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The generalized 3-connectivity of Cayley graphs on symmetric groups generated by trees and cycles

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Abstract The generalized connectivity of a graph is a natural generalization of the connectivity and can serve for measuring the capability of a network G to connect any k vertices in G. Given a graph $G = (V, E)$ and a subset $S \subseteq V$ of at least two vertices, we denote by $\kappa_G(S)$ the maximum number r of edge-disjoint trees $T_1, T_2, \cdots, T_r$ in G such that $V(T_i) \cap V(T_j) = S$ for any pair of distinct integers i, j, where $1 \leq i, j \leq r$. For an integer k with $2 \leq k \leq n$, the *generalized k-connectivity* is defined as $\kappa_k(G) = \min\{\kappa_G(S)|S \subseteq V(G) \textit{ and } |S| = k\}$. That is, $\kappa_k(G)$ is the minimum value of $\kappa_G(S)$ over all k-subsets S of vertices.

The study of Cayley graphs has many applications in the field of design and analysis of interconnection networks. Let $Sym(n)$ be the group of all permutations on $\{1, \ldots, n\}$ and $\mathcal{T}$ be a set of transpositions of $Sym(n)$. Let $G(\mathcal{T})$ be the graph on n vertices $\{1, 2, \ldots, n\}$ such that there is an edge ij in $G(\mathcal{T})$ if and only if the transposition $[ij] \in \mathcal{T}$. If $G(\mathcal{T})$ is a tree, we use the notation $\mathbb{T}_n$ to denote the Cayley graph $Cay(Sym(n), \mathcal{T})$ on symmetric groups generated by $G(\mathcal{T})$. If $G(\mathcal{T})$ is a cycle, we use the notation MB_n to denote the Cayley graph $Cay(Sym(n), \mathcal{T})$ on symmetric groups generated by $G(\mathcal{T})$. In this paper, we investigate the generalized 3-connectivity of $\mathbb{T}_n$ and MB_n and show that $\kappa_3(\mathbb{T}_n) = n - 2$ and $\kappa_3(MB_n) = n - 1$.

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1 Introduction

The connectivity $\kappa(G)$ of a graph G is one of the basic concepts of graph theory: it asks for the minimum number of vertices that need to be removed to disconnect the remaining vertices from each other. A graph G is *k-connected* if $\kappa(G) \geq k$. An equivalent definition of connectivity was given in [19]. For each 2-subset $S = \{u, v\}$ of vertices of G, let $\kappa_G(S)$ denote the maximum number of internally vertex-disjoint paths from u to v in G. Then $\kappa(G)$ =min$\{\kappa_G(S)|S \subseteq V$ and $|S| = 2\}$.

The connectivity of a graph is an important measure of its robustness as a network. The generalized k-connectivity was introduced in [4,5] in order to measure the capability of a network G to connect any k vertices in G and not just any two.

Given a graph $G = (V, E)$ and a vertex subset S of size at least 2, an *S-Steiner tree* is a subgraph $T = (V', E')$ of G that is a tree with $S \subseteq V'$. Two S-Steiner trees T and T' are said to be *internally disjoint* if $E(T) \cap E(T') = \emptyset$ and $V(T) \cap V(T') = S$. For a vertex subset S of size at least 2, we denote by $\kappa_G(S)$ the maximum number of internally disjoint S-Steiner trees in G. For any integer $k \geq 2$, the *generalized k-connectivity* of G, denoted by $\kappa_k(G)$, is the minimum value of $\kappa_G(S)$ when S runs over all k-subsets of $V(G)$, i.e., $\kappa_k(G) = \min\{\kappa_G(S)|S \subseteq V(G) \textit{ and } |S| = k\}$. Clearly, $\kappa_2(G) = \kappa(G)$. The generalized k-connectivity has been studied, see [9–15] and a survey [16].

Due to the development of parallel and distributed computing, the design and analysis of various interconnection networks have been a main topic of research for the past decade. Interconnection networks are often modelled by graphs (or digraphs). The vertices of the graph represent the nodes of the network, that is, processing elements, memory modules or switches, and the edges correspond to communication lines. Because Cayley graphs have a lot of properties which are desirable in an interconnection network, such as vertex transitivity, edge transitivity, hierarchical structure, high fault tolerance etc., a number of researchers have proposed Cayley graphs as models for interconnection networks (see [6] for details).

Let X be a group and S be a subset of X. The *Cayley digraph* $Cay(X, S)$ is a digraph with vertex set X and arc set $\{(g, gs)|g \in X, s \in S\}$. Clearly, if $S = S^{-1}$, where $S^{-1} = \{s^{-1}|s \in S\}$, then $Cay(X, S)$ can be considered as an undirected graph.

Now, we consider Cayley graphs $Cay(X, S)$ when the group X is a permutation group. Denote by $Sym(n)$ the group of all permutations on $\{1, \ldots, n\}$. For convenience, we use $(p_1 p_2 \ldots p_n)$ to denote the permutation $\binom{1\ 2\ \cdots\ n}{p_1\ p_2\ \cdots\ p_n}$, and $[ij]$ to denote the permutation $\binom{1\ \cdots\ i\ \cdots\ j\ \cdots\ n}{1\ \cdots\ j\ \cdots\ i\ \cdots\ n}$, which is called a *transposition.* We define the composition $\sigma\pi$ of two permutations σ and π as the function that maps any element i to $\sigma(\pi(i))$. Thus $(p_1 \ldots p_i \ldots p_j \ldots p_n)[ij] =$

$(p_1 \ldots p_j \ldots p_i \ldots p_n)$, which swaps the objects at positions i and j (not swapping element i and j). Let $\mathcal{T}$ be a set of transpositions and $G(\mathcal{T})$ be the graph on n vertices $\{1, 2, \ldots, n\}$ such that there is an edge ij in $G(\mathcal{T})$ if and only if the transposition $[ij] \in \mathcal{T}$. The graph $G(\mathcal{T})$ is called the *transposition generating graph* of $Cay(Sym(n), \mathcal{T})$. It is well known that the Cayley graph $Cay(Sym(n), \mathcal{T})$ is connected if and only if the transposition generating graph $G(\mathcal{T})$ is connected (see [3]).

Moreover, if $G(\mathcal{T})$ is a tree, we call $G(\mathcal{T})$ a *transposition tree* and denote $Cay(Sym(n), \mathcal{T})$ by $\mathbb{T}_n$. Specially, if $G(\mathcal{T}) \cong K_{1,n-1}$, then $Cay(Sym(n), \mathcal{T})$ is called a *star graph* S_n; and $Cay(Sym(n), \mathcal{T})$ is called a *bubble-sort graph* B_n if $G(\mathcal{T}) \cong P_n$. If $G(\mathcal{T})$ is a unicyclic graph, $Cay(Sym(n), \mathcal{T})$ is denoted by UG_n. In particular, if $G(\mathcal{T}) \cong C_n$, UG_n is called a *modified bubble-sort graph* MB_n. Here, Cayley graphs generated by trees and cycles means that the transposition generating graphs of the Cayley graphs are trees and cycles.

Recently, Li et al. [17] investigated the generalized 3-connectivity of S_n and B_n, and showed that $\kappa_3(S_n) = n-2$ and $\kappa_3(B_n) = n-2$. In this paper, we further study the generalized 3-connectivity of $\mathbb{T}_n$ and obtain a more general result: $\kappa_3(\mathbb{T}_n) = n-2$. Moreover, we also study the generalized 3-connectivity of the modified bubble-sort graph MB_n, and show that $\kappa_3(MB_n) = n-1$. The results can be seen as a generalization of [2] and [20].

2 Preliminaries

We first introduce some notation and results about connectivity that will be used throughout the paper.

