CONGRUENCE CLASSES OF PRESENTATIONS FOR THE COMPLEX REFLECTION GROUPS $G(m,p,n)$

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Abstract. We give an explicit description in terms of rooted graphs for representatives of all the congruence classes of presentations (or r.c.p. for brevity) for the imprimitive complex reflection group $G(m,p,n)$. Also, we show that (S, P_S) forms a presentation of $G(m,p,n)$, where S is any generating reflection set of $G(m,p,n)$ of minimally possible cardinality and P_S is the set of all the basic relations on S.

Introduction.

In [7], we describe r.c.p. for two special families of imprimitive complex reflection groups $G(m,1,n)$ and $G(m,m,n)$ in terms of graphs. In the present paper, we shall extend the results to the imprimitive complex reflection group $G(m,p,n)$ for any $m,p,n \in \mathbb{N}$ with $p|m$ (read " p divides m ") and $1 < p < m$.

Let $\Sigma(m,p,n)$ be the set of all the reflection sets S of $G(m,p,n)$ such that (S,P) forms a presentation of $G(m,p,n)$ for some relation set P on S (see 1.7 and 2.6). We associate each $S \in \Sigma(m,p,n)$ to a connected rooted graph Γ_S^r with exactly one circle (see 1.6 and Lemma 2.2). Also, we define a value $\delta(S) \in \mathbb{N}$ for each $S \in \Sigma(m,p,n)$

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(see 2.3), which satisfies the condition $\gcd\{\delta(S), p\} = 1$ by Theorem 2.4. Then our first main result (i.e., Theorem 2.9) asserts that the congruence of a presentation (S, P) for $G(m, p, n)$ is entirely determined by the isomorphism class of the rooted graph Γ_S^r if the circle of Γ_S^r contains more than two nodes, and by the isomorphism class of Γ_S^r and the value $\gcd\{\delta(S), m\}$ if the circle of Γ_S^r contains only two nodes.

Next we introduce the set P_S of the basic relations on any $S \in \Sigma(m, p, n)$ (see (A)-(M) in 4.2-4.3). Our second main result (i.e., Theorem 6.2) asserts that (S, P_S) forms a presentation of $G(m, p, n)$. There are two crucial steps in proving this result: one is to apply a circle operation on the set $\Sigma(m, p, n)$, as we did before on the set $\Sigma(m, 1, n)$ and $\Sigma(m, m, n)$ in [7]; the other is a transition between certain pair $S, X \in \Sigma(m, p, n)$, which is new, where both Γ_S^r and Γ_X^r are a string with a two-nodes circle at one end and with the rooted node on the circle and not adjacent to any node outside the circle; X differs from S only by one reflection of type I whose corresponding edges are on the circles of the respective graphs (see 6.3).

Comparing with the cases of $G(m, 1, n)$ and $G(m, m, n)$, the cardinality of any $S \in \Sigma(m, p, n)$ is $n+1$, rather than n. Hence there are more basic relations on S. This makes our treatment for $G(m, p, n)$ more complicated than that for $G(m, 1, n)$ and $G(m, m, n)$. We introduce the concept of a generalized circle sequence and a root-circle sequence in the graph Γ_S^r (see 4.3). We use them to simplify our discussion for the circle relations, root-circle relations and circle-root relations on S (see relations (J), (K), (L) in 4.2).

Let X be the subset of S containing the reflection of type II such that Γ_X^r is the subgraph of Γ_S^r corresponding to the root-circle sequence. Then by Theorem 2.4, we see that the subgroup $\langle X \rangle$ of $G(m, p, n)$ generated by X is isomorphic to $G(m, p, |X| - 1)$. Moreover, it is easily seen by Theorem 2.4 that a subset X' of S satisfies the condition $\langle X' \rangle \cong G(m, p, |X'| - 1)$ if and only if $X' \supseteq X$ and the graph $\Gamma_{X'}$ is connected. Denote X by S_0. It looks likely that a presentation (S, P) has a simpler relation set P when the graph Γ_S^r has no branching node and when the subgraph obtained from Γ_S^r by removing

$\Gamma_{S_0}^r$ is either a string or empty. In particular, two cases are worthy to be mentioned: one is when Γ_S^r is a string with a two-nodes circle at one end (the presentation given in [1, Appendix II] belongs to such a case, see 4.1); the other is when Γ_S^r is a rooted circle. The latter may convenience us to associate $G(m,p,n)$ with the extended affine Weyl group of type $\widetilde{A}_{n-1}$ in the study of the group $G(m,p,n)$.

Theorems 2.9 and 6.2 suggest an effective way to find a representative (S,P) for any congruence class of presentations of $G(m,p,n)$, see Remark 2.10 (1) for getting S, and see relations (A)–(M) in 4.2–4.3 and Remark 6.9 (1) for getting P.

It is interesting to find an essential presentation (see 1.7) by removing some redundant relations from any presentation of $G(m,p,n)$ given in the paper. However, this has not yet been solved in general (see Remark 6.9 (1)). We can deal with it only in some special cases or when the given numbers m,p,n are smaller.

The contents of the paper are organized as follows. Section 1 is the preliminaries, some concepts, notations and known results are collected there. We show the first main result in Section 2. Then Sections 3-6 are served to show the second main result. More precisely, we introduce the circle operations on the set $\Sigma(m,p,n)$ in Section 3; Basic relations on any $S\in\Sigma(m,p,n)$ are introduced in Section 4; In Section 5, we consider the equivalence of the basic relation sets P_S on $S\in\Sigma(m,p,n)$ when S is changed by a circle operation; finally, we consider the equivalence of the basic relation sets P_S on $S\in\Sigma(m,p,n)$ when Γ_S has a two-nodes circle with the rooted node on it and when the value $\delta(S)$ is changed.

§1. Preliminaries.

1.1. Let V be a complex vector space of dimension n. A *reflection* on V is a linear transformation on V of finite order with exactly $n-1$ eigenvalues equal to 1. A *reflection group* G on V is a finite group generated by reflections on V. A reflection group G on V is called a *real* group or a *Coxeter* group if there is a G-invariant $\mathbb{R}$-subspace V_0 of

V such that the canonical map $\mathbb{C} \otimes_{\mathbb{R}} V_0 \to V$ is bijective. If this is not the case, G will be called *complex*. (Note that, according to this definition, a real reflection group is not complex.)

Since G is finite, there exists a unitary inner product $(\ ,\ )$ on V invariant under G. From now on we fix such an inner product.

1.2. A reflection group G in V is *imprimitive*, if G acts on V irreducibly and there exists a decomposition $V = V_1 \oplus ...... \oplus V_r$ of nontrivial proper subspaces V_i, $1 \leqslant i \leqslant r$, of V such that G permutes the set $\{V_i \mid 1 \leqslant i \leqslant r\}$ (see [2]).

1.3. Let S_n be the symmetric group on n letters $1, 2, ..., n$. For $\sigma \in S_n$, we denote by $[(a_1, ..., a_n)|\sigma]$ the $n \times n$ monomial matrix with non-zero entries a_i in the $(i, (i)\sigma)$-positions. For $p|m$ in $\mathbb{N}$, we set

$$G(m,p,n) = \left\{ [(a_1,...,a_n)|\sigma] \,\middle|\, a_i \in \mathbb{C}, a_i^m = 1\ \forall\ 1 \leqslant i \leqslant n; \left(\prod_{j=1}^{n} a_j\right)^{m/p} = 1; \sigma \in S_n \right\}.$$

$G(m,p,n)$ is the matrix form of an imprimitive reflection group acting on V with respect to an orthonormal basis $e_1, e_2, ..., e_n$, which is Coxeter only when either $m \leqslant 2$ or $(m,p,n) = (m,m,2)$. We have $G(m,p,n) = G(1,1,n) \ltimes A(m,p,n)$, where $A(m,p,n)$ consists of all the diagonal matrices of $G(m,p,n)$, and $G(1,1,n) \cong S_n$.

There are two special imprimitive reflection groups $G(m,1,n)$ and $G(m,m,n)$ with the inclusions $G(m,m,n) \subseteq G(m,p,n) \subseteq G(m,1,n)$, where the smaller ones are normal subgroups of the bigger ones. We described r.c.p. for these two families of groups in [7]. In the present paper, we shall deal with the imprimitive group $G(m,p,n)$ for any $m,p,n \in \mathbb{N}$ with $p|m$ and $1 < p < m$.

From now on, we shall always assume $n \geqslant 2$, $p|m$ and $1 < p < m$ when we consider the group $G(m,p,n)$ unless otherwise specified.

1.4. For an orthonormal basis $e_1, ..., e_n$ of V, $\zeta_m = e^{2\pi i/m}$, $\mu_m = \{\zeta_m^k \mid k \in \mathbb{Z}\}$, and $q = p^{-1}m \in \mathbb{N} \setminus \{1\}$, put

$$R_1 = \mu_2 \cdot \mu_m \cdot \{\zeta_m^k e_i - e_j \mid i, j, k \in \mathbb{Z}, 1 \le i \ne j \le n\},$$

$$R_2 = \mu_q \cdot \{e_k \mid 1 \le k \le n\},$$

$$R(m, p, n) = R_1 \cup R_2.$$

Then $R(m, p, n)$ is a root system of the group $G(m, p, n)$ (refer [2] for the definition of a root system).

1.5. There are two kinds of reflections in the group $G(m, p, n)$ as follows.

(i) One is with respect to a root in R_1. It is of the form

$$s(i, j; k) := [(1, ..., 1, \zeta_m^{-k}, 1, ..., 1, \zeta_m^k, 1, ..., 1)|(i, j)],$$

where ζ_m^{-k}, ζ_m^k with some $k \in \mathbb{Z}$ are the ith, resp. jth components of the n-tuple and (i, j) is the transposition of i and j for some $1 \leqslant i < j \leqslant n$. We call $s(i, j; k)$ *a reflection of type I*. Clearly, any reflection of type I has order 2. We also set $s(j, i; k) = s(i, j; -k)$.

(ii) The other is with respect to a root in R_2. It is of the form

$$s(i; k) := [(1, ..., 1, \zeta_q^k, 1, ..., 1)|\mathbf{1}]$$

for some $k \in \mathbb{Z}$, where ζ_q^k occurs as the ith component of the n-tuple and $\mathbf{1}$ is the identity element of S_n. We call $s(i; k)$ *a reflection of type II*. $s(i; k)$ has order $q/\gcd\{q, k\}$.

