On the Enumeration and Congruences for m-ary Partitions

Lisa Hui Sun[1] and Mingzhi Zhang[2]

Center for Combinatorics, LPMC
Nankai University, Tianjin 300071, P. R. China

[1]sunhui@nankai.edu.cn, [2]mingzhi@mail.nankai.edu.cn

Abstract. Let $m \geq 2$ be a fixed integer. Suppose that n is a positive integer such that $m^j \leq n < m^{j+1}$ for some integer $j \geq 0$. Denote $b_m(n)$ the number of m-ary partitions of n, where each part of the partition is a power of m. In this paper, we show that $b_m(n)$ can be represented as a j-fold summation by constructing a one-to-one correspondence between the m-ary partitions and a special class of integer sequences relying only on the base m representation of n. It directly reduces to Andrews, Fraenkel and Sellers' characterization of the values $b_m(mn)$ modulo m. Moreover, denote $c_m(n)$ the number of m-ary partitions of n without gaps, wherein if m^i is the largest part, then m^k for each $0 \leq k < i$ also appears as a part. We also obtain an enumeration formula for $c_m(n)$ which leads to an alternative representation for the congruences of $c_m(mn)$ modulo m due to Andrews, Fraenkel and Sellers.

Keywords: m-ary partition, base m representation, congruence

AMS Classification: 05A17, 11P83

1 Introduction

The arithmetic properties for partition functions have been extensively studied since the discoveries of Ramanujan [12]. In this paper, we are mainly concerned with the enumeration of m-ary partitions which leads to the congruence properties given by Andrews, Fraenkel and Sellers [3, 4].

Let $m \geq 2$ be a fixed integer. An m-ary partition of a positive integer n is a partition of n such that each part is a power of m. The number of m-ary partitions of n is denoted by $b_m(n)$. For example, there are five 3-ary partitions for $n = 10$:

$$9+1,\ 3+3+3+1,\ 3+3+1+1+1+1,\ 3+1+1+1+1+1+1+1,\ 1+1+1+1+1+1+1+1+1+1.$$

Thus, $b_3(10) = 5$. Denote an m-ary partition of n by a sequence $\lambda = (a_\ell, a_{\ell-1}, \ldots, a_0)$ such that

$$n = a_\ell m^\ell + a_{\ell-1} m^{\ell-1} + \cdots + a_0,$$

where $a_\ell > 0$ and $a_i \geq 0$ for $0 \leq i \leq \ell - 1$. Denote the set of all the m-ary partitions of n by $\mathcal{B}_m(n)$. It is known that the generating function of $b_m(n)$ is given by

$$B_m(q) = \sum_{n=0}^{\infty} b_m(n) q^n = \prod_{k=0}^{\infty} \frac{1}{1 - q^{m^k}}. \tag{1.1}$$

For the case $m = 2$, Churchhouse [6] conjectured the following congruences for the binary partition function $b_2(n)$:

$$\begin{aligned} b_2(2^{2k+2}n) &\equiv b_2(2^{2k}n) \pmod{2^{3k+2}}, \\ b_2(2^{2k+1}n) &\equiv b_2(2^{2k-1}n) \pmod{2^{3k}}, \end{aligned} \tag{1.2}$$

where $n, k \geq 1$. The conjecture was first proved by Rødseth [13] and further studied by Hirschhorn and Loxton [9]. Later, it was extended to m-ary partitions by Andrews [1], Gupta [8], and Rødseth and Sellers [14].