In this paper, we consider finite, undirected and simple graphs G with vertex set $V(G)$ and edge set $E(G)$. For $v \in V(G)$, let $N_G(v)$ denote the set of neighbors of v in G and $d_G(v)$ denote the *degree* of v in G. For a subset of vertices $U \subseteq V$, let $N(U) := (\cup_{u \in U} N(u)) \setminus U$, and the subgraph induced by U is denoted by $G[U]$. For simplicity, we sometimes use a graph itself to represent its vertex set; for instance, $N(G_1)$ means $N(V(G_1))$, where G_1 is a subgraph of G.

Lemma 1 ([15]) *Let G be a connected graph and δ be its minimum degree. Then $\kappa_3(G) \leq \delta$. Further, if there are two adjacent vertices of degree δ, then $\kappa_3(G) \leq \delta - 1$.*

Lemma 2 ([15]) *Let G be a connected graph with n vertices. If $\kappa(G) = 4k+r$, where k and r are two integers with $k \geq 0$ and $r \in \{0, 1, 2, 3\}$, then $\kappa_3(G) \geq 3k + \lceil \frac{r}{2} \rceil$. Moreover, the lower bound is sharp.*

Lemma 3 ([1]-p.214) *Let G be a k-connected graph, and let X and Y be two vertex subsets of size at least k. Then there exists a family of k pairwise disjoint (X, Y)-paths in G.*

Lemma 4 (The Fan Lemma [1]-p.214) *Let $G = (V, E)$ be a k-connected graph, x be a vertex of G, and $Y \subseteq V \setminus \{x\}$ be a set with at least k vertices.*

Then there exists a k-fan in G from x to Y, that is, there exists a family of k internally vertex-disjoint (x, Y)-paths whose terminal vertices are distinct in Y.

3 $\kappa_3(\mathbb{T}_n)$

We first determine $\kappa_3(\mathbb{T}_n)$. Recall that $\mathbb{T}_n = Cay(Sym(n), \mathcal{T})$ represents the Cayley graph generated by some transposition tree $G(\mathcal{T})$. Without loss of generality, we assume that $V(G(\mathcal{T})) = \{1, 2, \cdots, n\}$ and n is a leaf of $G(\mathcal{T})$ with $n-1$ being its only neighbor.

The Cayley graphs $\mathbb{T}_n$ are $(n-1)$-regular bipartite graphs and have $n!$ vertices; see [7] for the details. More useful properties are given below, which can be found in [2,17,18,20].

Lemma 5 ([2,20]) $\kappa(\mathbb{T}_n) = n - 1$.

Property 1 [18] For $\mathbb{T}_n$, the vertex set $V(\mathbb{T}_n)$ can be partitioned into n parts, say $V(\mathbb{T}^1_{n-1}), V(\mathbb{T}^2_{n-1}), \ldots, V(\mathbb{T}^n_{n-1})$, where $\mathbb{T}^i_{n-1}$ is an induced subgraph on vertex set $\{(p_1p_2 \ldots p_{n-1}i)|(p_1 \ldots p_{n-1})$ ranges over all permutations of $\{1, \ldots, n\}\backslash\{i\}\}$. Obviously, for each $1 \leq i \leq n$, $\mathbb{T}^i_{n-1}$ is isomorphic to $\mathbb{T}_{n-1}$. We let $\mathbb{T}_n = \mathbb{T}^1_{n-1} \oplus \mathbb{T}^2_{n-1} \oplus \ldots \oplus \mathbb{T}^n_{n-1}$.

Property 2 [2] Consider the Cayley graphs $\mathbb{T}_n$. Let n be a leaf of $G(\mathcal{T})$ with $n-1$ being its only neighbor. For any vertex u of $\mathbb{T}^i_{n-1}$, $u[(n-1)n]$, the unique neighbor of u outside of $\mathbb{T}^i_{n-1}$, is called the *out-neighbor* of u, written u'. We call the neighbors of u in $\mathbb{T}^i_{n-1}$ the *in-neighbors* of u. Any two distinct vertices of $\mathbb{T}^i_{n-1}$ have different out-neighbors. Hence, there are exactly $(n-2)!$ independent edges between $\mathbb{T}^i_{n-1}$ and $\mathbb{T}^j_{n-1}$ if $i \neq j$, that is, $|N(\mathbb{T}^i_{n-1}) \cap V(\mathbb{T}^j_{n-1})| = (n-2)!$ if $i \neq j$.

Lemma 6 *[17] For $\mathbb{T}_n$, let n be a leaf of $G(\mathcal{T})$ with $n-1$ being its only neighbor and $\mathbb{T}_n = \mathbb{T}^1_{n-1} \oplus \mathbb{T}^2_{n-1} \oplus \ldots \oplus \mathbb{T}^n_{n-1}$. For every $i \in \{1, 2, \ldots, n\}$, let $\overline{\mathbb{T}_i} := \mathbb{T}_n[V(\mathbb{T}_n) \setminus V(\mathbb{T}^i_{n-1})]$. If $n \geq 3$, then for every $i \in \{1, 2, \ldots, n\}$,*

$$\kappa(\overline{\mathbb{T}_i}) = n - 2.$$

Theorem 7 *[17] Let S_n be a star graph and B_n be a bubble-sort graph. Then $\kappa_3(S_n) = n - 2$ and $\kappa_3(B_n) = n - 2$.*

Now, we give the first main result.

Theorem 8 $\kappa_3(\mathbb{T}_n) = n - 2$*, for any integer $n \geq 3$.*

Proof. Since $\mathbb{T}_n$ is an $(n-1)$-regular graph, by Lemma 1, $\kappa_3(\mathbb{T}_n) \leq \delta - 1 = n-2$. Thus we just need to prove that $\kappa_3(\mathbb{T}_n) \geq n-2$. We prove by induction on n.

For $n = 3$, obviously $\mathbb{T}_n$ is connected, and hence $\kappa_3(\mathbb{T}_n) \geq 1 = n - 2$.

For $n = 4$, $\mathbb{T}_n$ is a star graph or a bubble-sort graph, so by Theorem 7, $\kappa_3(\mathbb{T}_n) = 2 = n - 2$.

Suppose the claim is true for all integers $4 \leq n' < n$. Now consider n. let n be a leaf of $G(\mathcal{T})$ with $n-1$ being its only neighbor and $\mathbb{T}_n = \mathbb{T}^1_{n-1} \oplus \mathbb{T}^2_{n-1} \oplus \ldots \oplus \mathbb{T}^n_{n-1}$. Let v_1, v_2 and v_3 be any three vertices of $\mathbb{T}_n$, and $H := \{v_1, v_2, v_3\}$.

We distinguish three cases:

Case 1: v_1, v_2 and v_3 belong to the same part $V(\mathbb{T}^i_{n-1})$.

Note that $\mathbb{T}^i_{n-1} \cong \mathbb{T}_{n-1}$. By the inductive hypothesis, $\kappa_3(\mathbb{T}^i_{n-1}) \geq n - 3$. That is to say, there are at least $n - 3$ internally disjoint trees connecting H in $\mathbb{T}^i_{n-1}$.

Let v'_1, v'_2 and v'_3 be the out-neighbors of v_1, v_2 and v_3, respectively. By Lemma 6, $\mathbb{T}_n[V(\mathbb{T}_n) \setminus V(\mathbb{T}^i_{n-1})]$ is connected, and hence contains a tree T connecting $\{v'_1, v'_2, v'_3\}$. The tree T' obtained by adding three pendant edges $v_1v'_1, v_2v'_2, v_3v'_3$ to T is a tree connecting H and $V(T') \cap V(\mathbb{T}^i_{n-1}) = H$.

Now, in this case there are at least $n-2$ internally disjoint trees connecting H in $\mathbb{T}_n$, and hence $\kappa_{\mathbb{T}_n}(H) \geq n - 2$.

Case 2: v_1, v_2 and v_3 belong to two parts.

