

On some q -identities related to divisor functions

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Abstract: We give generalizations and simple proofs of some q -identities of Dilcher, Fu and Lascoux related to divisor functions.

Let a_1, \dots, a_N be N indeterminates. It is easy to see that

$$\frac{1}{(1 - a_1 z)(1 - a_2 z) \dots (1 - a_N z)} = \sum_{k=1}^N \frac{\prod_{j=1, j \neq k}^N (1 - a_j/a_k)^{-1}}{1 - a_k z}. \quad (1)$$

The coefficient of z^τ ($\tau \geq 0$) in the left side of (1) is usually called the τ -th *complete symmetric function* $h_\tau(a_1, \dots, a_N)$ of a_1, \dots, a_N . Clearly, we have $h_0(a_1, \dots, a_N) = 1$ and equating the coefficients of z^τ ($\tau \geq 1$) in two sides of (1) yields

$$h_\tau(a_1, \dots, a_N) := \sum_{1 \leq i_1 \leq i_2 \leq \dots \leq i_\tau \leq N} a_{i_1} a_{i_2} \dots a_{i_\tau} = \sum_{k=1}^N \prod_{j=1, j \neq k}^N (1 - a_j/a_k)^{-1} a_k^\tau. \quad (2)$$

In particular, if $a_k = \frac{a - bq^{k+i-1}}{c - zq^{k+i-1}}$ ($1 \leq k \leq N$) for a fixed integer i ($1 \leq i \leq n$), then formula (2) with $N = n - i + 1$ reads

$$h_\tau \left(\frac{a - bq^i}{c - zq^i}, \frac{a - bq^{i+1}}{c - zq^{i+1}}, \dots, \frac{a - bq^n}{c - zq^n} \right) = \frac{c^{n-i+1} (zq^i/c)_{n-i+1}}{(q)_{n-i+1} (az - bc)^{n-i}} \cdot \sum_{k=i}^n (-1)^{k-i} \begin{bmatrix} n - i + 1 \\ n - k \end{bmatrix} q^{\binom{k-i+1}{2} - k(n-i)} \frac{(1 - q^{k-i+1})(a - bq^k)^{\tau+n-i}}{(c - zq^k)^{\tau+1}}, \quad (3)$$

where $(x)_n = (1 - x)(1 - xq) \dots (1 - xq^{n-1})$ and $\begin{bmatrix} n \\ i \end{bmatrix} = (q^{n-i+1})_i / (q)_i$ with $(x)_0 = 1$.

The aim of this note is to show that (3) turns out to be a common source of several q -identities surfacing recently in the literature.

First of all, the $i = 1$ case of formula (3) with $\tau = m - n + 1$ corresponds to an identity of Fu and Lascoux [3, Prop. 2.1]:

$$h_\tau \left(\frac{a - bq}{c - zq}, \frac{a - bq^2}{c - zq^2}, \dots, \frac{a - bq^n}{c - zq^n} \right) = \frac{c^n (zq/c)_n}{(q)_n (az - bc)^{n-1}} \sum_{k=1}^n \begin{bmatrix} n \\ k \end{bmatrix} (-1)^{k-1} q^{\binom{k+1}{2} - nk} \frac{(1 - q^k)(a - bq^k)^m}{(c - zq^k)^{\tau+1}}. \quad (4)$$

Next, for $i = 1, \dots, n$ and $m \geq 1$ set

$$A_i(z) := \frac{q^i(zq)_{i-1}(q)_n}{(q)_i(zq)_n} h_{m-1} \left(\frac{q^i}{1-zq^i}, \dots, \frac{q^n}{1-zq^n} \right). \quad (5)$$

Then we have the following polynomial identity in x :

$$\sum_{k=1}^n \begin{bmatrix} n \\ k \end{bmatrix} \frac{(x-1) \cdots (x-q^{k-1})}{(1-zq^k)^m} q^{mk} = \sum_{k=1}^n (-1)^k \begin{bmatrix} n \\ k \end{bmatrix} \frac{q^{\binom{k}{2}+mk}}{(1-zq^k)^m} + \sum_{i=1}^n A_i(z) x^i. \quad (6)$$

Indeed, using the q -binomial formula [1, p. 36]:

$$(x-1)(x-q) \cdots (x-q^{N-1}) = \sum_{j=0}^N \begin{bmatrix} N \\ j \end{bmatrix} (-1)^{N-j} x^j q^{\binom{N-j}{2}},$$

we see that the coefficient of x^i ($1 \leq i \leq n$) in the left side of (6) is equal to

$$\sum_{k=i}^n (-1)^{k-i} \begin{bmatrix} n \\ k \end{bmatrix} \begin{bmatrix} k \\ i \end{bmatrix} \frac{q^{mk+\binom{k-i}{2}}}{(1-zq^k)^m} = \frac{q^i(zq)_{i-1}(q)_n}{(q)_i(zq)_n} h_{m-1} \left(\frac{q^i}{1-zq^i}, \dots, \frac{q^n}{1-zq^n} \right), \quad (7)$$

where the last equality follows from (3) with $a = 0$, $c = 1$, $b = -1$ and $\tau = m - 1$.

Now, with $z = i = 1$ and m shifted to $m + 1$, formula (7) reduces to Dilcher's identity [2]:

$$\sum_{k=1}^n \begin{bmatrix} n \\ k \end{bmatrix} \frac{(-1)^{k-1} q^{\binom{k}{2}+mk}}{(1-q^k)^m} = h_m \left(\frac{q}{1-q}, \dots, \frac{q^n}{1-q^n} \right) = \sum_{i=1}^n A_i(1).$$

On the other hand, formula (1) with $N = n + 1$ and $a_i = q^{i-1}$ ($1 \leq i \leq N$) yields

$$\sum_{k=0}^n (-1)^k \begin{bmatrix} n \\ k \end{bmatrix} \frac{q^{\binom{k}{2}+k}}{1-zq^k} = \frac{(q)_n}{(z)_{n+1}}.$$

Hence, setting, respectively, $z = 1$ and $m = 1$ in formula (6) we recover two recent formulae of Fu and Lascoux [4] (see also [5]):

$$\sum_{k=1}^n \begin{bmatrix} n \\ k \end{bmatrix} \frac{(x-1) \cdots (x-q^{k-1})}{(1-q^k)^m} q^{mk} = \sum_{i=1}^n (x^i - 1) A_i(1), \quad (8)$$

and

$$\sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix} \frac{(x-1) \cdots (x-q^{k-1})}{1-zq^k} q^k = \frac{(q)_n}{(z)_{n+1}} \sum_{i=0}^n \frac{(z)_i}{(q)_i} x^i q^i. \quad (9)$$

Acknowledgement

The author is partially supported by EC's IHRP Programme, within the Research Training Network "Algebraic Combinatorics in Europe", grant HPRN-CT-2001-00272.

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