Throughout this paper, without specification, we set n to be a positive integer such that $m^j \leq n < m^{j+1}$ for some integer $j \geq 0$. Recall that the base m representation of n is the unique expression of n which can be written as follows

$$n = \alpha_j m^j + \alpha_{j-1} m^{j-1} + \cdots + \alpha_1 m + \alpha_0, \tag{1.3}$$

where $\alpha_j > 0$ and $0 \leq \alpha_i \leq m - 1$ for $0 \leq i \leq j - 1$. Denote the base m representation of n by

$$r_m(n) = (\alpha_j, \alpha_{j-1}, \ldots, \alpha_1, \alpha_0). \tag{1.4}$$

Based on the base m representation of n, Andrews, Fraenkel and Sellers [3, Theorem 1] provided the following modulo m characterization of $b_m(mn)$:

$$b_m(mn) \equiv \prod_{i=0}^{j} (\alpha_i + 1) \pmod{m}. \tag{1.5}$$

In this paper, by establishing a bijection between the set $\mathcal{B}_m(n)$ of m-ary partitions of n and the set of integer sequences given in the following theorem, we derive a j-fold summation formula for $b_m(n)$. It will directly lead to Andrews, Fraenkel and Sellers' congruence (1.5).

Theorem 1.1. *There is a one-to-one correspondence between the set $\mathcal{B}_m(n)$ of m-ary partitions of n and the following set of integer sequences*

$$\mathcal{S}_m(n) = \{(\beta_j, \beta_{j-1}, \ldots, \beta_1) \,|\, 0 \leq \beta_j \leq \alpha_j \text{ and } 0 \leq \beta_t \leq \alpha_t + m\beta_{t+1} \text{ for } 1 \leq t \leq j-1\}.$$

Based on the above bijection, we provide a combinatorial approach to derive the following j-fold summation formula for $b_m(n)$.

Theorem 1.2. *Let $r_m(n)=(\alpha_j, \alpha_{j-1}, \ldots, \alpha_1, \alpha_0)$ be the base m representation of n. We have*

$$b_m(n) = \sum_{k_j=0}^{\alpha_j} \sum_{k_{j-1}=0}^{\alpha_{j-1}+mk_j} \cdots \sum_{k_1=0}^{\alpha_1+mk_2} 1. \tag{1.6}$$

Obviously, $b_m(n) = 1$ when $j = 0$.

Notice that if $r_m(n)=(\alpha_j, \alpha_{j-1}, \cdots, \alpha_1, \alpha_0)$, then $r_m(mn)=(\alpha_j, \alpha_{j-1}, \cdots, \alpha_1, \alpha_0, 0)$. Thus the above theorem leads to that

$$b_m(mn) = \sum_{k_j=0}^{\alpha_j} \sum_{k_{j-1}=0}^{\alpha_{j-1}+mk_j} \cdots \sum_{k_1=0}^{\alpha_1+mk_2} \sum_{k_0=0}^{\alpha_0+mk_1} 1.$$

By taking modulo m on both sides of the above equation, it directly reduces to Andrews, Fraenkel and Sellers' congruence (1.5).

We also consider the cases for the m-ary partitions without gaps, wherein if m^i is the largest part, then m^k for each $0 \le k < i$ also appears as a part. The related works on such restricted m-ary partitions can be found in [2, 4, 11]. Moreover, in [5, 7, 10, 15], a general class of non-squashing partitions was introduced and studied, which contains m-ary partitions as a special case.

Let $c_m(n)$ denote the number of m-ary partitions without gaps of n. Based on the bijection given in Theorem 1.1, we also obtain the following enumeration formula for $c_m(n)$.

Theorem 1.3. *Let $r_m(n) = (\alpha_j, \alpha_{j-1}, \ldots, \alpha_1, \alpha_0)$ be the base m representation of n. We have*

$$c_m(n) = 1 + \sum_{r=1}^{j} \sum_{k_r=\chi_r}^{\lfloor \frac{n}{m^r} \rfloor -1} \cdots \sum_{k_1=\chi_1}^{\alpha_1-1+mk_2} 1, \tag{1.7}$$

where for $1 \le i \le r$,

$$\chi_i = \begin{cases} 0, & \text{if } \alpha_{i-1} > 0, \\ 1, & \text{if } \alpha_{i-1} = 0. \end{cases} \tag{1.8}$$

Applying formula (1.7), we obtain the following congruence property of $c_m(mn)$, which reveals the results given by Andrews, Fraenkel and Sellers [4, Theorem 2.1].