Without loss of generality, suppose that $v_1, v_2 \in V(\mathbb{T}^1_{n-1})$ and $v_3 \in V(\mathbb{T}^2_{n-1})$. By Lemma 5, $\kappa(\mathbb{T}^1_{n-1}) = n - 2$, and hence there are $n - 2$ internally vertex-disjoint (v_1, v_2)-paths $P_1, P_2, \ldots, P_{n-2}$ in $\mathbb{T}^1_{n-1}$. Choose $n-2$ distinct vertices $x_1, x_2, \ldots, x_{n-2}$ from $P_1, P_2, \ldots, P_{n-2}$ such that $x_i \in V(P_i)$, for $1 \leq i \leq n-2$. Note that at most one of these paths, say P_1, has length 1; if so, we can choose $x_1 = v_1$. Let x'_i be the out-neighbor of x_i, for all $i \in \{1, \ldots, n-2\}$. By Property 2, any two distinct vertices of $\mathbb{T}^1_{n-1}$ have different out-neighbors. So $X' = \{x'_1, x'_2, \ldots, x'_{n-2}\}$ is a set of size $n - 2$.

By Lemma 6 and Lemma 4, $\kappa(\overline{\mathbb{T}_1}) = n - 2$ and there exist $n - 2$ internally disjoint (v_3, X')-paths $P'_1, P'_2, \ldots, P'_{n-2}$ in $\mathbb{T}_n[V(\mathbb{T}_n) \setminus V(\mathbb{T}^1_{n-1})]$ whose terminal vertices are distinct in X'. Note that if $v_3 \in X'$, then there is a (v_3, X')-path that contains exactly one vertex v_3. Now, $T_1 = P_1 \cup x_1x'_1 \cup P'_1, \ldots, T_{n-2} = P_{n-2} \cup x_{n-2}x'_{n-2} \cup P'_{n-2}$ are $n - 2$ internally disjoint trees connecting H, and hence $\kappa_{\mathbb{T}_n}(H) \geq n - 2$.

Case 3: v_1, v_2 and v_3 belong to three different parts, respectively.

Without loss of generality, suppose that $v_1 \in V(\mathbb{T}^1_{n-1}), v_2 \in V(\mathbb{T}^2_{n-1})$ and $v_3 \in V(\mathbb{T}^3_{n-1})$.

Let $G(\mathcal{T})$ be a rooted tree with root n. For $1 \leq i \leq n - 1$, the *level* of i, denoted by $l(i)$, is the length of the path from n to i in $G(\mathcal{T})$. We renumber the vertices of $G(\mathcal{T})$ such that if $l(i) > l(j)$, then $i < j$. Recall that $n - 1$ is the only vertex whose level is 1. For example, there is a rooted tree $G(\mathcal{T})$ with 6 vertices in Figure 1.

We denote by P_i the unique path from i to $n-1$ in the tree $G(\mathcal{T})$. Consider the path $P_1 = 1x_1x_2 \ldots x_t(n-1)$ and the vertex v_1. Then $1 < x_1 < x_2 < \ldots < x_t < (n - 1)$. Let $v_1 := (i_1i_2i_3i_4 \cdots i_{n-1}1)$. Then,

$$v_1[1x_1] = (i_1i_2i_3i_4 \cdots i_{n-1}1)[1x_1] = (i_{x_1}i_2 \cdots i_{x_1-1}\underline{i_1}i_{x_1+1} \cdots i_{n-1}1) := w_1,$$
$$v_1[1x_1][x_1x_2] = (i_{x_1}i_2 \cdots i_{x_1-1}i_{x_2}i_{x_1+1} \cdots i_{x_2-1}\underline{i_1}i_{x_2+1} \cdots i_{n-1}1) := w_2,$$
$$\vdots$$

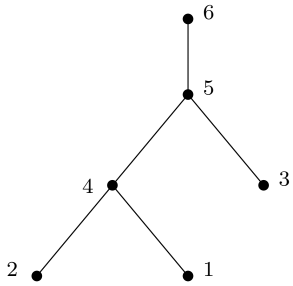

Fig. 1 An example of $G(\mathcal{T})$.

$$v_1[1x_1][x_1x_2]\cdots[x_tn-1] = (i_{x_1}i_2\cdots i_{x_1-1}i_{x_2}i_{x_1+1}\cdots i_{x_2-1}i_{x_3}i_{x_2+1}\cdots \underline{i_1}1) := w_t.$$

Thus we obtain a path $P_2^1 = v_1w_1w_2\cdots w_t$ in $\mathbb{T}_{n-1}^1$ starting at v_1 and ending at w_t. In the same way, according to P_2, P_3, $\cdots$, P_{n-1}, we can obtain another $n-2$ paths P_3^1, P_4^1, $\cdots$, P_n^1 in $\mathbb{T}_{n-1}^1$ starting at v_1. Note that P_n^1 contains only one vertex v_1.

Let $X^1 := \{w_i^1 | w_i^1$ is the terminal vertex of the path P_i^1 for $i \in \{2,\cdots,n\}\}$. Since $\{i_1,\cdots,i_{n-1}\} = \{2,\cdots,n\}$, it is easy to see that the out-neighbors of the vertices of X^1 are in $\mathbb{T}_{n-1}^2$, $\mathbb{T}_{n-1}^3$, $\ldots$, $\mathbb{T}_{n-1}^n$, respectively.

Fact 1: For every $k, l \in \{2,3,\ldots,n\}$ and $k \neq l$, $V(P_k^1) \cap V(P_l^1) = \{v_1\}$;
Proof. W.l.o.g., suppose that $k < l$. Consider the path $P_k := ky_1\cdots y_s(n-1)$ from k to $n-1$ in $G(\mathcal{T})$. As noted above, we have $k < y_1 < \cdots < y_s < (n-1)$. For every vertex $u \in V(P_k^1) \setminus \{v_1\}$, the element at position k of u is i_{y_1}. However, because $k < l$, the element at position k of any vertex $u' \in V(P_l^1)$ always is i_k. Thus, the fact indeed holds.

Now, in $\mathbb{T}_{n-1}^1$ there are $n-1$ internally vertex-disjoint paths $P_2^1, P_3^1, \cdots, P_n^1$ starting at v_1 and ending at w_2^1, w_3^1, $\cdots$, w_n^1 respectively. Furthermore, we can assume that the out-neighbor $(w_i^1)'$ of w_i^1 is in $\mathbb{T}_{n-1}^i$ for every $i \in \{2,3,\cdots,n\}$, otherwise we have to reorder these paths accordingly.

Similarly, in $\mathbb{T}_{n-1}^2$ there are $n-1$ internally vertex-disjoint paths P_1^2, P_3^2, $\cdots$, P_n^2 starting at v_2 and ending at w_1^2, w_3^2, $\cdots$, w_n^2 respectively. For every $i \in \{1,3,4,\cdots,n\}$, the terminal vertex of P_i^2 is w_i^2, and the out-neighbor $(w_i^2)'$ of w_i^2 is in $\mathbb{T}_{n-1}^i$.

In $\mathbb{T}_{n-1}^3$ there are $n-1$ internally vertex-disjoint paths P_1^3, P_2^3, P_4^3, $\cdots$, P_n^3 starting at v_3 and ending at w_1^3, w_2^3, w_4^3, $\cdots$, w_n^3 respectively. For every $i \in \{1,2,4,5,\cdots,n\}$, the terminal vertex of P_i^3 is w_i^3, and the out-neighbor $(w_i^3)'$ of w_i^3 is in $\mathbb{T}_{n-1}^i$.

Recall that $(w_1^3)'$ is the out-neighbor of w_1^3 and is in $\mathbb{T}_{n-1}^1$. There is a $((w_1^3)', v_1)$-path $\widetilde{P}$ in $\mathbb{T}_{n-1}^1$. Let t_1 be the first vertex of the path $\widetilde{P}$ which is in $\cup_{i\in\{2,\ldots,n\}}V(P_i^1)$.

Likewise, there is a $((w_2^3)', v_2)$-path $\widetilde{P'}$ in $\mathbb{T}_{n-1}^2$. Let t_2 be the first vertex of the path $\widetilde{P'}$ which is in $\cup_{i\in\{1,3,\ldots,n\}}V(P_i^2)$.