All the reflections of type I lie in the subgroup $G(m, m, n)$.

1.6. For any $Z \subseteq \{1, 2, ..., n\}$, let V_Z be the subspace of V spanned by $\{e_i \mid i \in Z\}$. Let $R_Z(m, p, n) = R(m, p, n) \cap V_Z$. Then $R_Z(m, p, n)$ is a root subsystem of $R(m, p, n)$. Let $G_Z(m, p, n)$ be the subgroup of $G(m, p, n)$ generated by the reflections with respect to the roots in $R_Z(m, p, n)$. Then $G_Z(m, p, n) \cong G(m, p, r)$ with $r = |Z|$. To any set $X = \{s(i_h, j_h; k_h) \mid h \in J\}$ of reflections of $G_Z(m, p, n)$ (J a finite index set), we associate a digraph $\overline{\Gamma}_{Z,X} = (N_X, \overline{E}_X)$ as follows. Its node set N_X is Z, and its arrow set $\overline{E}_X$ consists of all the ordered pairs (i,j), $i<j$, with labels k, where $s(i,j;k)\in X$ (hence, if $s(i,j;k)\in X$ and $i>j$, then $\overline{\Gamma}_{Z,X}$ contains an arrow (j,i) with the label $-k$). Denote by $\Gamma_{Z,X}$ the underlying graph of $\overline{\Gamma}_{Z,X}$, i.e., $\Gamma_{Z,X}=(Z,E_X)$ is obtained from $\overline{\Gamma}_{Z,X}$ by replacing all the labelled arrows (i,j) by unlabelled edges $\{i,j\}$, where E_X denotes the set of edges of $\Gamma_{Z,X}$.

We see that the graph $\Gamma_{Z,X}$ has no loop but may have multi-edges between two nodes.

The above definition of a graph can be extended: to any set X of reflections of $G_Z(m,p,n)$, we define a graph $\Gamma_{Z,X}$ to be $\Gamma_{Z,X'}$, where X' is the subset of X consisting of all the reflections of type I. When X contains exactly one reflection of type II (say $s(i;k)$), we define another graph, denoted by $\Gamma^r_{Z,X}$, which is obtained from $\Gamma_{Z,X}$ by rooting the node i, i.e., $\Gamma^r_{Z,X}$ is a rooted graph with the rooted node i. Sometimes we denote $\Gamma^r_{Z,X}$ by (Z,E_X,i).

When $Z=\{1,2,...,n\}$, we simply denote Γ_X (resp. Γ^r_X) for $\Gamma_{Z,X}$ (resp. $\Gamma^r_{Z,X}$).

Note that when X is the generator set in a presentation of $G(m,p,n)$, the graph Γ_X defined here is different from a Coxeter-like graph given in [1, Appendix 2]: in a Coxeter-like graph, all the generating reflections are represented by nodes; while in a graph defined here, most of the generating reflections are represented by edges.

Two graphs (N,E) and (N',E') are *isomorphic*, if there exists a bijection $\eta: N\rightarrow N'$ such that for any $v,w\in N$, $\{v,w\}$ is in E if and only if $\{\eta(v),\eta(w)\}$ is in E'.

Two rooted graphs (N,E,i) and (N',E',i') are *isomorphic*, if there exists a bijection $\eta: N\rightarrow N'$ with $\eta(i)=i'$ such that for any $v,w\in N$, $\{v,w\}$ is in E if and only if $\{\eta(v),\eta(w)\}$ is in E'.

1.7. For a reflection group G, *a presentation of G by generators and relations* (or *a presentation* in short) is by definition a pair (S,P), where

(1) S is a finite generator set for G which consists of reflections, and S has minimal cardinality with this property.

(2) P is a finite set of relations on S, and any other relation on S is a consequence of the relations in P.

A presentation (S, P) of G is *essential* if (S, P_0) is not a presentation of G for any proper subset P_0 of P.

Two presentations (S, P) and (S', P') for G are *congruent*, if there exists a bijection $\eta : S \longrightarrow S'$ such that for any $s, t \in S$,

$(*)$ $\langle s, t\rangle \cong \langle \eta(s), \eta(t)\rangle$, where the notation $\langle x, y\rangle$ stands for the group generated by x, y.

In this case, we see by taking $s = t$ that the order $o(s)$ of s is equal to the order $o(\eta(s))$ of $\eta(s)$ for any $s \in S$.

If there does not exist such a bijection η, then we say that these two presentations are *non-congruent*.

When a reflection group G is a Coxeter group, Coxeter system can be an example of the presentations of G. However, G may have some other presentations not congruent to (S, P). For example, let G be the symmetric group S_n. Then one can show that the set of all the congruence classes of presentations of S_n is in one-to-one correspondence to the set of isomorphism classes of trees of n nodes. The presentation of S_n as a Coxeter system corresponds to the string with n nodes.

Suppose that the structure of a reflection group G is known. Then by the above definition of a presentation, we see that for any generator set S of G with minimally possible cardinality, one can always find a relation set P on S such that (S, P) is a (essential) presentation of G. The congruence of the presentation (S, P) is entirely determined by the generator set S. So it makes sense to talk about the congruence relations among the generator sets of a reflection group G when a reflection group G is given.

1.8. For any non-zero vector $v \in V$, denote by l_v the one dimensional subspace $\mathbb{C}v$ of V spanned by v, we call it *a line*. In particular, denote $l_i := l_{e_i}$ for $1 \leqslant i \leqslant n$. Let

$L = \{l_i \mid 1 \leqslant i \leqslant n\}$. Then a reflection of the form $s(i,j;k)$ in $G(m,p,n)$ interchanges the lines l_i, l_j and leaves all the other lines l_h, $h \neq i,j$, in L stable. A reflection of the form $s(i;k)$ stabilizes all the lines in L. More generally, any element of $G(m,p,n)$ gives rise to a permutation on the set L, and the action of $G(m,p,n)$ on L is transitive.

Let X be a set of reflections of $G(m,p,n)$ and let $\langle X\rangle$ be the subgroup of $G(m,p,n)$ generated by X. Then the action of $\langle X\rangle$ on L is transitive if and only if the graph Γ_X is connected. In particular, the graph Γ_X must be connected when X is the generator set of a presentation of $G(m,p,n)$.

§2. The generator sets in the presentations for $G(m,p,n)$.

In the present section, we shall describe the generator set S in a presentation (S,P) for the group $G(m,p,n)$. Set $q = m/p$.

Let us first show

Lemma 2.1. *Let X be a subset of $G(m,p,n)$ consisting of $n-1$ reflections of type I and one reflection of type II and of order q such that the graph Γ_X is a tree. Then X generates a subgroup of $G(m,p,n)$ isomorphic to $G(q,1,n)$.*

Proof. We have a decomposition $X = X_1 \cup \{s(i;k)\}$, where X_1 is the set of $n-1$ reflections of type I in X, the integers i,k satisfy $1 \leqslant i \leqslant n$ and $\gcd\{k,q\} = 1$ (since the order of $s(i;k)$ is q). Then $\Gamma_{X_1} = \Gamma_X$ is a tree. Let $G_1 := \langle X_1\rangle$. Then $G_1 \cong S_n$ by [7, Lemma 2.7]. By the connectivity of the graph Γ_{X_1}, we see that for any $1 \leqslant j \leqslant n$, there exists a sequence of nodes $i_1 = i, i_2, ..., i_r = j$ in Γ_{X_1} such that X_1 contains the reflections $s_h = s(i_h, i_{h+1}; k_h)$ for any $1 \leqslant h < r$ and some $k_h \in \mathbb{Z}$. Thus $s(j;k) = s_{r-1}s_{r-2}...s_1 \cdot s(i;k) \cdot s_1...s_{r-1}$ is contained in the group $G := \langle X\rangle$. By the condition of $\gcd\{k,q\} = 1$, the subgroup D of $G(m,p,n)$ generated by $s(i;k)$, $1 \leqslant i \leqslant n$, consists of all the diagonal matrices of the form $[(\zeta_q^{k_1}, \zeta_q^{k_2}, ..., \zeta_q^{k_n})|\mathbf{1}]$, where $\zeta_q = e^{2\pi i/q}$, and $k_1, ..., k_n \in \mathbb{Z}$. So $D = A(q,1,n)$. It is easily seen that $G = G_1 \ltimes D$ and hence $G \cong G(q,1,n)$. $\square$

Next result is concerned with the members of the generator set in a presentation of $G(m,p,n)$.

Lemma 2.2. *The generator set S in a presentation (S,P) of the group $G(m,p,n)$ consists of n reflections of type I and one reflection of type II and of order q. Hence the graph Γ_S is connected with n edges.*

Proof. By the definition of a presentation and by [1, Appendix 2], the set S is of cardinality $\leqslant n+1$. Let a_1 (resp. a_2) be the number of reflections in S of type I (resp. of type II and of order q). Then we have $a_2 \geqslant 1$ and hence $a_1 \leqslant n$. So the graph Γ_S has at most n edges. Since the action of the group $G(m,p,n)$ on L is transitive (see 1.8), the graph Γ_S must be connected and hence has at least $n-1$ edges since it has n nodes. So $a_1 \geqslant n-1$ and hence $a_2 \leqslant 2$. Let X be a subset of S consisting of $n-1$ reflections of type I and one reflection of type II and of order q (say $s = s(i;k)$ for some integers i,k with $1 \leqslant i \leqslant n$ and $\gcd\{k,q\} = 1$) such that the graph Γ_X is connected. Let $G = \langle X \rangle$. Then by Lemma 2.1 and its proof, we see that $G \cong G(q,1,n)$, which contains all the reflections of $G(m,p,n)$ of type II. By comparing the orders of $G(m,p,n)$ and $G(q,1,n)$, we see that G is a proper subgroup of $G(m,p,n)$. Since S generates $G(m,p,n)$, S must contain one more reflection of type I than X. This proves our result. $\square$

2.3. Assume that X is a reflection set of $G(m,p,n)$ such that the graph Γ_X is connected and contains exactly one circle, say the edges of the circle are $\{a_h, a_{h+1}\}$, $1 \leqslant h \leqslant r$ (the subscripts are modulo r) for some integer $2 \leqslant r \leqslant n$. Then X contains the reflections $s(a_h, a_{h+1}; k_h)$ with some integers k_h for any $1 \leqslant h \leqslant r$ (the subscripts are modulo r). Denote $\delta(X) := |\sum_{h=1}^{r} k_h|$.

