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ON THE INDEX R FREE SEQUENCES OVER FINITE CYCLIC GROUPS

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Abstract. Let C_n be a finite cyclic group of order $n \geq 2$. Every sequence S over C_n can be written in the form $S = (n_1 g), \dots, (n_l g)$ where $g \in C_n$ and $n_1, \dots, n_l \in [1, \operatorname{ord}(g)]$, and the index $\operatorname{ind}(S)$ of S is defined as the minimum of $(n_1 + \dots + n_l)/\operatorname{ord}(g)$ over all $g \in C_n$ with $\operatorname{ord}(g) = n$. Let $d > 1$ and $r \geq 1$ be any fixed integers. We prove that, for every sufficiently large integer n divisible by d, there exists a sequence S over C_n of length $|S| \geq n + n/d + O(\sqrt{n})$ having no subsequence T of index $\operatorname{ind}(T) \in [1, r]$, which has substantially improved the previous results in this direction.

1. Introduction and Main Results

Throughout this paper, let C_n be an additively written finite cyclic group of order $|C_n| = n$, where $n \in \mathbb{Z}$ with $n > 1$. By a *sequence* S of *length* $|S| = \ell$ over C_n we mean an unordered sequence with ℓ terms from C_n and the repetition of terms is allowed. We call S a *zero-sum sequence* if the sum of S is zero. We let $\mathbb{Z}$ denote the integers, and $\mathbb{R}$ the real numbers. Given real numbers $a, b \in \mathbb{R}$, we use $[a, b] := \{u : u \in \mathbb{Z}, a \leq u \leq b\}$ to denote all integers between a and b. Recall that the index of a sequence S is defined as follows.

Definition 1.1. *For a sequence*

$$S = (n_1 g) \cdot \ldots \cdot (n_l g) \quad \textit{over} \quad C_n,$$

where $n_1, \dots, n_l \in [1, n]$ *and* $g \in C_n$ *with* $\operatorname{ord}(g) = |C_n|$, *we set*

$$\|S\|_g = \frac{n_1 + \ldots + n_l}{n},$$

and the index of S is defined by

$$\operatorname{ind}(S) = \min\{\|S\|_g \mid g \in C_n \textit{ with } \operatorname{ord}(g) = |C_n|\}.$$

The index of a sequence is a crucial invariant in the investigation of zero-sum sequences over cyclic groups. It was first addressed by Lemke and Kleitman ([9]), used as a key tool by Geroldinger ([7, page 736]), and then investigated by Gao [3] in a systematical way. And it has found a lot of attention in recent years (see [1, 2, 4, 6, 8, 10, 11, 13, 15, 16]). If S is a minimal zero-sum sequence, then $|S| \leq 3$, as well as $|S| \geq \lfloor \frac{n}{2} \rfloor + 2$, implies that $\operatorname{ind}(S) = 1$ (see [1], [12], [14]).

An important open problem (at the end of [5]) is to determine the maximum length of sequences over C_n without index 1 subsequences. Clearly, S is a zero-sum sequence if and only if $\operatorname{ind}(S)$ is an integer by definition 1.1. Hence we introduce the definitions of $\mathtt{t}_r(n)$ and *index-r-free* sequences.

Definition 1.2. *Let r be a positive integer, denote by $\mathsf{t}_r(n)$ the smallest integer ℓ such that every sequence S over C_n of length $|S| \geq \ell$ has a zero-sum subsequence T with* $\mathrm{ind}(T) \in [1, r]$.

Definition 1.3. *For any integer $r \geq 1$, a sequence S over C_n is called index-r-free, if S has no zero-sum subsequence T with* $\mathrm{ind}(T) \in [1, r]$.

In 1989, Lemke and Kleitman ([9, page 344]) conjectured that if S is a sequence over C_n of length $|S| = n$, then there exists a subsequence T of S such that $\mathrm{ind}(T) = 1$. That is to say, $\mathsf{t}_1(n) = n$. In 2011, Gao, Li, Peng, Plyley and Wang ([5]) gave a counterexample and proved that $\mathsf{t}_1(n) \geq n + \lfloor \frac{n}{4} \rfloor - 4$ for $n = 4k + 2 \geq 22$. In 2015, Zeng, Yuan and Li ([16]) promoted the former counterexample to general counterexamples, and by their results we could derive that $\mathsf{t}_1(n) \geq n + \lfloor \frac{n}{d^2} \rfloor - (d^3 - d^2 + d - 1)$ for $n > d^2(d^3 - d^2 + d + 1)$, where $d \in \mathbb{Z}$ with $d > 1$.