Now, we distinguish two subcases.

Subcase 3.1: $t_1 \in \cup_{i\in\{2,3\}} V(P_i^1)$ and $t_2 \in \cup_{i\in\{1,3\}} V(P_i^2)$.

In this subcase, the induced subgraph of $\mathbb{T}_n$ on $V(P_1^3) \cup V(\widetilde{P}[(w_1^3)', t_1]) \cup V(P_2^1) \cup V(P_3^1)$ contains a (v_3, v_1)-path, where $\widetilde{P}[(w_1^3)', t_1]$ is the subpath of the path $\widetilde{P}$ starting at $(w_1^3)'$ and ending at t_1.

Likewise, the induced subgraph of $\mathbb{T}_n$ on $V(P_2^3) \cup V(\widetilde{P'}[(w_2^3)', t_2]) \cup V(P_1^2) \cup V(P_3^2)$ contains a (v_3, v_2)-path, where $\widetilde{P'}[(w_2^3)', t_2]$ is the subpath of the path $\widetilde{P'}$ starting at $(w_2^3)'$ and ending at t_2.

The union of the (v_3, v_1)-path and (v_3, v_2)-path forms a tree connecting H.

On the other hand, for every $j \in \{4, \ldots, n\}$, there exists a tree connecting $V(P_j^1) \cup V(P_j^2) \cup V(P_j^3) \cup V(\mathbb{T}_{n-1}^j)$.

Hence in this subcase we conclude that there are $n-2$ internally disjoint trees connecting H in $\mathbb{T}_n$, that is, $\kappa_{\mathbb{T}_n}(H) \geq n-2$.

Subcase 3.2: $t_1 \in \cup_{i\in\{4,5,\ldots,n\}} V(P_i^1) \setminus \{v_1\}$ or $t_2 \in \cup_{i\in\{4,5,\ldots,n\}} V(P_i^2) \setminus \{v_2\}$.

W.l.o.g., suppose that $t_1 \in V(P_4^1) \setminus \{v_1\}$. Recall that $v_1 = (i_1 i_2 \ldots i_{n-1} 1)$. Since the out-neighbor of the terminal vertex w_4^1 of P_4^1 is in $\mathbb{T}_{n-1}^4$, $i_{n-1} \neq 4$ (otherwise, $V(P_4^1) = \{v_1\}$). Moreover, at least one of $i_{n-1} \neq 2$ and $i_{n-1} \neq 3$ must hold. W.l.o.g., we assume that $i_{n-1} \neq 2$.

Consider the path P_2^1. Recall that w_2^1 is the terminal vertex of P_2^1 and we can assume that $w_2^1 = (j_1 j_2 \cdots j_{n-2} 21)$. Suppose that $j_l = 4$ and $P_l = l x_1 x_2 \ldots x_t (n-1)$ is the path from l to $n-1$ in the tree $G(\mathcal{T})$. Then,

$$\begin{aligned}
w_2^1[lx_1] &= (j_1 \cdots j_{l-1} j_{x_1} j_{l+1} \cdots j_{x_1-1} \underline{4} j_{x_1+1} \cdots 21) := u_1,\\
w_2^1[lx_1][x_1x_2] &= (j_1 \cdots j_{l-1} j_{x_1} j_{l+1} \cdots j_{x_1-1} j_{x_2} j_{x_1+1} \cdots \underline{4} \cdots 21) := u_2,\\
&\vdots\\
w_2^1[lx_1][x_1x_2]\cdots[x_{t-1}x_t] &= (j_1 \cdots j_{l-1} j_{x_1} j_{l+1} \cdots j_{x_t-1} \underline{4} j_{x_t+1} \cdots 21) := u_t,\\
w_2^1[lx_1][x_1x_2]\cdots[x_t(n-1)] &= (j_1 \cdots j_{l-1} j_{x_1} j_{l+1} \cdots j_{x_t-1} 2 j_{x_t+1} \cdots \underline{4}1) := u_{t+1}.
\end{aligned}$$

Consider the vertex u_{t+1}. If $u_{t+1} = w_4^1$, where w_4^1 is the terminal vertex of the path P_4^1, then choose an edge kh of $G(\mathcal{T})$ such that $\{k, h\} \cap \{x_t, n-1\} = \emptyset$, and let $u_t' := u_t[kh]$, $u_{t+1}' = u_t'[x_t(n-1)]$. Now, $u_{t+1}' \neq w_4^1$.

If $u_{t+1} \neq w_4^1$, we denote by $\overline{P_2^1}$ a path $w_2^1 u_1 \cdots u_{t+1}$ starting at w_2^1 and ending at u_{t+1} in $\mathbb{T}_{n-1}^1$, otherwise, we denote by $\overline{P_2^1}$ a path $w_2^1 u_1 \cdots u_t u_t' u_{t+1}'$ starting at w_2^1 and ending at u_{t+1}' in $\mathbb{T}_{n-1}^1$.

Obviously, the terminal vertex of $\overline{P_2^1}$ is not w_4^1 and the out-neighbor of the terminal vertex of the path $\overline{P_2^1}$ is in $\mathbb{T}_{n-1}^4$. Let $\widehat{P_2^1} := P_2^1 w_2^1 \overline{P_2^1}$ be an extended path starting at v_1 and ending at u_{t+1} or u_{t+1}'. Next we prove the following fact.

Fact 2: $V(\widehat{P}_2^1) \cap V(P_i^1) = \{v_1\}$, for any $i \in \{3, 4, \ldots, n\}$.

Proof. Proof by contradiction. Suppose that there exists an integer $k \in \{3, 4, \ldots, n\}$ such that $|V(\widehat{P}_2^1) \cap V(P_k^1)| \geq 2$.

We assume that $w \in V(\widehat{P}_2^1) \cap V(P_k^1)$ and $w \neq v_1$. By Fact 1, $w \in V(\overline{P_2^1})$. If w is not the terminal vertex of $\overline{P_2^1}$, then the element at position $n-1$ of w is 2. However, the element at position $n-1$ of each vertex in $V(P_k^1)$ is i_{n-1}

or k, a contradiction. If w is the terminal vertex of $\overline{P_2^1}$, then the element at position $n-1$ of w is 4. Thus $i_{n-1}=4$ or $w=w_4^1$, a contradiction.

The proof of the claim is complete.

Similarly, if $t_2 \in V(P_l^2)$ for some $l \in \{4,5,\cdots,n\}$, we can extend the path P_1^2 or the path P_3^2 to obtain an extended path such that the out-neighbor of the terminal vertex of the extended path is in $\mathbb{T}_{n-1}^l$ and there is only one common vertex v_2 between the extended path and the other paths.

Now, if $V(\widetilde{P}[(w_1^3)', t_1]) \cap V(\overline{P_2^1}) = \emptyset$, then the induced subgraph of $\mathbb{T}_n$ on $V(P_1^3) \cup V(\widetilde{P}[(w_1^3)', t_1]) \cup V(P_4^1)$ contains a (v_3, v_1)-path, which is internally disjoint from $\widehat{P}_2^1$ and $P_5^1 \ldots, P_n^1$. Moreover, the out-neighbor of the terminal vertex of $\widehat{P}_2^1$ is in $\mathbb{T}_{n-1}^4$. Otherwise, $V(\widetilde{P}[(w_1^3)', t_1]) \cap V(\overline{P_2^1}) \neq \emptyset$ and let t_1' be the first vertex of the path $\widetilde{P}$ which is in $\cup_{i\in\{2,\ldots,n\}} V(P_i^1) \cup V(\overline{P_2^1})$. Then $t_1' \in V(\overline{P_2^1})$ and the induced subgraph of $\mathbb{T}_n$ on $V(P_1^3) \cup V(\widetilde{P}[(w_1^3)', t_1']) \cup V(\widehat{P}_2^1)$ contains a (v_3, v_1)-path, which is internally disjoint from $P_4^1 \ldots, P_n^1$. Similarly, we can obtain a (v_3, v_2)-path. The union of the (v_3, v_1)-path and (v_3, v_2)-path forms a tree connecting H. At the same time, for every $j \in \{4,5,\ldots,n\}$, there exists a tree connecting $H \cup V(\mathbb{T}_{n-1}^j)$. The most important thing is that we can guarantee that these $n-2$ trees connecting H are internally disjoint from the previous discussions.