Now we can characterize a reflection set of $G(m,p,n)$ to be the generator set of a presentation as follows.

Theorem 2.4. *Let X be a subset of $G(m,p,n)$ consisting of n reflections of type I and one reflection of type II and of order q such that the graph Γ_X is connected. Then X is* *the generator set of a presentation of* $G(m,p,n)$ *if and only if* $\gcd\{p,\delta(X)\}=1$.

Proof. We have a decomposition $X=X_1\cup\{s(i;k)\}$, where X_1 is the set of n reflections of X of type I. By Lemma 2.1 and its proof, we see that the group $G:=\langle X\rangle$ contains all the reflections of type II in $G(m,p,n)$.

Let $d=\gcd\{m,\delta(X_1)\}$ (note that $\delta(X_1)=\delta(X)$). Let D_1 (resp., D_2) be the set of all the diagonal matrices of the form $[(\zeta^{k_1d},...,\zeta^{k_nd})|\mathbf{1}]$ (resp., $[(\zeta^{h_1p},...,\zeta^{h_np})|\mathbf{1}]$), where $\zeta=e^{2\pi i/m}$, $k_i,h_j\in\mathbb{Z}$ and $k_1+...+k_n=0$. Let $D=\langle D_1,D_2\rangle$. Let X_2 be a subset of X_1 with $|X_2|=n-1$ and Γ_{X_2} connected. Let $G_1=\langle X_1\rangle$ and $G_2=\langle X_2\rangle$. Then by [7, Lemmas 2.13 and 2.16], we have $G_1=D_1\rtimes G_2$ with $G_2\cong S_n$. Let $G_3=\langle X_2,s(i;k)\rangle$. By Lemma 2.1, we have $G_3=D_2\rtimes G_2\cong G(q,1,n)$. Hence $G:=\langle X\rangle=D\rtimes G_2$. We see that $G=G(m,p,n)$ if and only if $D=A(m,p,n)$. We know that $A(m,p,n)$ is the set of all the diagonal matrices of the form $[(\zeta^{l_1},...,\zeta^{l_n})|\mathbf{1}]$ with $p|\sum_{i=1}^n l_i$. It is easily seen that $D=A(m,p,n)$ if and only if $\gcd\{d,p\}=1$ if and only if $\gcd\{p,\delta(X)\}=1$. So our result follows by Lemma 2.2. □

Remark 2.5. Under the assumption of Theorem 2.4, if $p=1$ then the condition " $\gcd\{p,\delta(X)\}=1$ " trivially holds; on the other extreme, if $p=m$ then this condition becomes $\gcd\{m,\delta(X)\}=1$. By [7, Theorems 2.8 and 2.19], we see that Theorem 2.4 also holds in the case of $p=1,m$, provided that the sentence " X is the generator set of a presentation of $G(m,p,n)$ "is replaced by " X is a generator set of $G(m,p,n)$ ".

2.6. Let $\Sigma(m,p,n)$ be the set of all the reflection sets S, where each of those comes from a presentation (S,P) of the group $G(m,p,n)$. By Theorem 2.4, we know that any $S\in\Sigma(m,p,n)$ has a decomposition $S=S_1\cup\{s(i;k)\}$, where S_1 consists of n reflections of type I with Γ_{S_1} connected and $\gcd\{\delta(S_1),p\}=1$, and $s(i;k)$ satisfies $1\leqslant i\leqslant n$ and $\gcd\{k,q\}=1$, $q=m/p$. Thus we can define a rooted graph Γ_S^r for any $S\in\Sigma(m,p,n)$. Let $S_1=\{t_h\mid 1\leqslant h\leqslant n\}$ and $s=s(i;k)$. For $1\leqslant h\leqslant n$, denote by $e(t_h)$ the edge of Γ_S^r corresponding to t_h. Then we have the following relations:

(i) $s^q = 1$;

(ii) $t_h^2 = 1$ for $1 \leqslant h \leqslant n$;

(iii) $t_h t_l = t_l t_h$ if the edges $e(t_h)$ and $e(t_l)$ have no common end node;

(iv) $t_h t_l t_h = t_l t_h t_l$ if the edges $e(t_h)$ and $e(t_l)$ have exactly one common end node;

(v) $(t_h t_l)^{m/d} = 1$ if $t_h \neq t_l$ with $e(t_h)$ and $e(t_l)$ having two common end nodes, where $d = \gcd\{m, \delta(S)\}$;

(vi) $st_h st_h = t_h st_h s$ if i is an end node of $e(t_h)$;

(vii) $st_h = t_h s$ if i is not an end node of $e(t_h)$;

We call (i)-(ii) the *order relations* on S, and (iii)-(vii) the *braid relations* on S.

We see that the congruence of $S \in \Sigma(m, p, n)$ is entirely determined by the order and braid relations (or briefly, the *o.b. relations*) on S, the latter are determined in turn by the rooted graph Γ_S^r except for the case where Γ_S^r contains a two-nodes circle. In this exceptional case, they are determined by the graph Γ_S^r together with the value $\gcd\{m, \delta(S)\}$. So we have the following

Lemma 2.7. *Let $S, S' \in \Sigma(m, p, n)$.*

(1) Suppose that Γ_S^r contains a circle with more than two nodes. Then S and S' are congruent if and only if $\Gamma_S^r \cong \Gamma_{S'}^r$;

(2) Suppose that Γ_S^r contains a two-nodes circle. Then S and S' are congruent if and only if $\Gamma_S^r \cong \Gamma_{S'}^r$ and $\gcd\{m, \delta(S)\} = \gcd\{m, \delta(S')\}$.

2.8. Denote by $\Lambda(m, p)$ the set of all the numbers $d \in \mathbb{N}$ such that $d|m$ and $\gcd\{d, p\} = 1$. Let $\Gamma(m, p, n)$ be the set of all the connected rooted graphs with n nodes and n edges. Let $\Gamma_1(m, p, n)$ be the set of all the rooted graphs in $\Gamma(m, p, n)$, where each of those contains a two-nodes circle. Let $\Gamma_2(m, p, n)$ be the complement of $\Gamma_1(m, p, n)$ in $\Gamma(m, p, n)$. Denote by $\widetilde{\Gamma}(m, p, n)$ (resp., $\widetilde{\Gamma}_i(m, p, n)$) the set of the isomorphism classes in the set $\Gamma(m, p, n)$ (resp., $\Gamma_i(m, p, n)$) for $i = 1, 2$ (see 1.6).

Let $\widetilde{\Sigma}(m, p, n)$ be the set of congruence classes in $\Sigma(m, p, n)$. Then the following is the first main result of the paper, which describes all the congruence classes of presentations for $G(m,p,n)$ in terms of rooted graphs.

Theorem 2.9. *The map* $\psi : S \mapsto \Gamma_S^r$ *from* $\Sigma(m,p,n)$ *to* $\Gamma(m,p,n)$ *induces a surjection (denoted by* $\widetilde{\psi}$*) from the set* $\widetilde{\Sigma}(m,p,n)$ *to the set* $\widetilde{\Gamma}(m,p,n)$*. Denote* $\widetilde{\Sigma}_i(m,p,n) := \widetilde{\psi}^{-1}(\widetilde{\Gamma}_i(m,p,n))$ *for* $i = 1, 2$*. Then the map* $\widetilde{\psi}$ *gives rise to a bijection from* $\widetilde{\Sigma}_2(m,p,n)$ *to* $\widetilde{\Gamma}_2(m,p,n)$*; and also the map* $S \mapsto (\Gamma_S^r, \gcd\{m, \delta(S)\})$ *induces a bijection from* $\widetilde{\Sigma}_1(m,p,n)$ *to* $\widetilde{\Gamma}_1(m,p,n) \times \Lambda(m,p)$*.*

Proof. This follows by Theorem 2.4 and Lemma 2.7. □

Remark 2.10. (1) By Theorem 2.9, we have an effective way to find a representative S for any given congruence class in $\Sigma(m,p,n)$. Fix a connected rooted graph $\Gamma^r = ([n], E, a)$ with $|E| = n$ (hence Γ^r contains a unique circle) and a number $k \in \Lambda(m,p)$. We choose an arrow (say (h,l)) on the circle of Γ^r and take $S = \{t(h,l;k), s(a;1), t(i,j;0) \mid (i,j) \in E \setminus \{(h,l)\}\}$. Then we see by Theorem 2.4 that S is in $\Sigma(m,p,n)$ with $\Gamma_S^r \cong \Gamma^r$ and $\delta(S) = k$. If the circle of Γ^r contains more than two nodes, then the congruence class of S is determined by Γ^r alone. If Γ^r contains a two-nodes circle, then the congruence class of S is determined by both Γ^r and k.

(2) Comparing with Theorem 2.9, we showed in [7] the following results concerning the congruence classes of presentations for the groups $G(m,1,n)$ and $G(m,m,n)$:

(a) The map $(S,P) \to \Gamma_S^r$ induces a bijection from the set of all the congruence classes of presentations for the group $G(m,1,n)$ to the set of isomorphism classes of rooted trees with n nodes (see [7, Theorem 3.2]).

(b) The map $(S,P) \to \Gamma_S$ induces a bijection from the set of all the congruence classes of presentations for the group $G(m,m,n)$ to the set of isomorphism classes of connected graphs with n nodes and n edges (or equivalently with n nodes and exactly one circle) (see [7, Theorem 3.4]).

§3. Circle operations on the set $\Sigma(m,p,n)$.

In the subsequent sections of the paper, we want to find, for any $S \in \Sigma(m,p,n)$, a relation set P on S such that the pair (S,P) forms a presentation of $G(m,p,n)$. A crucial tool to do this is an operation, called a *circle operation* on the set $\Sigma(m,p,n)$. We shall introduce such an operation in the present section.