In this paper we give longer general structures (theorem 1.4) to the conjecture of Lemke and Kleitman, and prove that $\mathsf{t}_1(n) \geq n + \frac{n}{d} + O(\sqrt{n})$ for every sufficiently large integer n divisible by d, where $d \in \mathbb{Z}$ with $d > 1$ (theorem 1.5). It is a greater lower bound of $\mathsf{t}_1(n)$ than before, and we conjecture that it is the best possible bound when n is big enough. Furthermore, we promote the index 1 free sequences to index r free sequences, and show that $\mathsf{t}_{\mathbf{r}}(n) \geq n + \frac{n}{d} + O(\sqrt{n})$ for every sufficiently large integer n divisible by d, where constant $r \in \mathbb{Z}$ with $r \geq 2$. Here are our main results.

Theorem 1.4. *Let d, n be any integers with $1 < d | n$ and $n > d^2$, and $g \in C_n$ with* $\mathrm{ord}(g) = n$. *For every integer $r \in \left[1, \frac{n}{d^2}\right)$ and $k \in \left[0, log_d^{\frac{n}{r}} - 2\right)$,*

$$S = \prod_{(i,j) \in A} \Big(\left(im + d^j\right) g \Big)^{\left\lfloor \frac{m}{d^j} \right\rfloor - (dr-1)d^{k-j} - 1} \tag{1}$$

is an index-r-free sequence, where $m = \frac{n}{d}$ and $A = [1, d-1] \times [0, k] \bigcup \{(0,0)\}$.

Theorem 1.5. *Given any fixed integers $d > 1$ and $r \geq 1$, for every sufficiently large integer n with $d | n$, there exists an index-r-free sequence S over C_n such that $|S| \geq n + \frac{n}{d} + O(\sqrt{n})$.*

In the following sections we provide the preliminaries and the proofs of Theorem 1.4 and Theorem 1.5. We end the paper with a further conjecture and an open problem.

2. Notations and Preliminaries

We let n and d be any integers with $1 < d | n$ and $n > d^2$, and let $g \in C_n$ with $\mathrm{ord}(g) = n$. For every integer $r \in \left[1, \frac{n}{d^2}\right)$ and $k \in \left[0, log_d^{\frac{n}{r}} - 2\right)$, let a sequence

$$S = \prod_{(i,j) \in A} \Big(\left(im + d^j\right) g \Big)^{\left\lfloor \frac{m}{d^j} \right\rfloor - (dr-1)d^{k-j} - 1},$$

where $m = \frac{n}{d}$ and $A = [1, d-1] \times [0, k] \bigcup \{(0,0)\}$.

Let T be a subsequence of S and $t_{ij} \in \mathbb{Z}$ be the multiplicity of $(im + d^j)g$ in T, where $(i, j) \in A$. If $(im + d^j)g \notin T$, we set $t_{ij} = 0$. That is,

$$T = \prod_{(i,j) \in A} \big((im + d^j) g \big)^{t_{ij}} \subset S,$$

where

$$0 \le t_{ij} \le \left\lfloor \frac{m}{d^j} \right\rfloor - (dr-1)d^{k-j} - 1. \tag{2}$$

We set $\mathrm{ind}(T) = \parallel T \parallel_{g_1}$, where $g_1 \in C_n$ with $\langle g_1 \rangle = C_n$. And we set $g = hg_1$, where $h \in [1, n-1]$ with $\gcd(h, n) = 1$. Then

$$T = \prod_{(i,j)\in A} \left((im + d^j)hg_1 \right)^{t_{ij}},$$

and

$$n \parallel T \parallel_{g_1} = \sum_{(i,j)\in A} t_{ij} \left|(im + d^j)h\right|_n, \tag{3}$$

where $\left|w\right|_n$ denotes the least positive residue of $w \in \mathbb{Z}$ modulo $n > 0$. We fix the notation concerning sequences over C_n. And let

$$B = \left\{ (i,j) \in A \,\middle|\, 0 < \left|(im + d^j)h\right|_n < m \right\},$$

and

$$C = \left\{ (i,j) \in A \,\middle|\, m < \left|(im + d^j)h\right|_n < n \right\}.$$

By next lemma we split A into two parts.