In conclusion, $\kappa_{\mathbb{T}_n}(H) \geq n-2$, and hence $\kappa_3(\mathbb{T}_n) = n-2$.

The proof is complete. ∎

4 $\kappa_3(MB_n)$

In this section, we consider the modified bubble-sort graphs MB_n, where the transposition generating graph $G(\mathcal{T})$ is a cycle. W.l.o.g., we assume that $\mathcal{T} = \{12, 23, \ldots, (n-1)n, 1n\}$. It is easy to see that MB_n are n-regular graphs.

Property 3 [21] For MB_n, the vertex set $V(MB_n)$ can be partitioned into n parts, say $V(B_{n-1}^1)$, $V(B_{n-1}^2)$, …, $V(B_{n-1}^n)$, where B_{n-1}^i is an induced subgraph on vertex set $\{(p_1p_2\ldots p_{n-1}i) | (p_1\ldots p_{n-1})$ ranges over all permutations of $\{1,\ldots,n\}\backslash\{i\}\}$. Obviously, for each $1 \leq i \leq n$, B_{n-1}^i is isomorphic to B_{n-1}. We let $MB_n = B_{n-1}^1 \otimes B_{n-1}^2 \otimes \ldots \otimes B_{n-1}^n$.

Moreover, for any $i \in \{1,2,\ldots,n\}$, each vertex u of B_{n-1}^i has two neighbors $u' = u[1n]$ and $u'' = u[(n-1)n]$ outside of B_{n-1}^i. Vertices u' and u'' are called the *out-neighbors* of u. We call the neighbors of u in B_{n-1}^i the *in-neighbors* of u. Any vertex u of B_{n-1}^i has its two out-neighbors in different parts. Any two distinct vertices of B_{n-1}^i have different out-neighbors. There are exactly $2(n-2)!$ independent edges between B_{n-1}^i and B_{n-1}^j if $i \neq j$, that is, $|N(B_{n-1}^i) \cap V(B_{n-1}^j)| = 2(n-2)!$ if $i \neq j$.

First, we give some lemmas.

Lemma 9 *[17] Let $MB_n = B_{n-1}^1 \otimes B_{n-1}^2 \otimes \cdots \otimes B_{n-1}^n$. For every $i \in \{1,2,\cdots,n\}$, $\kappa(B_{n-1}^i) = n-2$. If $n \geq 3$, then*

$$\kappa(B_{n-1}^i \otimes B_{n-1}^j) \geq n-2,$$

for any two distinct integers $i, j \in \{1, \cdots, n\}$, where $B^i_{n-1} \otimes B^j_{n-1}$ is the induced subgraph of MB_n on $V(B^i_{n-1}) \cup V(B^j_{n-1})$.

Lemma 10 *Let $MB_n = B^1_{n-1} \otimes B^2_{n-1} \otimes \cdots \otimes B^n_{n-1}$ and $G' = B^{i_1}_{n-1} \otimes B^{i_2}_{n-1} \otimes \cdots \otimes B^{i_t}_{n-1}$ be the induced subgraph of MB_n on $V(B^{i_1}_{n-1}) \cup V(B^{i_2}_{n-1}) \cup \cdots \cup V(B^{i_t}_{n-1})$, where $1 \leq i_1 < i_2 < \cdots < i_t \leq n$ and $t \geq 2$. Given a vertex $x \in V(G')$, if $d_{G'}(x) = k$ and $Y \subseteq V(G') \setminus \{x\}$ is a set of k vertices of G' such that $|Y \cap B^{i_j}_{n-1}| \leq n-2$ for each $j \in \{1, 2, \ldots, t\}$, then there exists a k-fan in G' from x to Y, that is, there exists a family of k internally disjoint (x, Y)-paths whose terminal vertices are distinct in Y.*

Proof. Obviously, $n-2 \leq k \leq n$.

We distinguish three cases:

Case 1: $k = n-2$.

By Lemmas 9 and 4, $\kappa(B^{i_1}_{n-1} \otimes B^{i_2}_{n-1} \otimes \cdots \otimes B^{i_t}_{n-1}) \geq n-2$ and the conclusion clearly holds for $k = n-2$.

Case 2: $k = n$, that is, $V(G')$ contains the two out-neighbors of x .

W.l.o.g., suppose that $x \in B^{i_1}_{n-1}$ and the two out-neighbors x' and x'' of x belong to $B^{i_2}_{n-1}$ and $B^{i_3}_{n-1}$, respectively.

Now let $Y \cap B^{i_j}_{n-1} = A_j$ and $|A_j| = a_j$, for $1 \leq j \leq t$. Clearly $0 \leq a_j \leq n-2$ and $\sum_{j=1}^{t} a_j = n$.

Subcase 2.1: $a_2 \geq 1$ and $a_3 \geq 1$.

Let $a'_j := a_j - 1$ for $j = 2, 3$ and $a'_j := a_j$ for $j \neq 2, 3$. Then $\sum_{j=1}^{t} a'_j = n-2$.

Select $t-1$ disjoint vertex sets $M_2, M_3, \ldots, M_t$ in $V(B^{i_1}_{n-1})$ such that

(1) M_j consists of a'_j vertices,

(2) for each vertex in M_j, one of the two out-neighbors of it belongs to $B^{i_j}_{n-1}$,

(3) and $M_j \cap (A_1 \cup \{x\}) = \emptyset$, for each $j \in \{2, 3, \cdots, t\}$.

This can be done because $2(n-2)! \geq (n-1)$. Let $M_1 := A_1$ and $M := M_1 \cup M_2 \cup \cdots \cup M_t$. By Lemma 4 and the facts that $|M| = n-2$, $x \notin M$ and $\kappa(B^{i_1}_{n-1}) = n-2$, there exist t fans F_1, F_2, $\cdots$ F_t in $B^{i_1}_{n-1}$ from x to M, where for each $j \in \{1, \cdots, t\}$, F_j is a family of a'_j internally disjoint (x, M_j)-paths whose terminal vertices are distinct in M_j.

For $2 \leq j \leq t$, let $M'_j := \{y' \mid y'$ is an out-neighbor of y such that $y' \in B^{i_j}_{n-1}$ for each $y \in M_j\}$, and $E_j := \{yy' \in E(MB_n) \mid y \in M_j$ and $y' \in M'_j\}$. Then add x' and x'' to M'_2 and M'_3 respectively, that is, $M'_2 := M'_2 \cup \{x'\}$ and $M'_3 := M'_3 \cup \{x''\}$. Now $|M'_j| = a_j = |A_j|$, for each $j \in \{2, 3, \ldots, t\}$. By Lemma 3, for each $j \in \{2, 3, \cdots, t\}$, since $\kappa(B^{i_j}_{n-1}) = n-2 \geq a_j$, and M'_j, A_j are two subsets of $B^{i_j}_{n-1}$ of cardinality a_j, there exists a family of a_j pairwise disjoint (M'_j, A_j)-paths F'_j in $B^{i_j}_{n-1}$.

Finally, it is not hard to see that combining the t fans $F_1, \cdots, F_t$, the edge sets $E_2, \cdots, E_t$, the edges xx', xx'' and the sets of paths $F'_2, \cdots, F'_t$, we can obtain an n-fan in G' from x to Y.

Subcase 2.2: $a_2 = 0$ or $a_3 = 0$.

W.l.o.g, $a_2 = 0$.