3.1. Assume that X is a reflection set of $G(m,p,n)$ such that Γ_X contains exactly one circle, say the edges of the circle are $\{c_h, c_{h+1}\}$, $1 \leqslant h \leqslant r$ (the subscripts are modulo r) for some integer $2 \leqslant r \leqslant n$. Then X contains the reflections $s(c_h, c_{h+1}; k_h)$ with some integers k_h, $1 \leqslant h \leqslant r$ (the subscripts are modulo r). Denote $\delta(X) := |\sum_{h=1}^{r} k_h|$. We see that $\delta(X)$ is independent of the choice of an orientation for the circle in Γ_X.

3.2. Suppose that Γ_X in 3.1 also contains an edge $\{c_0, c_1\}$ with $c_0 \neq c_2, c_r$. Hence X contains a reflection $s(c_0, c_1; k_0)$ for some $k_0 \in \mathbb{Z}$. Let $Y = (X \setminus \{s(c_r, c_1; k_r)\}) \cup \{s(c_r, c_0; k_r - k_0)\}$ Then the graph Γ_Y can be obtained from Γ_X by replacing the edge $\{c_r, c_1\}$ by $\{c_r, c_0\}$. We see that the graph Γ_Y also contains exactly one circle with $\delta(Y) = \delta(X)$. We call the transformation $X \mapsto Y$ *a circle expansion* and the reverse transformation $Y \mapsto X$ *a circle contraction.* We call both transformations *circle operations.* It is easily seen that the graph Γ_X is connected if and only if so is Γ_Y. Since $s(c_r, c_0; k_r - k_0) = s(c_0, c_1; k_0)s(c_r, c_1; k_r)s(c_0, c_1; k_0)$, we have $\langle Y \rangle = \langle X \rangle$.

We see that a circle contraction on X is applicable whenever X has a circle with at least three nodes. Also, a circle expansion on X is applicable whenever there exist a circle and an edge in Γ_X which have a unique common end node.

Recall the notation $\Sigma(m,p,n)$ defined in 2.6. The following result shows that a circle operation, when applicable, stabilizes the set $\Sigma(m,p,n)$.

Lemma 3.3. *For $X \in \Sigma(m,p,n)$, let Y be obtained from X by a sequence of circle operations. Then $\langle Y \rangle = \langle X \rangle$ and $\delta(Y) = \delta(X)$. Hence $Y \in \Sigma(m,p,n)$.*

Proof. This follows by the above discussion and by Theorem 2.4. □

Next two results are concerned with the action of circle operations on $\Sigma(m,p,n)$.

Lemma 3.4. *Any* $X \in \Sigma(m,p,n)$ *can be transformed to some* X' *in* $\Sigma(m,p,n)$ *by applying a sequence of circle expansions so that the graph* $\Gamma_{X'}$ *becomes a circle.*

Proof. We know by Lemma 2.2 that the graph Γ_X is connected and contains exactly one circle. If Γ_X is itself a circle then there is nothing to do. Otherwise, let $c_1, c_2, ..., c_r$ be the nodes on the circle of Γ_X such that X contains reflections $t_h = s(c_h, c_{h+1}; k_h)$ for any $1 \leqslant h \leqslant r$ (the subscripts are modulo r) and some $k_h \in \mathbb{Z}$. Then X also contains a reflection $t = s(c_j, c; k)$ for some $1 \leqslant j \leqslant r$, $c \in \{1, 2, ..., n\} \setminus \{c_1, ..., c_r\}$ and $k \in \mathbb{Z}$. Let $s = tt_{j-1}t$ and let $X'' = (X \setminus \{t_{j-1}\}) \cup \{s\}$. Then X'' is obtained from X by a circle expansion with $\delta(X'') = \delta(X)$. Hence $X'' \in \Sigma(m,p,n)$ by the assumption $X \in \Sigma(m,p,n)$ and by Theorem 2.4. There are $r+1$ nodes on the circle of $\Gamma_{X''}$. By induction on $n - r \geqslant 0$, we can eventually transform X to some $X' \in \Sigma(m,p,n)$ with $\Gamma_{X'}$ a circle by successively applying circle expansions on X. □

Lemma 3.5. *Any* $X \in \Sigma(m,p,n)$ *can be transformed to some* X' *in* $\Sigma(m,p,n)$ *by a sequence of circle operations such that the graph* $\Gamma^r_{X'}$ *is a string with a two-nodes circle at one end and with the rooted node on the circle and not adjacent to any node outside the circle.*

Proof. By Lemma 3.4, we may assume without loss of generality that the graph Γ_X is a circle, say $c_1, c_2, ..., c_n$ are nodes of Γ_X such that X consists of the reflections $t_h = s(c_h, c_{h+1}; k_h)$ for $1 \leqslant h \leqslant n$ (the subscripts are modulo n) and $s = s(c_1; k)$. Let $t'_{n-1} = t_n t_{n-1} t_n$ and $t'_j = t'_{j+1} t_j t'_{j+1}$ for $2 \leqslant j < n-1$. Let $X_{n-1} = (X \setminus \{t_n\}) \cup \{t'_{n-1}\}$ and $X_j = (X_{j+1} \setminus \{t'_{j+1}\}) \cup \{t'_j\}$ for $2 \leqslant j < n-1$. Denote $X_n = X$. Then X_j is obtained from X_{j+1} by a circle contraction and $X_j \in \Sigma(m,p,n)$ for $2 \leqslant j < n$. Hence $X' = X_2$ is a required element in $\Sigma(m,p,n)$. □

§4. The basic relations on any $S \in G(m,p,n)$.

In the present section, we introduce the concept of basic relations on any $S \in \Sigma(m,p,n)$ and discuss some relations among these basic relations.

4.1. It is well known that the group $G(m, p, n)$ has a presentation (S, P), where $S = \{s(h, h+1; 0), s(1, 2; 1), s(1; 1) \mid 1 \leqslant h < n\}$, and P consists of the following relations: denote $t_h = s(h, h+1; 0)$, $1 \leqslant h < n$, $t'_1 = s(1, 2; 1)$, $s = s(1; 1)$, and $q = m/p$.

(i) $s^q = 1$;

(ii) $t_h^2 = t'^2_1 = 1$ for $1 \leqslant h < n$;

(iii) $t_i t_j = t_j t_i$ if $j \neq i \pm 1$;

(iv) $t_i t_{i+1} t_i = t_{i+1} t_i t_{i+1}$ for $1 \leqslant i < n-1$;

(v) $st_i = t_i s$ for $i > 1$;

(vi) $t'_1 t_1 t_2 t'_1 t_1 t_2 = t_2 t'_1 t_1 t_2 t'_1 t_1$;

(vii) $t'_1 t_i = t_i t'_1$ for $i > 2$;

(viii) $t'_1 t_2 t'_1 = t_2 t'_1 t_2$;

(ix) $st'_1 t_1 = t'_1 t_1 s$;

(x) $t_1 s t'_1 t_1 t'_1 t_1 ...... = s t'_1 t_1 t'_1 t_1 ......$, where each side contains $p+1$ factors.

Relation (x) can be rewritten as

$$(t'_1 t_1)^{p-1} = s^{-1} t_1 s t'_1. \tag{4.1.1}$$

By (ix), this implies that $(t'_1 t_1)^{p-1} = t_1 s t'_1 s^{-1}$ and hence $s^{-1} t_1 s t'_1 = t_1 s t'_1 s^{-1}$, i.e.,

$$t_1 s t_1 s = s t'_1 s t'_1. \tag{4.1.2}$$

So $t_1 s t_1 s t_1 = s t'_1 s t'_1 t_1 = s t_1 s$ by (ix). We get

$$t_1 s t_1 s = s t_1 s t_1. \tag{4.1.3}$$

By (4.1.2) and (4.1.3), we further deduce

$$t'_1 s t'_1 s = s t'_1 s t'_1. \tag{4.1.4}$$

More generally, we can show that for any $a, b \in \mathbb{Z}$, the relations

(4.1.5) $$t_1' s^a t_1' s^b = s^b t_1' s^a t_1' \quad \text{and} \quad t_1 s^a t_1 s^b = s^b t_1 s^a t_1$$

hold.

By (i), (ix), (x) and (4.1.5), we have

$$(t_1' t_1)^{(p-1)q} = (s^{-1} t_1 s t_1')^q = s^{-q} t_1 s^q t_1' (t_1 t_1')^{q-1} = (t_1 t_1')^q.$$

Then we get

(4.1.6) $$(t_1' t_1)^m = 1.$$

4.2. Recall that in Section 2 we listed all the o.b. relations on any $S \in \Sigma(m, p, n)$. Let $S = \{s = s(a; k), t_h \mid 1 \leqslant h \leqslant n\}$. Note that by Lemma 2.2, all the t_h's are reflections of type I. Also, a is the rooted node of Γ_S^r. In the present section, we shall list some more relations on S.

(A) $s^{m/p} = 1$;

(B) $t_i^2 = 1$ for $1 \leqslant i \leqslant n$;

(C) $t_i t_j = t_j t_i$ if the edges $e(t_i)$ and $e(t_j)$ have no common end node;

(D) $t_i t_j t_i = t_j t_i t_j$ if the edges $e(t_i)$ and $e(t_j)$ have exactly one common end node;

(E) $(t_i t_j)^{m/d} = 1$ if $t_i \neq t_j$ with $e(t_i)$ and $e(t_j)$ having two common end nodes, where $d = \gcd\{m, \delta(S)\}$ (comparing with relation (4.1.6) by noting that the edges $e(t_1)$ and $e(t_1')$ have two common end nodes, and $\delta(S) = 1$ in (4.1.6));

(F) $s t_i s t_i = t_i s t_i s$ if a is an end node of $e(t_i)$ (comparing with relations (4.1.3)-(4.1.4));

(G) $s t_i = t_i s$ if a is not an end node of $e(t_i)$;

(H) $t_i \cdot t_j t_l t_j = t_j t_l t_j \cdot t_i$ for any triple $X = \{t_i, t_j, t_l\} \subseteq S$ with Γ_X having a branching node (by a branching node, we mean a node of Γ_X such that there are more than two other nodes of Γ_X connecting this node by edges);

(I) $s \cdot t_i t_j t_i = t_i t_j t_i \cdot s$, if $e(t_i)$ and $e(t_j)$ have exactly one common end node a (note that the node a is rooted);

We call relations (A)-(B) the *order relations*, (C)-(G) the *braid relations*, (H) the *branching relations*, and (I) the *root-braid relations* on S.