Lemma 2.1. *$B \cup C = A$.*

Proof. For every $(i, j) \in A$, combining $A = [1, d-1] \times [0, k] \bigcup \{(0, 0)\}$, $r \in \left[1, \frac{n}{d^2}\right)$ with $k \in \left[0, log_d^{\frac{n}{r}} - 2\right)$, we derive $0 < d^j < m$. Then by $\gcd(h, n) = 1$ and $dm = n$, we have $0 < \left|(im + d^j)h\right|_n < n$ and $\left|(im + d^j)h\right|_n \neq m$ for every $(i, j) \in A$. Then by the definitions of B and C, we have $B \cup C = A$. □

Lemma 2.2. *For any integer $j \in [0, k]$, we have*

$$\left\{ \left|(im + d^j)h\right|_n \,\middle|\, i \in [0, d-1] \right\} = \left\{ im + \left|hd^j\right|_m \,\middle|\, i \in [0, d-1] \right\},$$

and there exists only one element $i_0 \in [0, d-1]$ such that $0 < \left|(i_0 m + d^j)h\right|_n < m$.

Proof. By

$$\left|\left|(im + d^j)h\right|_n\right|_m = \left|hd^j\right|_m, \text{ where } i \in [0, d-1],$$

we have

$$\left\{ \left|(im + d^j)h\right|_n \,\middle|\, i \in [0, d-1] \right\} \subset \left\{ im + \left|hd^j\right|_m \,\middle|\, i \in \mathbb{Z} \right\}.$$

For any $j \in [0, k]$, by the relevant definitions we have $0 < d^j < m$, then $0 < \left|(im + d^j)h\right|_n < n$. So we have

$$\left\{ \left|(im + d^j)h\right|_n \,\middle|\, i \in [0, d-1] \right\} \subset \left\{ im + \left|hd^j\right|_m \,\middle|\, i \in [0, d-1] \right\}.$$

By $\gcd(h, n) = 1$, we derive that $\left\{ \left|(im + d^j)h\right|_n \,\middle|\, i \in [0, d-1] \right\}$ have d distinct elements. Since these two sets both have d elements, we have

$$\left\{ \left|(im + d^j)h\right|_n \,\middle|\, i \in [0, d-1] \right\} = \left\{ im + \left|hd^j\right|_m \,\middle|\, i \in [0, d-1] \right\},$$

and there exists only one element $i_0 \in [0, d-1]$ such that

$$0 < \left|(i_0 m + d^j)h\right|_n < m.$$

□

By lemma 2.1, we rewrite Eq. (3) as

$$n \parallel T \parallel_{g_1} = \left(\sum_{(i,j)\in B} + \sum_{(i,j)\in C} \right) t_{ij} \left| \left(im + d^j\right) h \right|_n . \tag{4}$$

We consider the d elements of A, $(i,0)$, where $i \in [0, d-1]$. By lemma 2.2, we have

$$\left\{ \left|(im + d^0)h\right|_n \,\middle|\, i \in [0, d-1] \right\} = \left\{ im + \left|hd^0\right|_m \,\middle|\, i \in [0, d-1] \right\}.$$

Then for some $i_0 \in [0, d-1]$, one has $\left|(i_0 m + d^0)h\right|_n = |h|_m$, so $(i_0, 0) \in B$. For some $i_1 \in [0, d-1]$, one has $\left|(i_1 m + d^0)h\right|_n = m + |h|_m$, so $(i_1, 0) \in C$. Then we derive that $B, C \neq \emptyset$. Here we set $|B| = x$ and sort the elements in B as

$$B = \{ (\mu_1, \tau_1), (\mu_2, \tau_2), \cdots, (\mu_x, \tau_x) \},$$

where μ_*, τ_* and x are integers with $\mu_* \in [0, d-1]$, $0 = \tau_1 \leq \tau_2 \leq \cdots \leq \tau_x \leq k$ and $x \geq 1$.

By lemma 2.2, we derive that for any integer τ_*, there exists at most one element $\mu_* \in [0, d-1]$ such that $0 < \left|(\mu_* m + d^{\tau_*})h\right|_n < m$. By the enumeration of the elements of B, we know that actually $0 = \tau_1 < \tau_2 < \cdots < \tau_x \leq k$.