If $a_2 = 0$ and $a_3 \geq 2$, then find a (x', w)-path P' in $B^{i_2}_{n-1}$ such that one of the two out-neighbors of w, denoted by w', is in $B^{i_3}_{n-1}$ and $w' \notin \{x''\} \cup M'_3$. Next, Let $a'_j := a_j - 2$ for $j = 3$ and $a'_j := a_j$ for $j \neq 3$. The proof is similar except that add $\{w', x''\}$ to M'_3 instead of adding x' and x'' to M'_2 and M'_3. Now M_2, M'_2 and E_2 are empty sets, F_2 and F'_2 do not exist.

Combining the $t-1$ fans F_1, F_3, $\cdots$, F_t, the edge sets E_3, $\cdots$, E_t, the edges xx', xx'', ww', the path P' and the sets of paths F'_3, $\cdots$, F'_t, we can obtain an n-fan in G' from x to Y.

If $a_2 = 0$ and $a_3 = 1$, since $a_1 \leq n-2$, there exists a part $V(B^{i_k}_{n-1})$ such that $a_k \geq 1$ and $k \neq 1, 3$. Find a (x', w)-path P' in $B^{i_2}_{n-1}$ such that one of the two out-neighbors of w, denoted by w', is in $B^{i_k}_{n-1}$ and $w' \notin M'_k$. Next, let $a'_j := a_j - 1$ for $j \in \{3, k\}$ and $a'_j := a_j$ for $j \neq 3, k$. The proof is similar to the previous proof and we can obtain an n-fan in G' from x to Y.

If $a_2 = a_3 = 0$, then there exists a part $V(B^{i_k}_{n-1})$ such that $a_k \geq 2$ and $k \in \{4, \cdots, t\}$, or there exist two parts $V(B^{i_h}_{n-1})$ and $V(B^{i_r}_{n-1})$ such that $a_h, a_r \geq 1$ and $h, r \in \{4, \cdots, t\}$. Similarly, we can obtain an n-fan in G' from x to Y.

Hence, for $k = n$ the conclusion holds.

Case 3: $k = n-1$, that is, $V(G')$ contains only one of the two out-neighbors of x .

This case can be handled similarly to Case 2 and more simply. So we omit the proof.

In conclusion, in any case, there always exists a k-fan in G' from x to Y.

The proof is complete. ∎

Lemma 11 *Let $G' = B^{i_1}_{n-1} \otimes B^{i_2}_{n-1} \otimes \ldots \otimes B^{i_t}_{n-1}$ be the induced subgraph of MB_n on $V(B^{i_1}_{n-1}) \cup V(B^{i_2}_{n-1}) \cup \ldots \cup V(B^{i_t}_{n-1})$, where $t \geq 2$. Then for any two vertices x and y of G', $\kappa_{G'}(x, y) = \min\{d_{G'}(x), d_{G'}(y)\}$, that is, there exist $\min\{d_{G'}(x), d_{G'}(y)\}$ internally vertex-disjoint (x, y)-paths in G'.*

Proof. W.l.o.g., assume that $\min\{d_{G'}(x), d_{G'}(y)\} = d_{G'}(x) = k$. Then $d_{G'}(y) \geq k$, $n-2 \leq k \leq n$ and $\kappa_{G'}(x, y) \leq k$.

If $d_{G'}(y) > k$ or x and y are not adjacent, then we can always find a subset Y of $N_{G'}(y)$ such that $|Y| = k$ and $x \notin Y$. Clearly, $|N_{G'}(y) \cap B^{i_j}_{n-1}| \leq n-2$ and $|Y \cap B^{i_j}_{n-1}| \leq n-2$, for each $j \in \{1, 2, \ldots, t\}$. By Lemma 10, there exists a k-fan in G' from x to Y. Combining the edges from y to Y, we can obtain k internally disjoint (x, y)-paths in G', that is, $\kappa_{G'}(x, y) \geq k$.

If $d_{G'}(y) = k$ and x and y are adjacent, let $Y := (N_{G'}(y) \cup \{y\}) \setminus \{x\}$. Then $|Y| = k$ and $x \notin Y$. If $|Y \cap B^{i_j}_{n-1}| \leq n-2$ for each $j \in \{1, 2, \ldots, t\}$, by Lemma 10, there exists a k-fan in G' from x to Y. Similarly, combining the edges from y to $Y \setminus \{y\}$, we can obtain k internally disjoint (x, y)-paths in G'. If $|Y \cap B^{i_j}_{n-1}| \geq n-1$ for some $j \in \{1, 2, \ldots, t\}$, then $B^{i_j}_{n-1}$ contains y and the $n-2$ in-neighbors of y and x is one of the two out-neighbors of y. Choose one in-neighbor z of y such that one out-neighbor z' of z belongs to $V(G')$.

Let $Y' := Y \backslash \{z\} \cup \{z'\}$. It is easy to see that $|Y' \cap B_{n-1}^{i_j}| \leq n-2$ for each $j \in \{1, 2, \ldots, t\}$. By Lemma 10, there exists a k-fan F in G' from x to Y'. Combining the fan F, the edges from y to Y and the edge zz', we can obtain k internally disjoint (x, y)-paths in G'.

Now we can conclude that $\kappa_{G'}(x, y) = k$. The proof is complete. ∎

The following result can be obtained immediately by letting $G' = MB_n$ in Lemma 11.

Lemma 12 $\kappa(MB_n) = n$.

Now, we give the generalized 3-connectivity of the modified bubble-sort graph MB_n.

Theorem 13 $\kappa_3(MB_n) = n - 1$, *for any integer* $n \geq 3$.

Proof. By Lemma 1, $\kappa_3(MB_n) \leq \delta(MB_n) - 1 = n - 1$. Thus we just need to prove that $\kappa_3(MB_n) \geq n - 1$.

For $n = 3, 4, 5$, by Lemmas 12 and 2, it is easy to check that $\kappa_3(MB_n) \geq n - 1$.

Now suppose that $n \geq 6$. Let $MB_n = B_{n-1}^1 \otimes B_{n-1}^2 \otimes \cdots \otimes B_{n-1}^n$. Let v_1, v_2 and v_3 be any three vertices of MB_n, and $H := \{v_1, v_2, v_3\}$.

We distinguish three cases:

Case 1: v_1, v_2 and v_3 belong to the same part.

W.l.o.g., assume that $v_1, v_2, v_3 \in V(B_{n-1}^1)$. By Theorem 8, $\kappa_3(B_{n-1}^1) = \kappa_3(B_{n-1}) = n - 3$, and hence there are at least $n - 3$ internally disjoint trees $T_1, T_2, \ldots, T_{n-3}$ connecting H in B_{n-1}^1. Note that by property 3, for every $1 \leq i \leq 3$, v_i has two out-neighbors v_i', v_i'' in different parts, and any two distinct vertices of B_{n-1}^1 have different out-neighbors. Hence $N = \{v_1', v_1'', v_2', v_2'', v_3', v_3''\}$ is a set of size 6 and each part contains at most three vertices of N.

Subcase 1.1: there exists a part which contains three vertices of N.

W.l.o.g., suppose that $v_1', v_2', v_3' \in V(B_{n-1}^2)$. Then there is a tree T_{n-2} connecting $H \cup \{v_1', v_2', v_3'\}$ in the induced subgraph on $V(B_{n-1}^2) \cup H$. On the other hand, there is a tree T_{n-1} connecting $H \cup \{v_1'', v_2'', v_3''\}$ in the induced subgraph on $(\cup_{i=3}^n V(B_{n-1}^i)) \cup H$.

Now, we obtain $n - 1$ internally disjoint H-Steiner trees $T_1, T_2, \cdots, T_{n-1}$ in MB_n.

Subcase 1.2: there exists a part which contains two vertices of N and all other parts contain at most two vertices of N.

W.l.o.g., suppose that $v_1', v_2' \in V(B_{n-1}^2)$.