4.3. Given $S \in \Sigma(m,p,n)$ and take any node a of Γ_S, there exists a sequence of nodes $\xi_a : a_0 = a, a_1, ..., a_r = a$ with $r > 1$ and $a_h \neq a_{h+1}$ for $0 \leqslant h < r$ such that S contains reflections $t_h = s(a_{h-1}, a_h; k_h)$ for $1 \leqslant h \leqslant r$ with some integers k_h, where $t_l \neq t_{l+1}$ for $1 \leqslant l < r$. Since the graph Γ_S is connected and contains a unique circle, the sequence ξ_a always exists, which contains all the nodes on the circle of Γ_S and is uniquely determined by the set S and the node a up to an orientation of the circle. We call ξ_a a *generalized circle sequence* (or *g.c.s.* in short) of S at the node a for a fixed orientation of the circle of Γ_S. In particular, when a is the rooted node of Γ_S^r, ξ_a is also called a *root-circle sequence* of S. It is easily seen that the node a is on the circle of Γ_S if and only if $t_1 \neq t_r$.

Let c, c' be the smallest, resp., the largest integer with the node a_c, resp., $a_{c'}$ lying on the circle of Γ_S. Then a is on the circle of Γ_S if and only if $c = 0$ and $c' = r$. Denote by s_{hj} the element $t_h t_{h+1} ... t_{j-1} t_j t_{j-1} ... t_h$ for $1 \leqslant h < j \leqslant r$.

The following relation is called a *circle relation* on S (at the node pair $\{a, a_j\}$):

(J) $(s_{1j} s_{j+1,r})^{\frac{m}{\gcd\{m,\delta(S)\}}} = 1$.

Note that relation (E) can be regarded as a special case of (J), where the circle of Γ_S contains only two nodes and the node a is on the circle.

Assume that a is the rooted node of Γ_S^r. Then the relation (K) below is called a *root-circle relation* on S (at the node a_j).

(K) $s s_{1j} s_{j+1,r} = s_{1j} s_{j+1,r} s$,

and the relation (L) below is called a *circle-root relation* on S (at the node a_j).

(L) $(s_{j+1,r} s_{1j})^{p-1} = s^{-\delta(S)} s_{1j} s^{\delta(S)} s_{j+1,r}$.

In any of the above relations (J)-(L), the integer j is required to satisfy $c < j < c'$, i.e., the node a_j is on the circle of Γ_S but is not the node at the entry for the path from the node a to the circle. Then $\{a, a_j\}$ is called an *admissible node pair* of Γ_S^r, at which we are allowed to talk about relations (J)-(L) on S, where a is required to be the rooted node of Γ_S^r for relations (K)-(L).

The following relations are called the *branching-circle relations* on S (at the nodes a, a_j for (M) (a), at the node a for (M) (b) and at the node a_j for (M) (c)):

(M) (a) $us_{1j}u \cdot vs_{j+1,r}v = vs_{j+1,r}v \cdot us_{1j}u$,

(b) $us_{1j}s_{j+1,r}us_{1j}s_{j+1,r} = s_{1j}s_{j+1,r}us_{1j}s_{j+1,r}u$, and

(c) $vs_{1j}s_{j+1,r}vs_{1j}s_{j+1,r} = s_{1j}s_{j+1,r}vs_{1j}s_{j+1,r}v$,

if there are some $u, v \in S$ with $e(u)$, $e(v)$ incident to the g.c.s. ξ_a of Γ_S at the nodes a, a_j respectively for some $c < j < c'$ with c, c' defined as above.

We call all the relations (A)-(M) above the *basic relations* on S.

In the remaining part of the section, we always assume that the o.b. relations on S hold.

4.4. Note that the validity of relations (J), (K) and (M) on S at an admissible node pair $\{a, a_j\}$ is independent of the choice of an orientation of the circle of Γ_S in the following sense: any of such relations is true for one orientation of the circle if and only if it is true for the other orientation of the circle. The reasons for this are based on the following facts:

(1) $(s_{1j}s_{j+1,r})^{-1} = s_{j+1,r}s_{1j}$;

(2) anyone of s^{-1} and s can be expressed as a positive power of the other.

(3) $uv = vu$ in the case of (M)(a).

However, the relations (L) on S may hold only for one orientation of the circle of Γ_S.

When $c+1 < j < c'$ (resp., $c < j < c'-1$), by left-multiplying and right-multiplying simultaneously both sides of (J) by the reflection t_j (resp., t_{j+1}), we get the corresponding circle relation on S at the node pair $\{a, a_{j-1}\}$ (resp., $\{a, a_{j+1}\}$). This implies that the circle relation on S holds at one node pair $\{a, a_j\}$ for some $c < j < c'$ if and only if they hold at the node pairs $\{a, a_j\}$ for all $c < j < c'$. Similar assertion is true concerning the relations (K) and (L) on S.

Assume $c > 0$. Thus we have a g.c.s. $\xi_{a_1} : a_1, a_2, ..., a_{r-1} = a_1$ of Γ_S at the node a_1. We can talk about the circle relations on S at the node pair $\{a_1, a_j\}$ for $c < j < c'$. Since $s_{2j}s_{j+1,r-1} = t_1 \cdot s_{1j}s_{j+1,r} \cdot t_1$, we see that the circle relation $(s_{1j}s_{j+1,r})^{\frac{m}{\gcd\{m,\delta(S)\}}} = 1$ holds if and only if $(s_{2j}s_{j+1,r-1})^{\frac{m}{\gcd\{m,\delta(S)\}}} = 1$ holds.

Next assume $c = 0$. Then a is on the circle of Γ_S and hence all the node pairs $\{a_i, a_j\}$, $0 \leqslant i \neq j < r$, are admissible for Γ_S. It is easily seen that the circle relations on S at all these node pairs are mutually equivalent.

The above discussion implies the following

Lemma 4.5. *Assume that $S \in \Sigma(m, p, n)$ satisfies all the o.b. relations. Then relation (J) on S holds at one admissible node pair if and only if it holds at all the admissible node pairs.*

The above discussion also implies the following

Lemma 4.6. *Assume that $S \in \Sigma(m, p, n)$ satisfies all the o.b. relations. Then in the setup of 4.3 with a the rooted node of Γ_S^r, relations (K) (resp., (L)) on $S \in \Sigma(m, p, n)$ for different j, $c < j < c'$, are mutually equivalent.*

Next result asserts that relation (J) is a consequence of some other basic relations.

Lemma 4.7. *In the setup of 4.3 with a the rooted node of Γ_S^r, the o.b. relations together with relations (K) and (L) on S imply relation (J) on S.*

Proof. Denote $\delta = \delta(S)$, $q = \frac{m}{p}$ and $d = \gcd\{m, \delta\}$. The o.b. relations on S implies

$$ss_{1j}ss_{1j} = s_{1j}ss_{1j}s \quad \text{and} \quad ss_{j+1,r}ss_{j+1,r} = s_{j+1,r}ss_{j+1,r}s. \tag{4.7.1}$$

This, together with the o.b. relations and relations (K), (L) on S, implies

$$(s_{j+1,r}s_{1j})^{(p-1)\frac{q}{d}} = (s^{-\delta}s_{1j}s^{\delta}s_{j+1,r})^{\frac{q}{d}} = s^{-\frac{q\delta}{d}}s_{1j}s^{\frac{q\delta}{d}}s_{j+1,r}(s_{1j}s_{j+1,r})^{\frac{q}{d}-1} = (s_{1j}s_{j+1,r})^{\frac{q}{d}}.$$

So we get relation (J): $(s_{1j}s_{j+1,r})^{\frac{m}{d}} = 1$ on S. □

Next two results are concerned with the branching relations (H) and the branching-circle relations (M) on $S \in \Sigma(m,p,n)$, which can be shown by the same arguments as those in [7].

Lemma 4.8. (see [7, Lemma 4.8]) *For $S \in \Sigma(m,p,n)$, assume that all the o.b. relations on S hold. For any branching node v of Γ_S, fix some $t_v \in S$ of type I with $e(t_v)$ incident to v. Then the branching relations (H) on S is equivalent to the following relations:*

(H′) The relation $t_v \cdot tt't = tt't \cdot t_v$ holds for any $t \neq t'$ in $S \setminus \{t_v\}$ of type I with $\Gamma_{\{t_v,t,t'\}}$ having v as a branching node.

Lemma 4.9. (see [7, Lemma 4.12]) *For $S \in \Sigma(m,p,n)$ with the circle of Γ_S containing more than two nodes, the branching-circle relations (M) on S are a consequence of the o.b. and branching relations on S.*

Now consider the root-braid relations (I) on $S \in \Sigma(m,p,n)$.

Lemma 4.10. *For $S \in \Sigma(m,p,n)$, assume that all the o.b. and branching relations on S hold and that the rooted node v of Γ_S^r (hence $s = s(v;k) \in S$ for some $k \in \mathbb{Z}$) is also a branching node. Fix some $t_v \in S$ of type I with $e(t_v)$ incident to v. Then the root-braid relations (I) on S are equivalent to the following relations:*

(I′) $s \cdot t_v t t_v = t_v t t_v \cdot s$ for any $t \in \Gamma_S \setminus \{t_v\}$ of type I with $e(t)$, $e(t_v)$ having just one common end node v.

Proof. It is clear that relations (I) imply (I′). Now assume relations (I′). We have to show the relation $s \cdot tt't = tt't \cdot s$ for any $t' \neq t$ in $S \setminus \{t_v\}$ of type I with $e(t')$ and $e(t)$ having just one common end node v. Indeed, we have

$$s \cdot tt't = tt't \cdot s \iff s \cdot t_v tt'tt_v = t_v tt'tt_v \cdot s \iff s \cdot t_v tt_v \cdot t_v t't_v \cdot t_v tt_v = t_v tt_v \cdot t_v t't_v \cdot t_v tt_v \cdot s,$$

where the two equivalences follow by a branching relation, resp., an order relation on S, while the last equation is a consequence of (I$'$). $\square$

§5. Equivalence of the basic relation sets under circle operations.

In the present section, we want to show that if $S, S' \in \Sigma(m,p,n)$ can be obtained from one to another by a circle operation then S satisfies all the basic relations if and only if so does S'.