Next we will prove another quality of the sorted elements in B when $x \geq 2$.

Lemma 2.3. *When $|B| = x \geq 2$, for every integer $a \in [1, x-1]$, we have*

$$m < \left|(\mu_a m + d^{\tau_a})h\right|_n d^{\tau_{a+1} - \tau_a} < n.$$

Proof. **Case 1.** $\tau_{a+1} - \tau_a = 1$.

By the definition of B we have $0 < \left|(\mu_a m + d^{\tau_a})h\right|_n < m$, thus $0 < \left|(\mu_a m + d^{\tau_a})h\right|_n d < n$. It is clear that $\left|(\mu_a m + d^{\tau_a})h\right|_n d \neq m$. Assuming that $0 < \left|(\mu_a m + d^{\tau_a})h\right|_n d < m$, by the definition of B we also have $0 < \left|(\mu_{a+1} m + d^{\tau_{a+1}})h\right|_n < m$. Thus

$$\left|(\mu_a m + d^{\tau_a})h\right|_n d - \left|(\mu_{a+1} m + d^{\tau_{a+1}})h\right|_n \in (-m, m). \tag{5}$$

But we have

$$\left| \left|(\mu_a m + d^{\tau_a})h\right|_n d - \left|(\mu_{a+1} m + d^{\tau_{a+1}})h\right|_n \right|_n = \left| -\mu_{a+1} hm \right|_n = \left| -\mu_{a+1} h \right|_d m.$$

Since $\mu_{a+1} \in [1, d-1]$ and $\gcd(h, n) = 1$, we have $\left| -\mu_{a+1} h \right|_d \neq d$. Hence

$$\left|(\mu_a m + d^{\tau_a})h\right|_n d - \left|(\mu_{a+1} m + d^{\tau_{a+1}})h\right|_n = ym \text{ with integer } y \neq 0,$$

a contradiction to Eq. (5). So that $m < \left|(\mu_a m + d^{\tau_a})h\right|_n d < n$.

Case 2. $\tau_{a+1} - \tau_a \geq 2$.

First, for any integers $v \in [\tau_a + 1, \tau_{a+1} - 1]$ and $i \in [1, d-1]$, we have $(i, v) \in A$ by the definition of A. By definition of B, $(i, v) \notin B$. By lemma 2.1, we have $(i, v) \in C$. Then by the definition of C, we have

$$m < \left|(im + d^v)h\right|_n < n, \tag{6}$$

where $v \in [\tau_a + 1, \tau_{a+1} - 1]$ and $i \in [1, d-1]$.

Second, for every $z \in [0\,,\, \tau_{a+1} - \tau_a - 2]$, we will prove that, if $0 < \left|(\mu_a m + d^{\tau_a})h\right|_n d^z < m$, then $0 < \left|(\mu_a m + d^{\tau_a})h\right|_n d^{z+1} < m$.

For every $z \in [0\,,\, \tau_{a+1} - \tau_a - 2]$, we let $v = \tau_a + z + 1$, and suppose that

$$0 < \left|(\mu_a m + d^{\tau_a})h\right|_n d^z < m.$$

Then we have

$$0 < \left|(\mu_a m + d^{\tau_a})h\right|_n d^{z+1} < n.$$

Therefore,

$$\left|(\mu_a m + d^{\tau_a})h\right|_n d^{z+1} = \left|(\mu_a m + d^{\tau_a})hd^{z+1}\right|_n = \left|d^{\tau_a + z+1}h\right|_n = \left|hd^v\right|_n . \tag{7}$$

By lemma 2.2, we have

$$\left\{\left|(im + d^v)h\right|_n \,\middle|\, i \in [\mathbf{0}, d-1]\right\} = \left\{im + \left|hd^v\right|_m \,\middle|\, i \in [\mathbf{0}, d-1]\right\}. \tag{8}$$

Note that $v = \tau_a + z + 1 \in [\tau_a + 1,\, \tau_{a+1} - 1]$. By Eq. (6), we have

$$\left\{\left|(im + d^v)h\right|_n \,\middle|\, i \in [\mathbf{1}, d-1]\right\} \subset \left\{im + \left|hd^v\right|_m \,\middle|\, i \in [\mathbf{1}, d-1]\right\}.$$