If there is a part $V(B_{n-1}^k)(k \neq 1, 2)$ such that $V(B_{n-1}^k) \cap N = \{v_3'\}$ or $\{v_3''\}$ (w.l.o.g., $=\{v_3'\}$), then $MB_n[V(B_{n-1}^2) \cup V(B_{n-1}^k) \cup H]$ contains a tree T_{n-2} connecting $H \cup \{v_1', v_2', v_3'\}$, and $MB_n[(\cup_{i \in \{1, \cdots, n\} \text{ and } i \neq 1,2,k} V(B_{n-1}^i)) \cup H]$ contains a tree T_{n-1} connecting $H \cup \{v_1'', v_2'', v_3''\}$.

Otherwise, there must exist another two parts such that both of them contain two vertices of N. W.l.o.g., suppose that $v_3', v_1'' \in V(B_{n-1}^3)$ and $v_3'', v_2'' \in V(B_{n-1}^4)$. Since $v_3' = v_3[1n]$, one of the two out-neighbors of v_3' is $v_3 \in B_{n-1}^1$. let $x = v_3'[(n-1)n]$ be the other out-neighbor of v_3'.

If $x \notin V(B_{n-1}^4)$, then $MB_n[\cup_{i\in\{1,\cdots,n\} \text{ and } i\neq 1,3,4} V(B_{n-1}^i) \cup H \cup \{v_3'\}]$ contains an H-Steiner tree T_{n-2} connecting $H \cup \{v_1', v_2', v_3', x\}$. By Lemma 9, $\kappa(B_{n-1}^3 \otimes B_{n-1}^4) \geq n-2$ and $MB_n[(V(B_{n-1}^3) \cup V(B_{n-1}^4) \cup H) \setminus \{v_3'\}]$ contains an H-Steiner tree T_{n-1} connecting $H \cup \{v_1'', v_2'', v_3''\}$.

Otherwise, $x \in V(B_{n-1}^4)$. Let $y := v_3'[(n-2)(n-1)]$ be an in-neighbor of v_3'. Clearly, $y[1n]$, an out-neighbor of y, belongs to B_{n-1}^1. Let $z := y[(n-1)n]$ be the other out-neighbor of y. It is easy to see that $z \notin B_{n-1}^4$. Hence $MB_n[\cup_{i\in\{1,\cdots,n\} \text{ and } i\neq 1,3,4} V(B_{n-1}^i) \cup H \cup \{v_3', y\}]$ contains an H-Steiner tree T_{n-2} connecting $H \cup \{v_1', v_2', v_3', z, y\}$. On the other hand, $MB_n[(V(B_{n-1}^3) \cup V(B_{n-1}^4) \cup H) \setminus \{v_3', y\}]$ contains an H-Steiner tree T_{n-1} connecting $H \cup \{v_1'', v_2'', v_3''\}$.

Now, we can always obtain $n-1$ internally disjoint trees connecting H in MB_n.

Subcase 1.3: each part contains at most one vertex of N.

W.l.o.g., suppose that $B_{n-1}^2, B_{n-1}^3, B_{n-1}^4$ contain v_1', v_2', v_3', respectively, and $B_{n-1}^5, B_{n-1}^6, B_{n-1}^7$ contain v_1'', v_2'', v_3'', respectively. Then the induced subgraph of MB_n on $V(B_{n-1}^2) \cup V(B_{n-1}^3) \cup V(B_{n-1}^4) \cup H$ contains an H-Steiner tree T_{n-2} connecting $H \cup \{v_1', v_2', v_3'\}$, and the induced subgraph of MB_n on $V(B_{n-1}^5) \cup V(B_{n-1}^6) \cup V(B_{n-1}^7) \cup H$ contains an H-Steiner tree T_{n-1} connecting $H \cup \{v_1'', v_2'', v_3''\}$. Thus we obtain $n-1$ internally disjoint H-Steiner trees in MB_n.

Thus, in this case, $\kappa_{MB_n}(H) \geq n-1$.

Case 2: v_1, v_2 and v_3 belong to two parts.

W.l.o.g., assume that $v_1, v_2 \in V(B_{n-1}^1)$ and $v_3 \in V(B_{n-1}^2)$. By Lemma 5, $\kappa(B_{n-1}^1) = n-2$, and hence there are $n-2$ internally disjoint (v_1, v_2)-paths $P_1, P_2, \ldots, P_{n-2}$ in B_{n-1}^1. Let $G' := B_{n-1}^2 \otimes B_{n-1}^3 \otimes \ldots \otimes B_{n-1}^n$. Then, $v_3 \in V(G')$.

Subcase 2.1: neither of the two out-neighbours of v_3 belongs to B_{n-1}^1, that is, $d_{G'}(v_3) = n$.

Choose $n-2$ distinct vertices $x_1, x_2, \ldots, x_{n-2}$ from $P_1, P_2, \ldots, P_{n-2}$ such that $x_i \in V(P_i)$, for $1 \leq i \leq n-2$. Note that at most one of these paths, say P_1, has length 1. If so, we can let $x_1 := v_1$ and make sure that $x_i \neq v_2$ for any $i \in \{2, \cdots, n-2\}$. Let $Y := \{x_1, \cdots, x_{n-2}\} \cup \{v_1, v_2\}$. If $x_1 \neq v_1$, we let

$$Y' := \{x' | x' \text{ is an out-neighbor of } x \text{ and } x \in Y\};$$

otherwise,

$$Y' := \{x' | x' \text{ is an out-neighbor of } x \text{ and } x \in Y\} \cup \{v_1''\};$$

where v_1', v_1'' are the two out-neighbors of v_1. Clearly, $|Y| \geq n-1$, and $|Y'| = n$. On the other hand, we can make sure that $|Y' \cap B_{n-1}^j| \leq n-2$ for each $j \in \{2, 3 \ldots, n\}$, otherwise we choose the other out-neighbor of x for some $x \in Y$.

By Lemma 10 and the fact that $d_{G'}(v_3) = n$, there exist n internally disjoint (v_3, Y')-paths $R_1, R_2, \cdots, R_n$ in G' such that the terminal vertex of R_i is x_i' for each $i \in \{1, \ldots, n-2\}$, the terminal vertex of R_{n-1} is v_1' or v_1'', and the terminal vertex of R_n is v_2'. Now, $T_1 = P_1 \cup R_1 \cup x_1x_1'$, $\cdots$, $T_{n-2} = P_{n-2} \cup R_{n-2} \cup x_{n-2}x_{n-2}'$ and $T_{n-1} = R_{n-1} \cup R_n \cup \{v_2v_2'\} \cup \{v_1v_1'\}($ or $\{v_1v_1''\})$ are $n-1$ internally disjoint trees connecting H, and hence $\kappa_{MB_n}(H) \geq n-1$.

Subcase 2.2: one of the two out-neighbours of v_3 belongs to B_{n-1}^1, that is, $d_{G'}(v_3) = n-1$.

Assume that the out-neighbour v_3' of v_3 belongs to B_{n-1}^1. Since B_{n-1}^1 is connected, there is a (v_3', v_1)-path $\widetilde{P}$ in B_{n-1}^1. Let t be the first vertex of the path $\widetilde{P}$ which is in $\cup_{k\in\{1,2,\ldots,n-2\}}V(P_k)$. W.l.o.g., suppose that $t \in V(P_{n-2})$. Clearly, $P_{n-2} \cup \widetilde{P}[v_3', t] \cup \{v_3v_3'\}$ contains a tree connecting H, denoted by T_{n-1}.

Now, choose $n-3$ distinct vertices $x_1, x_2, \ldots, x_{n-3}$ from $P_1, P_2, \ldots, P_{n-3}$ such that $x_i \in V(P_i)$, for $1 \leq i \leq n-3$. Similarly, by Lemma 10 and the fact that $d_{G'}(v_3) = n-1$, there exist $n-1$ internally disjoint paths $R_1, R_2, \cdots, R_{n-1}$ starting at v_3 in G' such that the terminal vertex of R_i is x_i' for each $i \in \{1, \ldots, n-3\}$, the terminal vertex of R_{n-2} is v_1' or v_1'', and the terminal vertex of R_{n-1} is v_2'. Now, $T_1 = P_1 \cup R_1 \cup x_1x_1'$, $\cdots$, $T_{n-3} = P_{n-3} \cup R_{n-3} \cup x_{n-3}x_{n-3}'$, $T_{n-2} = R_{n-2} \cup R_{n-1} \cup \{v_2v_2'\} \cup \{v_1v_1'\}($ or $\{v_1v_1''\})$ and T_{n-1} are $n-1$ internally disjoint trees connecting H, and hence $\kappa_{MB_n}(H) \geq n-1$.