Theorem 1.4. *Let χ_i be defined by (1.8) for $1 \le i \le r$, then we have*

$$c_m(mn) \equiv \alpha_0 + (\alpha_0 - 1) \sum_{i=1}^{j} (\alpha_1 - \chi_1)(\alpha_2 - \chi_2) \cdots (\alpha_i - \chi_i) \pmod{m}. \tag{1.9}$$

2 The enumeration formula for $b_m(n)$

In this section, we provide a bijection between the set $\mathcal{B}_m(n)$ of m-ary partitions of n and the set

$$\mathcal{S}_m(n) = \{(\beta_j, \beta_{j-1}, \ldots, \beta_1) \,|\, 0 \le \beta_j \le \alpha_j \text{ and } 0 \le \beta_t \le \alpha_t + m\beta_{t+1} \text{ for } 1 \le t \le j-1\},$$

which relies only on the base m representation of n. It will lead to the enumeration formula (1.6) for the m-ary partitions.

To this end, we first define the following subtraction between the base m representation of n and an ordinary m-ary partition of n.

Definition 2.1. *Let $r_m(n) = (\alpha_j, \alpha_{j-1}, \ldots, \alpha_0)$ be the base m representation of n and $\lambda = (\lambda_\ell, \lambda_{\ell-1}, \ldots, \lambda_0)$ be an m-ary partition of n. Then subtracting λ from $r_m(n)$ is given as follows*

$$r_m(n) - \lambda = (\beta_j, \beta_{j-1}, \ldots, \beta_1), \tag{2.1}$$

where for $1 \le i \le j$,

$$\beta_i = \sum_{k=i}^{j} m^{k-i}(\alpha_k - \lambda_k)$$

provided that $\lambda_k = 0$ for $\ell < k \le j$.

We further show that the subtraction (2.1) defined above gives a bijection between $\mathcal{B}_m(n)$ and $\mathcal{S}_m(n)$.

Theorem 2.2. *Let $r_m(n) = (\alpha_j, \alpha_{j-1}, \ldots, \alpha_0)$ be the base m representation of n and $\lambda = (\lambda_\ell, \lambda_{\ell-1}, \ldots, \lambda_0)$ be an arbitrary m-ary partition of n. Define a map φ from $\mathcal{B}_m(n)$ to $\mathcal{S}_m(n)$ by $\varphi(\lambda) = r_m(n) - \lambda$. Then φ is a bijection between $\mathcal{B}_m(n)$ and $\mathcal{S}_m(n)$.*

Proof. Denote $\beta = \varphi(\lambda) = (\beta_j, \beta_{j-1}, \ldots, \beta_1)$. First, we proceed to show that $\beta \in \mathcal{S}_m(n)$ and thereby φ is well defined. Following Definition 2.1, it is easy to see that

$$\beta_j = \alpha_j - \lambda_j, \tag{2.2}$$

$$\beta_t = \alpha_t - \lambda_t + m\beta_{t+1}, \tag{2.3}$$

where $1 \le t \le j-1$ and $\lambda_k = 0$ for $\ell < k \le j$. Since $\lambda_k \ge 0$ for $0 \le k \le j$, we see that $\beta_j \le \alpha_j$ and $\beta_t \le \alpha_t + m\beta_{t+1}$ for $1 \le t \le j-1$.

It is obvious that $\lambda_j \le \alpha_j$, so that $\beta_j \ge 0$. From the fact that

$$\lambda_j m^j + \lambda_{j-1} m^{j-1} + \cdots + \lambda_0 = \alpha_j m^j + \alpha_{j-1} m^{j-1} + \cdots + \alpha_0,$$

we are led to that for $1 \le t \le j-1$,

$$\lambda_j m^j + \lambda_{j-1} m^{j-1} + \cdots + \lambda_t m^t \le \alpha_j m^j + \alpha_{j-1} m^{j-1} + \cdots + \alpha_0.$$