5.1. Keep the setup of 4.3 on $S \in \Sigma(m,p,n)$: the node a, the g.c.s. ξ_a, the numbers c, c', j with $c < j < c'$ in Γ_S, and the reflections s, t_h for $1 \leqslant h \leqslant r$. Assume that a is the rooted node of Γ_S^r. Let $S' \in \Sigma(m,p,n)$ be obtained from S by a circle operation such that the node a_j is still on the circle of $\Gamma_{S'}$. Then a is also the rooted node of $\Gamma_{S'}^r$. Concerning the root-circle relations (K) and the circle-root relations (L) on both S and S', we need only consider the following five cases:

(1) $S' = (S \setminus \{t_h\}) \cup \{t\}$, where $c+1 < h \leqslant j$ and $t = t_h t_{h-1} t_h$.

(2) $S' = (S \setminus \{t_h\}) \cup \{t\}$, where $c < h < j$ and $t = t_h t_{h+1} t_h$.

(3) $S' = (S \setminus \{t_{c+1}\}) \cup \{t\}$, where $t = t_{c+1} t_{c'} t_{c+1}$.

(4) $S' = (S \setminus \{t_{c+1}\}) \cup \{t\}$, where $t = t_c t_{c+1} t_c$.

(5) $S' = (S \setminus \{t_h\}) \cup \{t\}$, where $c < h \leqslant j$, $t = t_h t' t_h$ for some $t' = s(a', a''; k') \in S$ with $|\{a', a''\} \cap \{a_{h-1}, a_h\}| = 1$.

In any of the above five cases, one can check easily the equations $s_{a,a_j} = s'_{a,a_j}$, $s_{a_j,a} = s'_{a_j,a}$ and $\delta(S) = \delta(S')$ by assuming the o.b. relations and the branching relations on both S and S', where $s_{a,a_j} := s_{1j}$ and $s_{a_j,a} := s_{j+1,r}$; and then s'_{a,a_j} (resp., $s'_{a_j,a}$) is defined for S' in the same way as s_{a,a_j} (resp., $s_{a_j,a}$) for S in 4.3. This implies that at the node a_j, the root-circle relation (K) (resp., the circle-root relation (L)) on S at the node a_j holds if and only if the corresponding root-circle relation (resp., circle-root relation) on S' at the node a_j holds.

The following examples illustrate the above discussion.

Examples 5.2. Assume that $S \in \Sigma(m,p,n)$ contains $\{s = s(1;h), t_i = s(a_{i-1}, a_i; h_i) \mid 1 \leqslant i \leqslant 6\}$ for some $h, h_i \in \mathbb{Z}$ such that $(a_0, a_1, ..., a_6) = (a, b, c, d, e, b, a)$ is a root-circle sequence of S. Then b, c, d, e are the nodes on the circle of Γ_S^r and $t_1 = t_6$. We have $s_{a,d} = t_1t_2t_3t_2t_1$ and $s_{d,a} = t_4t_5t_6t_5t_4$

(i) Let $t = t_2t_3t_2$ and $S^{(1)} = (S \setminus \{t_3\}) \cup \{t\}$. Then $S^{(1)}$ is obtained from S by a circle contraction. A root-circle sequence for S^1 is $(a_0^{(1)}, ..., a_5^{(1)}) = (a, b, d, e, b, a)$ with the nodes b, d, e on the circle of $\Gamma_{S^{(1)}}^r$. we have $s_{a,d}^1 = t_1tt_1$ and $s_{d,a}^{(1)} = t_4t_5t_6t_5t_4$.

(ii) Let $t = t_1t_2t_1$ and $S^{(2)} = (S \setminus \{t_2\}) \cup \{t\}$. Then $S^{(2)}$ is obtained from S by a circle expansion. A root-circle sequence for $S^{(2)}$ is $(a_0^{(2)}, ..., a_5^{(2)}) = (a, c, d, e, b, a)$ all of whose terms are on the circle of $\Gamma_{S^{(2)}}^r$. We have $s_{a,d}^{(2)} = tt_3t$ and $s_{d,a}^{(2)} = t_4t_5t_6t_5t_4$.

(iii) Let $t = t_2t_5t_2$ and $S^{(3)} = (S \setminus \{t_2\}) \cup \{t\}$. Then $S^{(3)}$ is obtained from S by a circle contraction. A root-circle sequence for $S^{(3)}$ is $(a_0^{(3)}, ..., a_7^{(3)}) = (a, b, e, c, d, e, b, a)$ with c, d, e on the circle of $\Gamma_{S^{(3)}}^r$. We have $s_{a,d}^{(3)} = t_1t_5tt_3tt_5t_1$ and $s_{d,a}^{(3)} = t_4t_5t_6t_5t_4$.

We see that a is the rooted node in any of Γ_S^r and $\Gamma_{S^{(i)}}^r$, $i = 1, 2, 3$. Then by the facts of $s_{a,d} = s_{a,d}^{(i)}$, $s_{d,a} = s_{d,a}^{(i)}$ and $\delta(S) = \delta(S^{(i)})$, $i = 1, 2, 3$, it is easily seen that at the node d, the circle-root relation $(s_{d,a}s_{a,d})^{p-1} = s^{-\delta(S)}s_{a,d}s^{\delta(S)}s_{d,a}$ on S holds if and only if the circle-root relation $(s_{d,a}^{(i)}s_{a,d}^{(i)})^{p-1} = s^{-\delta(S^{(i)})}s_{a,d}^{(i)}s^{\delta(S^{(i)})}s_{d,a}^{(i)}$ on $S^{(i)}$ holds for any $i = 1, 2, 3$. Also, at the node d, the root-circle relation $ss_{a,d}s_{d,a} = s_{a,d}s_{d,a}s$ on S holds if and only if the root-circle relation $ss_{a,d}^{(i)}s_{d,a}^{(i)} = s_{a,d}^{(i)}s_{d,a}^{(i)}s$ on $S^{(i)}$ holds for any $i = 1, 2, 3$.

Hence we get the following

Lemma 5.3. *Assume that $S, S' \in \Sigma(m,p,n)$ can be obtained from one to the other by a circle operation. Then*

(1) Γ_S^r and $\Gamma_{S'}^r$ have the same rooted node, say a.

(2) S satisfies the circle-root relation (L) (resp., the root-circle relation (K)) at a node v if and only if S' satisfies the corresponding circle-root (resp., root-circle) relation

at the node v, provided that $\{a, v\}$ is an admissible node pair of both Γ_S and $\Gamma_{S'}$.

5.4. Suppose that $S \in \Sigma(m, p, n)$ contains the reflections $t_h = s(c_h, c_{h+1}; k_h)$ (the subscripts are modulo r) for $1 \leqslant h \leqslant r$ and some integers k_h, where $r > 2$, and $c_1, ..., c_r$ are the nodes on the circle of Γ_S. Let $t = t_1 t_r t_1$ and let $S' = (S \setminus \{t_r\}) \cup \{t\}$. Then S' is obtained from S by a circle contraction.

Proposition 5.5. *In the above setup, the reflection set S satisfies all the basic relations (see 4.2–4.3) if and only if so does the reflection set S'.*

Proof. Under the assumption of the o.b. and branching relations on both S and S', the root-circle relations (K) (resp., the circle-root relations (L)) on S' are equivalent to those on S by Lemma 5.3.

(I) First assume S satisfies all the basic relations. We want to show that S' also satisfies all the basic relations. Since any basic relation on S' not involving t is just a basic relation on S, we need only to check all the basic relations on S' involving t. Note $e(t) = \{c_2, c_r\}$.

The order relation $t^2 = 1$ follows by the order relations $t_1^2 = 1 = t_r^2$ on S.

Let $s \in S' \setminus \{t\}$ be of type I with the edge $e(s)$ not incident to $e(t)$. We must show $st = ts$. We see that $e(s)$ is incident to either both or none of $e(t_1), e(t_r)$. The result is obvious if $e(s)$ is incident to none of $e(t_1), e(t_r)$. In the case when $e(s)$ is incident to both $e(t_1)$ and $e(t_r)$, we see that c_1 is a branching node of Γ_S to which the edges $e(t_1), e(t_r), e(s)$ are incident. Then we have $ts = t_1 t_r t_1 s = s t_1 t_r t_1 = st$ by a branching relation on S.

Let $s \in S' \setminus \{t\}$ be of type I with $e(s)$ incident to $e(t)$ at exactly one node in $\Gamma_{S'}$. We want to show $sts = tst$, i.e., $s t_1 t_r t_1 s = t_1 t_r t_1 s t_1 t_r t_1$. This can be shown by the o.b. relations on S and by the fact that either the relations $st_1 = t_1 s$, $st_r s = t_r s t_r$, or $st_1 s = t_1 s t_1$, $st_r = t_r s$ hold. When $r = 3$, $e(t_2)$ and $e(t)$ form the two-nodes circle of $\Gamma_{S'}$. The braid relation $(t_2 t)^{m/d} = 1$ on S' is the same as the circle relation $(t_2 t_1 t_3 t_1)^{m/d} = 1$

on S since $\gcd\{m,\delta(S')\}=d=\gcd\{m,\delta(S)\}$.

Let $s\in S'\setminus\{t\}$ be of type II. When $e(t)$ is not incident to the rooted node of $\Gamma^r_{S'}$, relation $st=ts$ follows by relations $t_1s=st_1$ and $t_rs=st_r$ on S if the node c_1 is not rooted, or by the root-braid relation $t_1t_rt_1s=st_1t_rt_1$ on S if c_1 is rooted. Now assume that $e(t)$ is incident to the rooted node of $\Gamma_{S'}$, i.e., either the node c_2 or c_r is rooted. If c_2 is rooted then we have

$$stst=tsts\Longleftrightarrow st_rt_1t_rst_rt_1t_r=t_rt_1t_rst_rt_1t_rs\Longleftrightarrow st_1st_1=t_1st_1s.$$

The last equation is just a braid relation on S. So we get relation $stst=tsts$. Similarly we can show the relation $stst=tsts$ if the node c_r is rooted.

So we have shown all the o.b. relations on S' involving t.