Thus, in this case, $\kappa_{MB_n}(H) \geq n-1$.

Case 3: v_1, v_2 and v_3 belong to different parts, respectively.

W.l.o.g., suppose that $v_1 \in V(B_{n-1}^1)$, $v_2 \in V(B_{n-1}^2)$ and $v_3 \in V(B_{n-1}^3)$.

Let $W := \{v_1', v_1'', v_2', v_2'', v_3', v_3''\}$, where v_i' and v_i'' are the two out-neighbors of v_i for $i \in \{1, 2, 3\}$. We distinguish two subcases.

Subcase 3.1: $W \subseteq V(B_{n-1}^1) \cup V(B_{n-1}^2) \cup V(B_{n-1}^3)$.

Let $\widehat{G} = B_{n-1}^1 \otimes B_{n-1}^2$. Since one of the two out-neighbors of v_1 belongs to B_{n-1}^2 and one of the two out-neighbors of v_2 belongs to B_{n-1}^1, $d_{\widehat{G}}(v_1) = n-1$ and $d_{\widehat{G}}(v_2) = n-1$. Therefore, by Lemma 11, we have $\kappa_{\widehat{G}}(v_1, v_2) = \min\{d_{\widehat{G}}(v_1), d_{\widehat{G}}(v_2)\} = n-1$. Hence, there are $n-1$ internally disjoint (v_1, v_2)-paths $P_1, P_2, \ldots, P_{n-1}$ in $\widehat{G}$.

Let v_3' be an out-neighbour of v_3. Then we have $v_3' \in V(\widehat{G})$. Since $\widehat{G}$ is connected, there is a (v_3', v_1)-path $\widetilde{P}$ in $\widehat{G}$. Let t be the first vertex of the path $\widetilde{P}$ which is in $\cup_{k\in\{1,2,\ldots,n-1\}}V(P_k)$. W.l.o.g., suppose that $t \in V(P_{n-1})$. Clearly, $P_{n-1} \cup \widetilde{P}[v_3', t] \cup \{v_3v_3'\}$ contains a tree connecting H, denoted by T_{n-1}.

Now, let x_i be a neighbour of v_1 such that $x_i \in V(P_i)$, for $1 \leq i \leq n-2$. Note that at most one of these vertices, say x_1, is an out-neighbour of v_1. If so, we can let $x_1 := v_1$. Then $\{x_1, x_2, \ldots, x_{n-2}\} \subseteq B_{n-1}^1$. Let $G' = B_{n-1}^3 \otimes B_{n-1}^4 \otimes \ldots \otimes B_{n-1}^n$ and let x_i' be one of the two out-neighbors of x_i such that $x_i' \in V(G')$ for all $i \in \{1, \ldots, n-2\}$. Clearly, $Y = \{x_1', x_2', \ldots, x_{n-2}'\}$ is a set of size $n-2$. Moreover, since $d_{G'}(v_3) = n-2$, by Lemma 10, there exist $n-2$ internally disjoint (v_3, Y)-paths $R_1, R_2, \cdots, R_{n-2}$ in G' such that the terminal vertex of R_i is x_i' for each $i \in \{1, \ldots, n-2\}$. Now, $T_1 = P_1 \cup R_1 \cup x_1x_1'$, $\cdots$, $T_{n-2} = P_{n-2} \cup R_{n-2} \cup x_{n-2}x_{n-2}'$ and T_{n-1} are $n-1$ internally disjoint trees connecting H, and hence $\kappa_{MB_n}(H) \geq n-1$.

Subcase 3.2: W.l.o.g., at least one out-neighbour of v_3 does not belong to $V(B^1_{n-1}) \cup V(B^2_{n-1})$.

Let $G' = B^3_{n-1} \otimes B^4_{n-1} \otimes \ldots \otimes B^n_{n-1}$. This subcase is similar to Case 2, since $d_{G'}(v_3) \geq n-1$.

Select $n-2$ vertices $x_1, x_2, \ldots, x_{n-2}$ in $B^1_{n-1} \setminus \{v_1\}$ such that for every vertex x_i $(1 \leq i \leq n-2)$, $x_i = ((i+2)\cdots 21)$, that is, the element at position 1 is $i+2$ and the element at position $n-1$ is 2. Further, we request that for any $i \in \{1, 2, \cdots, n-2\}$, x_i and v_2 have different out-neighbors, and v_2 is not the out-neighbor of x_i. This can be done because $(n-3)! \geq 2$ for $n \geq 6$. Let $S := \{x_1, x_2, \ldots, x_{n-2}\}$. Moreover, let $x'_i := x_i[(n-1)n]$ and $x''_i := x_i[1n]$ $(1 \leq i \leq n-2)$ and let $S' := \{x'_1, x'_2, \ldots, x'_{n-2}\}$. Obviously, $S' \subseteq V(B^2_{n-1})$. Since $\kappa(B^1_{n-1}) = \kappa(B^2_{n-1}) = n-2$, by Lemma 4, there exist $n-2$ internally disjoint (v_1, S)-paths $P_1, P_2, \cdots, P_{n-2}$ in B^1_{n-1} such that the terminal vertex of P_i is x_i, and there exist $n-2$ internally disjoint (v_2, S')-paths $P'_1, P'_2, \cdots, P'_{n-2}$ in B^2_{n-1} such that the terminal vertex of P'_i is x'_i, for every $i \in \{1, 2, \ldots, n-2\}$. Then, we can obtain $n-2$ internally disjoint (v_1, v_2)-paths in $B^1_{n-1} \otimes B^2_{n-1}$: $v_1P_1x_1x'_1P'_1v_2$, $v_1P_2x_2x'_2P'_2v_2$, $\ldots$, $v_1P_{n-2}x_{n-2}x'_{n-2}P'_{n-2}v_2$.

Now, let x''_{n-1} be one of the two out-neighbors of v_1 such that $x''_{n-1} \in V(G')$ and x''_n be one of the two out-neighbors of v_2 such that $x''_n \in V(G')$. Let $Y := \{x''_1, x''_2, \ldots, x''_{n-2}, x''_{n-1}, x''_n\}$. Obviously, $Y \subseteq V(G')$ and $|Y| = n$.

If neither of the two out-neighbour of v_3 belongs to $V(B^1_{n-1}) \cup V(B^2_{n-1})$, that is, $d_{G'}(v_3) = n$, the proof is similar to the proof of Subcase 2.1.

If one of the two out-neighbour of v_3 belongs to $V(B^1_{n-1}) \cup V(B^2_{n-1})$, that is, $d_{G'}(v_3) = n-1$, the proof is similar to the proof of Subcase 2.2.

In conclusion, $\kappa_{MB_n}(H) \geq n-1$. The proof is complete. ∎

5 Conclusion

The generalized k-connectivity is a natural generalization of the connectivity and can serve for measuring the capability of a network G to connect any k vertices in G. In this paper, we restrict our attention to two classes of Cayley graphs, the Cayley graphs generated by trees $\mathbb{T}_n$ and the modified bubble-sort graphs MB_n (i.e., the Cayley graphs generated by cycles). We investigate the generalized 3-connectivity of $\mathbb{T}_n$ and MB_n, and show that $\kappa_3(\mathbb{T}_n) = n-2$ and $\kappa_3(MB_n) = n-1$. For future work, it would be interesting to study the generalized connectivity of some other classes of Cayley graphs and some other network graphs.

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