Hence we obtain that

$$\Big((\lambda_t-\alpha_t)+\sum_{k=1}^{j-t}(\lambda_{t+k}-\alpha_{t+k})m^k\Big)m^t \le \alpha_{t-1}m^{t-1}+\alpha_{t-2}m^{t-2}+\cdots+\alpha_0.$$

Since $(\alpha_j,\alpha_{j-1},\ldots,\alpha_0)$ is the base m representation of n, it is obvious that

$$\alpha_{t-1}m^{t-1}+\alpha_{t-2}m^{t-2}+\cdots+\alpha_0<m^t,$$

which implies that

$$\Big((\lambda_t-\alpha_t)+\sum_{k=1}^{j-t}(\lambda_{t+k}-\alpha_{t+k})m^k\Big)m^t<m^t.$$

Note that λ_k and α_k are all integers for $0\le k\le j$, it follows that

$$\lambda_t-\alpha_t+\sum_{k=1}^{j-t}(\lambda_{t+k}-\alpha_{t+k})m^k\le 0$$

and thereby

$$\lambda_t\le\alpha_t+\sum_{k=1}^{j-t}(\alpha_{t+k}-\lambda_{t+k})m^k=\alpha_t+m\beta_{t+1}.$$

By (2.3), it directly leads to that $\beta_t\ge 0$ for $1\le t\le j-1$. Thus $\beta\in\mathcal{S}_m(n)$ and φ is well defined.

To prove that φ is a bijection, it is sufficient to show that there exists the inverse map of φ. For a given $\beta=(\beta_j,\beta_{j-1},\ldots,\beta_1)\in\mathcal{S}_m(n)$, let $\varphi^{-1}(\beta)$ be given by computing

$$\lambda'=(\alpha_j-\beta_j,\ \alpha_{j-1}-\beta_{j-1}+m\beta_j,\ \ldots,\ \alpha_1-\beta_1+m\beta_2,\ \alpha_0+m\beta_1)$$

and then deleting the preceding zeros. From the definition of $\mathcal{S}_m(n)$, we see that each element of λ' is nonnegative. Furthermore, it is easy to see that

$$(\alpha_j-\beta_j)m^j+(\alpha_{j-1}-\beta_{j-1}+m\beta_j)m^{j-1}+\cdots+\alpha_0+m\beta_1=n,$$

which implies that $\varphi^{-1}(\beta)$ is an m-ary partition of n and thereby $\varphi^{-1}(\beta)\in\mathcal{B}_m(n)$. It completes the proof of the bijection. ∎

For example, let $m=4$ and $n=36$, then the base 4 representation of 36 is $r_4(36)=(2,1,0)$. The correspondence between all the 4-ary partitions of 36 and the integer sequences belonging to $\mathcal{S}_4(36)$ can be seen in Table 2.1.

The above theorem directly leads to that $b_m(n)=|\mathcal{B}_m(n)|=|\mathcal{S}_m(n)|$. By studying the recursive properties of the sequences in $\mathcal{S}_m(n)$, we obtain the j-fold summation formula (1.6) of $b_m(n)$. Now we give the detailed proof of Theorem 1.2.

Table 2.1: The correspondence between $\lambda \in \mathcal{B}_4(36)$ and $\beta \in \mathcal{S}_4(36)$

λ	$(2,1,0)$	$(2,0,4)$	$(1,5,0)$	$(1,4,4)$	$(1,3,8)$	$(1,2,12)$	$(1,1,16)$	$(1,0,20)$	$(0,9,0)$
β	$(0,0)$	$(0,1)$	$(1,0)$	$(1,1)$	$(1,2)$	$(1,3)$	$(1,4)$	$(1,5)$	$(2,0)$
λ	$(0,8,4)$	$(0,7,8)$	$(0,6,12)$	$(0,5,16)$	$(0,4,20)$	$(0,3,24)$	$(0,2,28)$	$(0,1,32)$	$(0,0,36)$
β	$(2,1)$	$(2,2)$	$(2,3)$	$(2,4)$	$(2,5)$	$(2,6)$	$(2,7)$	$(2,8)$	$(2,9)$

