetic graphs with maximum degree 4 in the next section.



G_1 G_2

Figure 1. Two 4-regular L-borderenergetic graphs G_1 and G_2 of order 9.

3 Bounds on the size of L-borderenergetic graphs with maximum degree 4

First, we use the Koolen-Moulton type of inequality on Laplacian energy to obtain Theorem 3.

Theorem 3. *If G is an L-borderenergetic graph with maximum degree $\Delta = 4$, then*

$$m \leq \frac{1}{16} Z_g(G) + \frac{5n}{4} - \frac{(n-3)^2}{4(n-1)} - 1. \quad (5)$$

When G is 4-regular, the bound in (5) is asymptotically tight.

Proof. Let $f(x) = \frac{2x}{n} + \sqrt{(n-1)[2(x + \frac{1}{2}\sum_{i=1}^{n}(d_i - \frac{2x}{n})^2) - (\frac{2x}{n})^2]}$. Then we see that the function $f(x)$ is increasing as $x \in [m, 2n]$. Due to $m \leq 2n$, we have $f(m) \leq f(2n)$. Hence, by (1), we have

$$\begin{aligned} \mathcal{LE}(G) &= 2(n-1) \\ &\leq \frac{2m}{n} + \sqrt{(n-1)[2(m + \frac{1}{2}\sum_{i=1}^{n}(d_i - \frac{2m}{n})^2) - (\frac{2m}{n})^2]} \\ &\leq 4 + \sqrt{(n-1)[4n + \sum_{i=1}^{n}(d_i - 4)^2 - 16]}. \end{aligned} \quad (6)$$

From above inequality, it arrives at

$$\begin{aligned}(2n-6)^2 &\leq (n-1)[4n+\sum_{i=1}^{n}(d_i-4)^2-16]\\ &= (n-1)(4n+\sum_{i=1}^{n}d_i^2+16n-16m-16)\\ &= (n-1)(20n+Z_g(G)-16m-16).\end{aligned}$$

By above inequality, it is easy to get

$$m \leq \frac{1}{16}Z_g(G)+\frac{5n}{4}-\frac{(n-3)^2}{4(n-1)}-1.$$

When G is 4-regular, we have $m=2n$ and $Z_g(G)=16n$. Then by above inequality, we get

$$m \leq \frac{9n}{4}-\frac{(n-3)^2}{4(n-1)}-1.$$

Since

$$\lim_{n\to\infty}\frac{\frac{9n}{4}-\frac{(n-3)^2}{4(n-1)}-1}{2n}=1,$$

the bound in (5) is asymptotically tight when G is 4-regular. $\square$

Next we use the McClelland type of inequality on Laplacian energy to obtain another result.

Theorem 4. *If G is an L-borderenergetic graph with maximum degree $\Delta=4$, then*

$$m \leq \frac{1}{16}Z_g(G)+\frac{5n}{4}-\frac{(n-1)^2}{4n}. \tag{7}$$

When G is 4-regular, the bound in (7) is asymptotically tight.

Proof. Let $g(x)=\sqrt{2(x+\frac{1}{2}\sum_{i=1}^{n}(d_i-\frac{2x}{n})^2)n}$. Then we see that the function $g(x)$ is increasing as $x\in[m,2n]$. Due to $m\leq 2n$, we have $g(m)\leq g(2n)$. Thus, by (2), we have

$$\begin{aligned}\mathcal{LE}(G) &= 2(n-1)\\ &\leq \sqrt{2(m+\frac{1}{2}\sum_{i=1}^{n}(d_i-\frac{2m}{n})^2)n}\\ &\leq \sqrt{4n^2+n\sum_{i=1}^{n}(d_i-4)^2}.\end{aligned} \tag{8}$$

By above inequality, we obtain

$$\begin{aligned} 4(n-1)^2 &\leq 4n^2 + n(\sum_{i=1}^{n} d_i^2 + 16n - 16m) \\ &= 4n^2 + nZ_g(G) + 16n^2 - 16mn \end{aligned}$$

Hence, it is easy to get

$$m \leq \frac{1}{16} Z_g(G) + \frac{5n}{4} - \frac{(n-1)^2}{4n}.$$

When G is 4-regular, we have $m = 2n$ and $Z_g(G) = 16n$. Then by above inequality, we get

$$m \leq \frac{9n}{4} - \frac{(n-1)^2}{4n}.$$

Since

$$\lim_{n\to\infty} \frac{\frac{9n}{4} - \frac{(n-1)^2}{4n}}{2n} = 1,$$

the bound in (7) is asymptotically tight when G is 4-regular. □

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