Since these two sets both have $d-1$ elements, we have

$$\left\{\left|(im + d^v)h\right|_n \,\middle|\, i \in [\mathbf{1}, d-1]\right\} = \left\{im + \left|hd^v\right|_m \,\middle|\, i \in [\mathbf{1}, d-1]\right\}. \tag{9}$$

Then combining Eq. (8) with Eq. (9), we have

$$\left\{\left|(im + d^v)h\right|_n \,\middle|\, i = 0\right\} = \left\{im + \left|hd^v\right|_m \,\middle|\, i = 0\right\}.$$

That is, $\left|hd^v\right|_n = \left|hd^v\right|_m$. Then by $0 < \left|hd^v\right|_m < m$ and Eq. (7), we have

$$0 < \left|(\mu_a m + d^{\tau_a})h\right|_n d^{z+1} < m.$$

Last, thus we proceed by induction on $z \in [0\,,\, \tau_{a+1} - \tau_a - 2]$. Since $0 < \left|(\mu_a m + d^{\tau_a})h\right|_n d^z < m$ is true for $z = 0$ by the definition of B, we let $z = \tau_{a+1} - \tau_a - 2$ and derive that

$$0 < \left|(\mu_a m + d^{\tau_a})h\right|_n d^{\tau_{a+1} - \tau_a - 1} < m$$

is true. Thus $0 < \left|(\mu_a m + d^{\tau_a})h\right|_n d^{\tau_{a+1} - \tau_a} < n$. It is clear that $\left|(\mu_a m + d^{\tau_a})h\right|_n d^{\tau_{a+1} - \tau_a} \neq m$. Assuming that $0 < \left|(\mu_a m + d^{\tau_a})h\right|_n d^{\tau_{a+1} - \tau_a} < m$, by the definition of B we also have $0 < \left|(\mu_{a+1} m + d^{\tau_{a+1}})h\right|_n < m$. Thus

$$\left|(\mu_a m + d^{\tau_a})h\right|_n d^{\tau_{a+1} - \tau_a} - \left|(\mu_{a+1} m + d^{\tau_{a+1}})h\right|_n \in (-m, m). \tag{10}$$

But we have

$$\Big|\left|(\mu_a m + d^{\tau_a})h\right|_n d^{\tau_{a+1} - \tau_a} - \left|(\mu_{a+1} m + d^{\tau_{a+1}})h\right|_n\Big|_n = \left| -\mu_{a+1} h\right|_d m\,.$$

It is a contradiction to Eq. (10). So that $m < \left|(\mu_a m + d^{\tau_a})h\right|_n d^{\tau_{a+1} - \tau_a} < n$. □

3. Proof of Theorem 1.4 and Theorem 1.5

Proof of Theorem 1.4. Suppose to the contrary that there exists a subsequence $T \subset S$ with $T \neq \emptyset$ and $\mathrm{ind}(T) \in [1, r]$. We use the same relevant notions defined in last section. Without loss of generality, we assume that $|B| = x \geq 2$, because the following proof also holds true by some minor modifications (for example, we view all the $\sum_{l=1}^{x-1} f(l)$ as 0 when $x = 1$). We could rewrite Eq. (4) as

$$\begin{aligned} n \parallel T \parallel_{g_1} = & \sum_{l=1}^{x-1} t_{\mu_l \tau_l} \left|(\mu_l m + d^{\tau_l})h\right|_n + t_{\mu_x \tau_x} \left|(\mu_x m + d^{\tau_x})h\right|_n \\ & + \sum_{(i,j)\in C} t_{ij} \left|(im + d^j)h\right|_n . \end{aligned} \tag{11}$$

For $l \in [1, x-1]$, we set

$$t_{\mu_l \tau_l} = s_l \, d^{\tau_{l+1} - \tau_l} + t'_{\mu_l \tau_l} , \tag{12}$$

where $s_l \geq 0$ and $t'_{\mu_l \tau_l} \in [0 \, , \, d^{\tau_{l+1} - \tau_l} - 1]$. Then we use three steps to complete the proof.