Proof of Theorem 1.2. Denote the summation on the right hand side of (1.6) by

$$f(\alpha_j, \alpha_{j-1}, \ldots, \alpha_1, \alpha_0) = \sum_{k_j=0}^{\alpha_j} \sum_{k_{j-1}=0}^{\alpha_{j-1}+mk_j} \cdots \sum_{k_2=0}^{\alpha_2+mk_3} \sum_{k_1=0}^{\alpha_1+mk_2} 1.$$

We prove the theorem by induction. When $j = 0$, $r_m(n) = (\alpha_0)$ with $0 \le \alpha_0 \le m-1$. It is obvious that $b_m(n) = f(\alpha_0) = 1$.

Suppose that (1.6) holds for $j = i-1$. When $j = i$, we have $r_m(n) = (\alpha_i, \alpha_{i-1}, \ldots, \alpha_1, \alpha_0)$. By Theorem 2.2, it implies that $b_m(n) = |\mathcal{S}_m(n)|$ where

$$\mathcal{S}_m(n) = \{(\beta_i, \beta_{i-1}, \ldots, \beta_1) \,|\, 0 \le \beta_i \le \alpha_i, 0 \le \beta_t \le \alpha_t + m\beta_{t+1}, 1 \le t \le i-1\}.$$

For a fixed β_i with $0 \le \beta_i \le \alpha_i$, let us consider the subset of $\mathcal{S}_m(n)$ with β_i being the first entry. By deleting β_i in these sequences, it is easy to see that this subset is bijective with the following set

$$\mathcal{S}_{\beta_i} = \left\{(\beta_{i-1}, \beta_{i-2}, \ldots, \beta_1) \,|\, 0 \le \beta_t \le \alpha_t + m\beta_{t+1}, 1 \le t \le i-1\right\},$$

and therefore

$$\mathcal{S}_m(n) = \bigcup_{\beta_i=0}^{\alpha_i} \mathcal{S}_{\beta_i}.$$

By induction, we obtain that the cardinality of the set $\mathcal{S}_{\beta_i}$ is

$$|\mathcal{S}_{\beta_i}| = f(\alpha_{i-1} + m\beta_i, \alpha_{i-1}, \ldots, \alpha_0) = \sum_{k_{i-1}=0}^{\alpha_{i-1}+m\beta_i} \cdots \sum_{k_2=0}^{\alpha_2+mk_3} \sum_{k_1=0}^{\alpha_1+mk_2} 1.$$

Then by summing the above equation for β_i from 0 to α_i, we obtain

$$|\mathcal{S}_m(n)| = f(\alpha_i, \alpha_{i-1}, \ldots, \alpha_0) = \sum_{k_i=0}^{\alpha_i} \sum_{k_{i-1}=0}^{\alpha_{i-1}+mk_i} \cdots \sum_{k_2=0}^{\alpha_2+mk_3} \sum_{k_1=0}^{\alpha_1+mk_2} 1,$$

which completes the proof. ∎

Note that the j-fold summation formula (1.6) also can be derived from the generating function (1.1) of $b_m(n)$. Moreover, by setting $M = \{m, m, \ldots\}$ in the summation given by Folsom, Homma, Ryu and Tong [7, Theorem 1.5], it reduces to another j-fold summation expression for $b_m(n)$.

3 The m-ary partitions without gaps

In this section, based on the bijection given in Theorem 2.2, we derive an enumeration formula for the m-ary partitions without gaps. Denote $c_m(n)$ the number of this restricted m-ary partitions of n. We also obtain an alternative expression for the congruence properties of $c_m(mn)$ given by Andrews, Fraenkel and Sellers [4, Theorem 2.1].

Recall that by using the base m representation of n in the following form

$$n = \sum_{i=\ell}^{\infty} \alpha_i m^i$$

where $1 \leq \alpha_\ell < m$ and $0 \leq \alpha_i < m$ for $i > \ell$, Andrews, Fraenkel and Sellers obtained the following result.