Now we want to show the branching relations on S' involving t. If c_2 is a branching node in $\Gamma_{S'}$, then by Lemma 4.8, we need only show the relation $st_1tt_1=t_1tt_1s$ for any $s\in S'\setminus\{t_1,t\}$ of type I with $e(s)$ incident to c_2 and not to c_r. This follows by the relations $t_1^2=1$ and $st_r=t_rs$ on S. If c_r is a branching node in $\Gamma_{S'}$, then we must show the relation $tss's=ss'st$ for any $s,s'\in S'\setminus\{t\}$ of type I with $e(s)$, $e(s')$ incident to c_r and not to c_2. This follows by the braid relations $t_1s=st_1$, $t_1s'=s't_1$ and the branching relation $t_rss's=ss'st_r$ on S.

We need not check the circle relations on S' by Lemma 4.7.

Next we show the branching-circle relations on S' involving t. By Lemma 4.9, we need only consider the case of $r=3$. In this case, $e(t)$ and $e(t_2)$ form a two-nodes circle. If there exists some $u\in S'\setminus\{t,t_2\}$ of type I with $e(u)$ incident to c_2 then the branching-circle relation $ut_2tut_2t=t_2tut_2tu$ on S' is the same as the branching-circle relation $ut_2t_1t_3t_1ut_2t_1t_3t_1=t_2t_1t_3t_1ut_2t_1t_3t_1u$ on S. Similarly for the case when there exists some $v\in S'\setminus\{t,t_2\}$ of type I with $e(v)$ incident to c_r. If both of such u,v exist then the branching-circle relation $utuvt_2v=vt_2vutu$ on S' is also the same as the branching-circle relation $ut_1t_3t_1uvt_2v=vt_2vut_1t_3t_1u$ on S.

(II) Next assume all the basic relations on S'. We want to check all the basic relations on S. We need only deal with the ones involving $t_r = t_1 t t_1$. By the first paragraph of the proof and by Lemmas 4.8, 4.10, we need only check the following relations:

(1) $t_r^2 = 1$;

(2) $t_r s = s t_r$ for $s \in S \setminus \{t_r\}$ of type I with $e(s)$, $e(t_r)$ having no common end node;

(3) $t_r s t_r = s t_r s$ for $s \in S \setminus \{t_r\}$ of type I with $e(s)$, $e(t_r)$ having exactly one common end node;

(4) $s \cdot t_1 t_r t_1 = t_1 t_r t_1 \cdot s$ if $s \in S$ is of type I with $e(s)$ incident to the circle of Γ_S at the node c_1;

(5) $s \cdot t_r t_{r-1} t_r = t_r t_{r-1} t_r \cdot s$ if $s \in S$ is of type I with $e(s)$ incident to the circle of Γ_S at the node c_r.

(6) $s \cdot t_r t_{r-1} t_r = t_r t_{r-1} t_r \cdot s$ in the case of $s = s(c_r; k) \in S$.

(7) $s \cdot t_r t_1 t_r = t_r t_1 t_r \cdot s$ in the case of $s = s(c_1; k) \in S$.

(8) $s t_r s t_r = t_r s t_r s$ in either case (7) or (8).

The proof for the above relations is similar to what we did in part (I) and hence is left to the readers. Note that the branching-circle relations (M) on S is a consequence of the o.b. and branching relations on S by Lemma 4.9 and that the circle relation (J) on S is a consequence of the o.b. relations and relations (K), (L) on S by Lemma 4.7. Hence they need not be checked. $\square$

§6. The case of two-nodes circle containing the rooted node.

We shall show our second main result, i.e., Theorem 6.2, in the present section. To do so, we need the following result.

Proposition 6.1. *Let $S \in \Sigma(m, p, n)$ be such that Γ_S^r is a string with a two-nodes circle at one end and that the rooted node is on the circle and is not adjacent to any node outside the circle. Let P_S be the set of all the basic relations on S. Then (S, P_S) forms a presentation of $G(m, p, n)$.*

Let us first show our second main result under the assumption of Proposition 6.1.

Theorem 6.2. *Let $S \in \Sigma(m,p,n)$ and let P_S be the set of all the basic relations on S. Then (S,P) forms a presentation of the group $G(m,p,n)$.*

Proof. By Lemma 3.3, any $X \in \Sigma(m,p,n)$ can be transformed to some $X' \in \Sigma(m,p,n)$ by a sequence of circle operations, where $\Gamma^r_{X'}$ is a string with a two-nodes circle at one end and that the rooted node is on the circle and is not adjacent to any node outside the circle. By Proposition 6.1, we know that $(X', P_{X'})$ is a presentation of $G(m,p,n)$. Then by Proposition 5.5, this implies that (X, P_X) is a presentation of $G(m,p,n)$. □

We shall show Proposition 6.1 in the remaining part of the section.

[1, Appendix 2] tells us that the conclusion of Proposition 6.1 is true in the case of $\delta(S) = 1$ (see 4.1). Now we must show that our result holds in general case: $\delta(S) \in \mathbb{N}$ with $\gcd\{\delta(S), p\} = 1$.

6.3. Let $S = \{s, t'_1, t_h \mid 1 \leqslant h < n\} \in \Sigma(m,p,n)$ be given as in 4.1. So S satisfies all the relations (i)-(x) in 4.1 and hence also (4.1.1)-(4.1.6). Fix any $q \in \mathbb{N}$ with $\gcd\{p,q\} = 1$. Let $t' = (t'_1 t_1)^q t_1 = s(1,2;q)$ and let $X = (S \setminus \{t'_1\}) \cup \{t'\}$, i.e., from S, t'_1 is replaced by t'. Then $X \in \Sigma(m,p,n)$. It is easily seen that $\Gamma^r_X \cong \Gamma^r_S$ is a string with a two-nodes circle at one end: $e(t_1)$ and $e(t')$ are two edges of the circle of Γ^r_X with $\delta(X) = q$; the node corresponding to s is rooted which is on the circle and is not adjacent to any node outside the circle.

Lemma 6.4. *The reflection set X defined above satisfies relations (i)-(ix) in 4.1 with t' in the place of t'_1, and also the following relations:*

(x′) $(t't_1)^{p-1} = s^{-q} t_1 s^q t'$, *where $q = \delta(X)$ satisfies the condition* $\gcd\{p,q\} = 1$.

(xi′) $t'st's = st'st'$ *and* $t_1 s t_1 s = s t_1 s t_1$.

(xii′) $(t't_1)^{\frac{m}{d}}$, *where* $d = \gcd\{m,q\}$.

Proof. Relation ${t'}^2 = 1$ holds since t' is conjugate to t'_1 and ${t'_1}^2 = 1$. Then it remains to show the following relations:

(vi′) $t't_1t_2t't_1t_2 = t_2t't_1t_2t't_1$;

(vii′) $t't_i = t_it'$ for $i > 2$;

(viii′) $t't_2t' = t_2t't_2$;

(ix′) $st't_1 = t't_1s$;

and (x′)-(xii′).

(vi′) can be shown by repeatedly applying the relation:

$$t'_1t_1t_2(t'_1t_1)^kt_2 = t_2(t'_1t_1)^kt_2t'_1t_1 \qquad \text{for any } k \geqslant 0. \tag{6.4.1}$$

The latter can be shown by repeatedly applying 4.1 (vi).

(vii′) is an easy consequence of relations 4.1 (iii), (vii).

(viii′) is a special case of the following relation:

$$(t'_1t_1)^kt'_1t_2(t'_1t_1)^kt'_1 = t_2(t'_1t_1)^kt'_1t_2 \qquad \text{for any } k \geqslant 0. \tag{6.4.2}$$

Now we show (6.4.2) by induction on $k \geqslant 0$. When $k = 0$, it is just relation 4.1 (viii). Now assume $k > 0$. Then by inductive hypothesis, we get

$(t'_1t_1)^kt'_1t_2(t'_1t_1)^kt'_1 = t'_1t_1 \cdot t_2(t'_1t_1)^{k-1}t'_1t_2 \cdot t_1t'_1$.

Now we have

$$\begin{aligned} t'_1t_1 \cdot t_2(t'_1t_1)^{k-1}t'_1t_2 \cdot t_1t'_1 &= t_2(t'_1t_1)^kt'_1t_2 \\ \Longleftrightarrow t'_1t_1t_2(t'_1t_1)^{k-1}t'_1 &= t_2(t'_1t_1)^kt'_1t_2t'_1t_1t_2 \\ \Longleftrightarrow t'_1t_1t_2(t'_1t_1)^{k-1}t'_1 &= t_2(t'_1t_1)^kt_2t'_1t_1t_2t_1 \\ \Longleftrightarrow t'_1t_1t_2(t'_1t_1)^kt_2 &= t_2(t'_1t_1)^kt_2t'_1t_1 \end{aligned}$$

The last equation is just (6.4.1). So (viii′) is proved.

(ix′) can be shown by repeatedly applying relations 4.1 (ix) and (4.1.3).

Relation (x′) is amount to

$$(t'_1t_1)^{q(p-1)} = s^{-q}t_1s^q(t'_1t_1)^{q-1}t'_1. \tag{6.4.3}$$

Now (6.4.3) follows by 4.1 (x), (i), (ix) and (4.1.3)-(4,1,4). So we get (x′).

Concerning relations (xi′), $t_1st_1s = st_1st_1$ is just (4.1.3), and $t'st's = st'st'$ is amount to

(6.4.4) $$(t_1't_1)^{q-1}t_1's(t_1't_1)^{q-1}t_1's = s(t_1't_1)^{q-1}t_1's(t_1't_1)^{q-1}t_1'.$$

Now (6.4.4) follows by 4.1 (ii), (xi) and (4.1.4). Hence (xi′) follows.

Finally, we have $(t't_1)^{\frac{m}{d}} = (t_1't_1)^{\frac{qm}{d}} = 1$ by (4.1.6). Thus (xii′) is proved. □

6.5. The relations on X mentioned in Lemma 6.4 form the full set of basic relations on X.

By the basic relations on X, we can easily deduce the following relations

(6.5.1) $$s^at's^bt' = t's^bt's^a \quad \text{and} \quad s^at_1s^bt_1 = t_1s^bt_1s^a \quad \text{for any } a, b \in \mathbb{Z}.$$

(6.5.2)
$$(t't_1)^kt't_2(t't_1)^kt' = t_2(t't_1)^kt't_2 \text{ and } (t_1t')^kt_1t_2(t_1t')^kt_1 = t_2(t_1t')^kt_1t_2 \text{ for any } k \geqslant 0.$$

Since $\gcd\{p, q\} = 1$, there are some $a, b \in \mathbb{Z}$ such that the equation $ap + bq = 1$ holds.