First, we will prove that $\sum_{l=1}^{x-1} s_l + \sum_{(i,j)\in C} t_{ij} \leq dr - 1$. By Eqs. (11) and (12), we have

$$\begin{aligned} n \parallel T \parallel_{g_1} = & \sum_{l=1}^{x-1} (s_l \, d^{\tau_{l+1} - \tau_l} + t'_{\mu_l \tau_l}) \left|(\mu_l m + d^{\tau_l})h\right|_n \\ & + t_{\mu_x \tau_x} \left|(\mu_x m + d^{\tau_x})h\right|_n + \sum_{(i,j)\in C} t_{ij} \left|(im + d^j)h\right|_n \\ = & \sum_{l=1}^{x-1} t'_{\mu_l \tau_l} \left|(\mu_l m + d^{\tau_l})h\right|_n + t_{\mu_x \tau_x} \left|(\mu_x m + d^{\tau_x})h\right|_n \\ & + \Bigg(\sum_{l=1}^{x-1} s_l \, d^{\tau_{l+1} - \tau_l} \left|(\mu_l m + d^{\tau_l})h\right|_n \\ & + \sum_{(i,j)\in C} t_{ij} \left|(im + d^j)h\right|_n \Bigg). \end{aligned} \tag{13}$$

Hence we have

$$\begin{aligned} n \parallel T \parallel_{g_1} \geq & \sum_{l=1}^{x-1} s_l \, d^{\tau_{l+1} - \tau_l} \left|(\mu_l m + d^{\tau_l})h\right|_n + \sum_{(i,j)\in C} t_{ij} \left|(im + d^j)h\right|_n \\ > & \sum_{l=1}^{x-1} s_l \, m + \sum_{(i,j)\in C} t_{ij} \, m = \left(\sum_{l=1}^{x-1} s_l + \sum_{(i,j)\in C} t_{ij} \right) m . \end{aligned}$$

We suppose that $\sum_{l=1}^{x-1} s_l + \sum_{(i,j)\in C} t_{ij} > dr$, and derive

$$n \parallel T \parallel_{g_1} > \left(\sum_{l=1}^{x-1} s_l + \sum_{(i,j)\in C} t_{ij} \right) m > rn.$$

Thus $\text{ind}(T) = \| T \|_{g_1} > r$, a contradiction to $\text{ind}(T) \in [1, r]$. So we have

$$\sum_{l=1}^{x-1} s_l + \sum_{(i,j)\in C} t_{ij} \le dr - 1. \tag{14}$$

Next, we will prove that $\left| n \| T \|_{g_1} \right|_{\mathbf{m}} \neq m$. By Eq. (13), we have

$$\begin{aligned}
&\left| n \| T \|_{g_1} \right|_{\mathbf{m}} \\
&= \left| \sum_{l=1}^{x-1} t'_{\mu_l\,\tau_l}\, d^{\tau_l} h + t_{\mu_x\,\tau_x}\, d^{\tau_x} h + \sum_{l=1}^{x-1} s_l\, d^{\tau_{l+1}-\tau_l}\, d^{\tau_l} h + \sum_{(i,j)\in C} t_{ij}\, d^j h \right|_{\mathbf{m}} \\
&= \left| h\left(\sum_{l=1}^{x-1} t'_{\mu_l\,\tau_l}\, d^{\tau_l} + t_{\mu_x\,\tau_x}\, d^{\tau_x} + \sum_{l=1}^{x-1} s_l\, d^{\tau_{l+1}} + \sum_{(i,j)\in C} t_{ij}\, d^j \right) \right|_{\mathbf{m}} \\
&= \left| h(**) \right|_{\mathbf{m}}, \qquad (15)
\end{aligned}$$

where

$$\begin{aligned}
(**) &= \sum_{l=1}^{x-1} t'_{\mu_l\,\tau_l}\, d^{\tau_l} + t_{\mu_x\tau_x}\, d^{\tau_x} + \sum_{l=1}^{x-1} s_l\, d^{\tau_{l+1}} + \sum_{(i,j)\in C} t_{ij}\, d^j \\
&\le \sum_{l=1}^{x-1} (d^{\tau_{l+1}-\tau_l} - 1) d^{\tau_l} + t_{\mu_x\tau_x}\, d^{\tau_x} + \sum_{l=1}^{x-1} s_l\, d^{\mathbf{k}} + \sum_{(i,j)\in C} t_{ij}\, d^{\mathbf{k}} \\
&= -\, d^{\tau_1} + d^{\tau_x} + t_{\mu_x\tau_x}\, d^{\tau_x} + \left(\sum_{l=1}^{x-1} s_l + \sum_{(i,j)\in C} t_{ij} \right) d^k \\
&\le -\, d^{\tau_1} + d^{\tau_x} + \left(\left\lfloor \frac{m}{d^{\tau_x}} \right\rfloor - (dr-1) d^{k-\tau_x} - 1 \right) d^{\tau_x} + (dr-1) d^k \qquad (16) \\
&\le -\, d^{\tau_1} + d^{\tau_x} + m - (dr-1) d^k - d^{\tau_x} + (dr-1) d^k \\
&\le m - 1\,. \qquad (17)
\end{aligned}$$