Theorem 3.1 (Andrews, Fraenkel and Sellers [4, Theorem 2.1]). (1) *If ℓ is even, then*

$$c_m(mn) \equiv \alpha_\ell + (\alpha_\ell - 1) \sum_{i=\ell+1}^{\infty} \alpha_{\ell+1} \cdots \alpha_i \pmod{m}. \tag{3.1}$$

(2) *If ℓ is odd, then*

$$c_m(mn) \equiv 1 - \alpha_\ell - (\alpha_\ell - 1) \sum_{i=\ell+1}^{\infty} \alpha_{\ell+1} \cdots \alpha_i \pmod{m}. \tag{3.2}$$

Denote the floor function of a real number a by $\lfloor a \rfloor$, which is the largest integer less than or equal to a. To derive our expression of the congruences (3.1) and (3.2), first let us show how to derive the enumeration formula (1.7) for $c_m(n)$ as given in Theorem 1.3.

Proof of Theorem 1.3. Denote the set of all the m-ary partitions without gaps of n by $\mathcal{G}_m(n)$. We claim that for any $\lambda \in \mathcal{G}_m(n)$, it can be written as

$$\lambda = \left(\left\lfloor \frac{n}{m^r} \right\rfloor - \beta_r,\, \alpha_{r-1} - \beta_{r-1} + m\beta_r,\, \ldots,\, \alpha_0 + m\beta_1 \right),$$

where $0 \leq r \leq j$ and β_i are integers such that

$$\left\lfloor \frac{n}{m^r} \right\rfloor - \beta_r > 0,\ \alpha_{r-1} - \beta_{r-1} + m\beta_r > 0,\ \ldots,\ \alpha_0 + m\beta_1 > 0. \tag{3.3}$$

Specially, when $r = 0$, there is a unique m-ary partition without gaps, say, $\lambda = (n)$ which consists of n ones. For $r > 1$, as an example, we consider $m = 4$ and $n = 73$, then $r_4(73) = (1, 0, 2, 1)$. When $r = 2$, we can obtain that $(3, 6, 1)$ is a 4-ary partition without gaps which can be represented as $\left(\left\lfloor \frac{73}{4^2} \right\rfloor - 1, 2 - 0 + 4 \times 1, 1 + 4 \times 0 \right)$.

Recall that

$$n = \alpha_j m^j + \alpha_{j-1} m^{j-1} + \cdots + \alpha_0.$$

For $1 \le r \le j$, it follows that

$$\begin{aligned}
&\Big(\Big\lfloor \frac{n}{m^r} \Big\rfloor - \beta_r \Big) m^r + (\alpha_{r-1} - \beta_{r-1} + m\beta_r) m^{r-1} + \cdots + (\alpha_1 - \beta_1 + m\beta_2) m + \alpha_0 + m\beta_1 \\
&\quad = \Big\lfloor \frac{n}{m^r} \Big\rfloor m^r + \alpha_{r-1} m^{r-1} + \cdots + \alpha_1 m + \alpha_0 \\
&\quad = (\alpha_j m^{j-r} + \cdots + \alpha_r) m^r + \alpha_{r-1} m^{r-1} + \cdots + \alpha_1 m + \alpha_0 \\
&\quad = \alpha_j m^j + \cdots + \alpha_r m^r + \alpha_{r-1} m^{r-1} + \cdots + \alpha_1 m + \alpha_0 \\
&\quad = n,
\end{aligned}$$

which certifies that $\lambda \in \mathcal{G}_m(n)$. For a fixed integer r such that $0 \le r \le j$, by the bijection given in Theorem 2.2, we get

$$\varphi(\lambda) = \Big(\alpha_j,\ \alpha_{j-1} + m\alpha_j,\ \ldots,\ \sum_{k=r+1}^{j} m^{k-r-1} \alpha_k,\ \beta_r,\ \beta_{r-1},\ \ldots,\ \beta_1 \Big).$$