Lemma 6.6. $t_1' = s^{-a}t's^a(t_1t')^{b-1}$.

Proof.

$$\begin{aligned} s^{-a}t's^a(t_1t')^{b-1} &= s^{-a}(t_1't_1)^{q-1}t_1's^a(t_1t_1')^{q(b-1)} = s^{-a}t_1's^a(t_1t_1')^{(q-1)+q(b-1)} \\ &= s^{-a}t_1's^a(t_1't_1)^{pa} = s^{-a}t_1's^a\left((t_1't_1)^{p-1}\right)^a(t_1't_1)^a \\ &= s^{-a}t_1's^a(s^{-1}t_1st_1')^a(t_1't_1)^a = s^{-a}t_1's^a \cdot s^{-a}t_1s^at_1'(t_1t_1')^{a-1} \cdot (t_1't_1)^a \\ &= t_1'. \qquad \square \end{aligned}$$

Lemma 6.7. *In the setup of 6.4, the basic relations on X imply the basic relations on S under the transition $t_1' = s^{-a}t's^a(t_1t')^{b-1}$.*

Proof. We need only check all the basic relations on S involving t_1'. Since ${t'}^2 = 1$ and t_1' is conjugate to t', we have ${t_1'}^2 = 1$. By the commutativity of s, t', t_1 with t_i, $i > 2$, we get $t_1't_i = t_it_1'$ for any $i > 2$. Next we have

$$st_1't_1 = t_1't_1s \Longleftrightarrow s^{1-a}t's^a(t_1t')^{b-1}t_1 = s^{-a}t's^a(t_1t')^{b-1}t_1s \Longleftrightarrow st's^at_1 = t's^at_1s$$
$$\Longleftrightarrow t'st's^a = s^at_1st_1 \Longleftrightarrow s^at'st' = s^at_1st_1 \Longleftrightarrow t_1t's = st_1t'.$$

The last equation follows by the basic relations $st't_1 = t't_1s$ and ${t'}^2 = t_1^2 = 1$ on X. So we get relation $st_1't_1 = t_1't_1s$.

$$\begin{aligned}(t_1't_1)^{p-1} &= (s^{-a}t's^a(t_1t')^{b-1}t_1)^{p-1} = s^{-a(p-1)}t's^{a(p-1)}t_1(t't_1)^{b(p-1)-1}\\
&= s^{-a(p-1)}t's^{a(p-1)}t_1\left((t't_1)^{p-1}\right)^b(t_1t') = s^{-a(p-1)}t's^{a(p-1)}t_1(s^{-q}t_1s^qt')^b(t_1t')\\
&= s^{-a(p-1)}t's^{a(p-1)}t_1s^{-bq}t_1s^{bq}t'(t_1t')^b = s^{-a(p-1)}t'\cdot t_1s^{-bq}t_1s^{a(p-1)}\cdot s^{bq}t'(t_1t')^b\\
&= s^{a-1}t's^{1-a}t'(t_1t')^b = s^{-1}t_1t'\cdot s^at's^{1-a}t'(t_1t')^{b-1}\\
&= s^{-1}t_1t'\cdot t's^{1-a}t's^a(t_1t')^{b-1} = s^{-1}t_1s\cdot s^{-a}t's^a(t_1t')^{b-1} = s^{-1}t_1st_1'.\end{aligned}$$

This implies relation $(t_1't_1)^{p-1} = s^{-1}t_1st_1'$.

$$t_1't_2t_1' = t_2t_1't_2 \Longleftrightarrow s^{-a}t's^a(t_1t')^{b-1}t_2s^{-a}t's^a(t_1t')^{b-1} = t_2s^{-a}t's^a(t_1t')^{b-1}t_2$$
$$\Longleftrightarrow t'(t_1t')^{b-1}t_2t'(t_1t')^{b-1} = t_2t'(t_1t')^{b-1}t_2.$$

The last equation follows by (6.5.2). So relation $t_1't_2t_1' = t_2t_1't_2$ is proved.

Finally we want to show the relation

(6.7.1) $$t_1't_1t_2t_1't_1t_2 = t_2t_1't_1t_2t_1't_1.$$

To do so, we need the following result.

Lemma 6.8.

$$(6.8.1)\qquad (t't_1)^bt't_2t_1t't_2t_1s^a = s^at't_2t_1t't_2t_1(t't_1)^b \qquad \forall\ a,b\in\mathbb{Z}.$$

Now we show (6.7.1) by assuming Lemma 6.8. (6.7.1) is amount to the relation

(6.7.2)

$$(t't_1)^{b-1}s^at's^{-a}\cdot t_1t_2\cdot(t't_1)^{b-1}s^at's^{-a}\cdot t_1t_2 = t_2\cdot(t't_1)^{b-1}s^at's^{-a}\cdot t_1t_2\cdot(t't_1)^{b-1}s^at's^{-a}\cdot t_1.$$

We can show that (6.7.2) is equivalent to

$$(6.7.3)\qquad t's^{-a}t'(t_1t')^{b-2}t_2t_1t't_2 = t_2t_1t't_2(t_1t')^{b-2}t_1s^{-a}t_1$$

by repeatedly applying the o.b. relations on X, the relations $t't_1s = st't_1$ and (6.5.1)–(6.5.2). Finally, (6.7.3) follows by Lemma 6.8. □

Proof of Lemma 6.8. Since the orders of the elements $t't_1$ and s are finite, we need only show (6.8.1) in the case of $a,b\in\mathbb{N}$. First we show (6.8.1) in the case of $a=0$:

$$(6.8.2)\qquad (t't_1)^bt't_2t_1t't_2t_1 = t't_2t_1t't_2t_1(t't_1)^b \qquad \forall\ a,b\in\mathbb{N}.$$

Equation (6.8.2) is trivial when $b=0$. To show (6.8.2) for $b>0$, we need only show it in the case of $b=1$. But equation $t't_1t't_2t_1t't_2t_1 = t't_2t_1t't_2t_1t't_1$ follows by the basic relation $t_1t't_2t_1t't_2 = t_2t_1t't_2t_1t'$ on X.

Next we show (6.8.1) in the case of $a>0$. By the above discussion, we need only show the equation

$$(6.8.3)\qquad (t't_1)^bt't_2t_1t't_2t_1s = s(t't_1)^bt't_2t_1t't_2t_1,$$

which is equivalent to

$$t_1t't_1t't_2t_1t't_2 = t_2t_1t't_2t_1t't_1t'. \tag{6.8.4}$$

The last equation follows by the basic relation $t_2t_1t't_2t_1t' = t_1t't_2t_1t't_2$ on X. □

Remark 6.9. (1) Let $S = \{s, t_h \mid 1 \leqslant h \leqslant n\} \in \Sigma(m, p, n)$ be given as in 4.2 and let P_S be the set of all the basic relations (A)-(M) on S (see 4.2-4.3). Then the presentation (S, P_S) of $G(m, p, n)$ is not essential in general (see 1.7).

For example, let (B′) be any one of the relations in (B). Then (B′) is equivalent to (B) under the assumption of (D).

Let (K′) (resp., (L′)) be a relation in (K) (resp., (L)) at any one admissible node pair. Then Lemma 4.6 tells us that (K′) (resp., (L′)) is equivalent to (K) (resp., (L)) under the assumption of the o.b. relations on S.

A subset (H′) of (H) in Lemma 4.8 is equivalent to (H) under the assumption of the o.b. relations on S. Also, a subset (I′) of (I) in Lemma 4.10 is equivalent to (I) under the assumption of the o.b. and branching relations on S.

(E) is a special case of (J), while (J) is a consequence of the o.b. relations and relations (K)-(L) on S by Lemma 4.7.

Let (M′) be relations (M) if Γ_S^r has a two-nodes circle and be the empty set of relations if otherwise.

If Γ_S^r has a two-nodes circle with the rooted node on the circle and not adjacent to any node outside the circle and that $\gcd\{\delta(S), q\} = 1$, then by the arguments similar to those for (4.1.3)-(4.1.4), we can show that relation (F) is a consequence of (L) and the other o.b. relations on S. Let (F′) be the empty set of relations if Γ_S^r is in such a case and be relation (F) if otherwise.

Let P_S' be the collection of relations (A), (B′), (C), (D), (F′), (G), (H′), (I′), (K′), (L′), (M′). Then (S, P_S') is again a presentation of $G(m, p, n)$. It is interesting to ask if the presentation (S, P'_S) should always be essential.

(2) Among all the presentations (S, P_S) of $G(m, p, n)$, we would like to single out two kinds of presentations whose relation sets have simpler forms: one is as that in 4.1; the other is when Γ_S is a circle. In the latter case, the relation set P_S can only consist of some o.b. relations, one root-braid relation, one root-circle relation and one circle-root relation.

References

1. M. Broué, G. Malle and R. Rouquier, *Complex reflection groups, braid groups, Hecke algebras*, J. Reine Angew. Math. **500** (1998), 127-190.
2. A. M. Cohen, *Finite complex reflection groups*, Ann. scient. Éc. Norm. Sup. **4^e** série t. **9** (1976), 379-436.
3. J. E. Humphreys, *Reflection groups and Coxeter groups*, vol. 29, Cambridge Studies in Advanced Mathematics, 1992.
4. J. Y. Shi, *Certain imprimitive reflection groups and their generic versions*, Trans. Amer. Math. Soc. **354** (2002), 2115-2129.
5. J. Y. Shi, *Simple root systems and presentations for certain complex reflection groups*, to appear in Comm. in Algebra.
6. J. Y. Shi, *Congruence classes of presentations for the complex reflection groups* G_{11}*,* G_{19} *and* G_{32}, preprint (2003).
7. J. Y. Shi, *Congruence classes of presentations for the complex reflection groups* $G(m, 1, n)$ *and* $G(m, m, n)$, preprint (2003).
8. G. C. Shephard and J. A. Todd, *Finite unitary reflection groups*, Canad. J. Math. **6** (1954), 274-304.
9. L. Wang, *Simple root systems and presentations for the primitive complex reflection groups generated by involutive reflections*, Master thesis in ECNU, 2003.
10. P. Zeng, *Simple root systems and presentations for the primitive complex reflection groups containing reflections of order* > 2, Master thesis in ECNU, 2003.