It is clear that $(**) > 0$ by $T \neq \emptyset$. So we have $\left| n \| T \|_{g_1} \right|_{\mathbf{m}} = \left| h(**) \right|_{\mathbf{m}} \neq m$ by Eqs. (15) and (17).

Last, since $\left| n \| T \|_{g_1} \right|_{\mathbf{m}} \neq m$ and $m|n$, we have $\left| n \| T \|_{g_1} \right|_{\mathbf{n}} \neq n$. Hence $\text{ind}(T) = \| T \|_{g_1}$ is not an integer and T is not a zero-sum subsequence of S. It is a contradiction to $\text{ind}(T) \in [1, r]$. Thus S is an index-r-free sequence.

□

Proof of Theorem 1.5. Given any fixed integers $d > 1$ and $r \ge 1$, we take the same S defined in theorem 1.4 and let $n > rd^2$ with $d|n$. Then S is an index-r-free sequence for any $k \in$

$\Big[0, log_d^{\frac{n}{r}} - 2\Big)$ by theorem 1.4. Since $\lfloor \frac{m}{d^j} \rfloor > \frac{m}{d^j} - 1$, we calculate the length of S and have

$$\begin{aligned}
|S| &= \sum_{(i,j)\in A} \left(\left\lfloor \frac{m}{d^j} \right\rfloor - (dr-1)d^{k-j} - 1 \right) \\
&> \sum_{(i,j)\in[1,d-1]\times[0,k]} \left(\frac{m}{d^j} - (dr-1)d^{k-j} - 2 \right) + m - (dr-1)d^k - 1 \\
&= (d-1) \sum_{j\in[0,k]} \left(\frac{m}{d^j} - (dr-1)d^{k-j} - 2 \right) + m - (dr-1)d^k - 1 \\
&= \left(1 + \frac{1}{d} - \frac{1}{d^{k+1}} \right) n - (dr-1)(d^{k+1} + d^k - 1) - 2(k+1)(d-1) - 1 \,.
\end{aligned}$$

We let $k = \lfloor \frac{1}{2} \ln(n) \rfloor > 0$ and have

$$|S| > \left(1 + \frac{1}{d} \right) n + C_1\sqrt{n} + C_2 \ln(n) + C_3 \,,$$

where C_1, C_2 and C_3 are some constants determined by d and r. Thus we have proved the theorem. □

Therefore, $\mathsf{t}_{\mathbf{r}}(n) \geq n + \frac{n}{d} + O(\sqrt{n})$ for every sufficiently large integer n divisible by d, where $d > 1$ and $r \geq 1$ are constant integers.

4. Concluding Remarks

Given any fixed integers $d > 1$ and $r \geq 1$. Since $\lfloor \frac{m}{d^j} \rfloor \leq \frac{m}{d^j}$, we can also get upper bounds of $|S|$ in theorem 1.5. Let d be the least prime factor of n. Generally, $|S| < n + \frac{n}{d}$. So we have the following conjecture.

Conjecture 4.1. *Let n be a composite number, C_n a cyclic group of order n, and d the least prime factor of n. Then every sequence S of length $|S| = n + \frac{n}{d}$ over C_n has a zero-sum subsequence T with* $\mathrm{ind}(T) = 1$.

Open Problem. *Determine $\mathsf{t}_r(n)$ for all integers $n \geq 2$ and $r > 0$.*

Acknowledgments

We would like to thank the referee for his/her careful reading, valuable comments and suggestions on improving the manuscript. We also would like to thank our advisor Prof. W. Gao for his useful comments and suggestions. This work was supported by the 973 Program of China (Grant No.2013CB834204), the PCSIRT Project of the Ministry of Science and Technology of China, and the National Science Foundation of China (Grant No.11671218).

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