Note that for any $\lambda \in \mathcal{G}_m(n)$ with the given r, the first $j - r$ elements in $\varphi(\lambda)$ are the same, which only depend on $\alpha_{r+1}, \ldots, \alpha_j$. Then by deleting these terms, we find that the set of m-ary partitions without gaps $\mathcal{G}_m(n)$ is in one-to-one correspondence with the following set of integer sequences

$$\mathcal{R}_m(n) = \bigcup_{r=0}^{j} \Big\{ (\beta_r, \ldots, \beta_1) \ \Big| \ \Big\lfloor \frac{n}{m^r} \Big\rfloor - \beta_r > 0,\ \alpha_{r-1} - \beta_{r-1} + m\beta_r > 0,\ \ldots,\ \alpha_0 + m\beta_1 > 0 \Big\}.$$

It indicates that $c_m(n) = |\mathcal{R}_m(n)|$.

From the conditions (3.3), we see that if $\alpha_0 = 0$, then $\beta_1 > 0$, which means β_1 starts from 1. If $\alpha_0 > 0$, then $\beta_1 \ge 0$, which means β_1 starts from 0. For both cases, we denote β_1 starting from χ_1, which is defined by (1.8). By $\alpha_1 - \beta_1 + m\beta_2 > 0$ we have $\beta_1 < \alpha_1 + m\beta_2$. Thereby we see that β_1 ranges from χ_1 to $\alpha_1 + m\beta_2 - 1$. By similar arguments applying to β_i for $2 \le i \le r$, we have

$$c_m(n) = |\mathcal{R}_m(n)| = 1 + \sum_{r=1}^{j} \sum_{k_r = \chi_r}^{\lfloor \frac{n}{m^r} \rfloor - 1} \cdots \sum_{k_1 = \chi_1}^{\alpha_1 - 1 + mk_2} 1,$$

where for $1 \le i \le j$,

$$\chi_i = \begin{cases} 0, & \text{if } \alpha_{i-1} > 0, \\ 1, & \text{if } \alpha_{i-1} = 0. \end{cases}$$

This completes the proof. ∎

Noting that $r_m(mn) = (\alpha_j, \alpha_{j-1}, \ldots, \alpha_0, 0)$, then by applying the above result we have

$$c_m(mn) = 1 + \sum_{r=1}^{j+1} \sum_{k_r = \chi_{r-1}}^{\lfloor \frac{mn}{m^r} \rfloor - 1} \cdots \sum_{k_2 = \chi_1}^{\alpha_1 - 1 + mk_3} \sum_{k_1 = 1}^{\alpha_0 - 1 + mk_2} 1.$$

By taking modulo m on both sides of the above identity, we directly obtain the congruence property (1.9), namely,

$$\begin{aligned}c_m(mn) &\equiv 1+\sum_{r=1}^{j+1}(\alpha_0-1)(\alpha_1-\chi_1)(\alpha_2-\chi_2)\cdots(\alpha_{r-1}-\chi_{r-1}) \pmod m\\ &\equiv 1+(\alpha_0-1)+(\alpha_0-1)\sum_{r=2}^{j+1}(\alpha_1-\chi_1)(\alpha_2-\chi_2)\cdots(\alpha_{r-1}-\chi_{r-1}) \pmod m\\ &\equiv \alpha_0+(\alpha_0-1)\sum_{i=1}^{j}(\alpha_1-\chi_1)(\alpha_2-\chi_2)\cdots(\alpha_i-\chi_i) \pmod m. \qquad (3.4)\end{aligned}$$

As an example, let $m=5$ and $n=485=3\cdot 5^3+4\cdot 5^2+2\cdot 5$. Then $j=3$ and the base 5 representation of 485 is $r_5(485)=(\alpha_3,\alpha_2,\alpha_1,\alpha_0)=(3,4,2,0)$. Therefore

$$\chi_3=\chi_2=0,\ \chi_1=1,$$

and

$$\begin{aligned}c_5(5\cdot 485) &\equiv -\big((\alpha_1-1)+(\alpha_1-1)\alpha_2+(\alpha_1-1)\alpha_2\alpha_3\big) \pmod 5\\ &= -(1+1\cdot 4+1\cdot 4\cdot 3)\\ &= -17\equiv 3 \pmod 5.\end{aligned}$$

In fact, we have $c_5(5\cdot 485)=230358\equiv 3 \pmod 5$, which coincides with the above result.

To conclude this paper, we remark that the congruence (3.4) for $c_m(mn)$ is equivalent to Theorem 3.1 due to Andrews, Fraenkel and Sellers [4].

Proof of Theorem 3.1. Let $r_m(n)=(\alpha_j,\alpha_{j-1},\ldots,\alpha_1,\alpha_0)$ be the base m representation of n.

Following Lemma 2.9 of [4], we see that $c_m(m^3n)\equiv c_m(mn) \pmod m$ for all $n\geq 0$. Thereby to prove Theorem 3.1, it is sufficient to show the cases that $\ell=0$ and $\ell=1$, which correspond to $\alpha_0>0$ and $\alpha_0=0$ ($\alpha_1>0$), respectively.

If $\alpha_0>0$, then $\chi_1=0$. It leads to that

$$c_m(mn)\equiv \alpha_0+(\alpha_0-1)\sum_{i=1}^{j}\alpha_1(\alpha_2-\chi_2)\cdots(\alpha_i-\chi_i) \pmod m. \qquad (3.5)$$

We further consider the values of $\alpha_1,\alpha_2,\cdots,\alpha_j$. If $\alpha_i>0$ for $i\geq 1$, then $\chi_i=0$ for $2\leq i\leq j$. Thus (3.5) turns to be

$$c_m(mn)\equiv \alpha_0+(\alpha_0-1)\sum_{i=1}^{j}\alpha_1\cdots\alpha_i\equiv \alpha_0+(\alpha_0-1)\sum_{i=1}^{\infty}\alpha_1\cdots\alpha_i \pmod m,$$

where $\alpha_i = 0$ for $i > j$. Otherwise, suppose that α_k $(1 \le k \le j)$ is the first zero in the sequence $\alpha_1, \alpha_2, \cdots$, then $\chi_i = 0$ for $1 \le i \le k$. Noting that $\alpha_k = 0$, we obtain

$$\sum_{i=1}^{j} \alpha_1(\alpha_2 - \chi_2) \cdots (\alpha_i - \chi_i) = \sum_{i=1}^{k-1} \alpha_1\alpha_2 \cdots \alpha_i = \sum_{i=1}^{\infty} \alpha_1\alpha_2 \cdots \alpha_i,$$

where $\alpha_i = 0$ for $i > j$. Therefore, (3.5) leads to that

$$c_m(mn) \equiv \alpha_0 + (\alpha_0 - 1) \sum_{i=1}^{\infty} \alpha_1 \cdots \alpha_i \pmod{m},$$

and both cases coincide with (3.1) with $\ell = 0$.

If $\alpha_0 = 0$ and $\alpha_1 > 0$, following the same procedure, we obtain that

$$\begin{aligned} c_m(mn) &\equiv (-1) \sum_{i=1}^{j} (\alpha_1 - 1)\alpha_2(\alpha_3 - \chi_3) \cdots (\alpha_i - \chi_i) \pmod{m} \\ &\equiv 1 - \alpha_1 - (\alpha_1 - 1) \sum_{i=2}^{j} \alpha_2(\alpha_3 - \chi_3) \cdots (\alpha_i - \chi_i) \pmod{m} \\ &\equiv 1 - \alpha_1 - (\alpha_1 - 1) \sum_{i=2}^{\infty} \alpha_2 \cdots \alpha_i \pmod{m}, \end{aligned}$$

which coincides with (3.2) with $\ell = 1$. This completes the proof. ∎

Acknowledgments.

This work was supported by the National Science Foundation of China, and the Natural Science Foundation of Tianjin, China.

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