# Two new families of $q$-positive integers


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**Abstract.** Let $n, p$ and $k$ be three non negative integers. We prove that the apparently rational fractions of $q$:

$$\begin{bmatrix} n \\ k \end{bmatrix}_q {}_3\phi_2 \left[ \begin{matrix} q^{1-k}, q^{-p}, q^{p-n} \\ q, q^{1-n} \end{matrix} \middle| q; q^{k+1} \right] \quad \text{and} \quad q^{(n-p)p} \begin{bmatrix} n \\ k \end{bmatrix}_q {}_3\phi_2 \left[ \begin{matrix} q^{1-k}, q^{-p}, q^{p-n} \\ q, q^{1-n} \end{matrix} \middle| q; q \right]$$

are actually polynomials of $q$ with positive integer coefficients. This generalizes a recent result of Lassalle (Ann. Comb. 6(2002), no. 3-4, 399-405), in the same way as the classical $q$-binomial coefficients refine the ordinary binomial coefficients.



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## 1 Introduction

In [3] Lassalle introduced a new family of positive integers generalizing the classical binomial coefficients. We refer the readers to [4] for some motivations from the study of Jack polynomials and to [2] for some further results and extensions of these coefficients. In this paper we will present a different generalization of Lassalle's coefficients. More precisely, using basic hypergeometric functions we will show that all the results of Lassalle [3] have *natural* $q$-analogues.

In order to present its $q$-analogues we need first to introduce some notations. Throughout this paper $q$ is a complex variable such that $|q| < 1$. For any integer $k \geq 0$, the $q$-raising factorial is defined by

$$(a; q)_k = (1 - a)(1 - aq) \dots (1 - aq^{k-1}) \qquad \text{for} \quad k \geq 1,$$

and $(a; q)_0 = 1$. We shall also use the concise notation $(a_1, \dots, a_s; q)_k = (a_1; q)_k \dots (a_s; q)_k$ for $s \geq 1$. For any integer $n$ (not necessary positive) the $q$-binomial coefficients $\begin{bmatrix} n \\ k \end{bmatrix}_q$ are defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q^{n-k+1}; q)_k}{(q; q)_k}.$$


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So $\begin{bmatrix} n \\ k \end{bmatrix}_q = 0$ if $k > n$. Furthermore we will use the standard notation for basic hypergeometric series [1]:

$$ {}_r\phi_s \left[ \left. \begin{matrix} a_1, a_2, \ldots, a_r \\ b_1, b_2, \ldots, b_s \end{matrix} \right| q; z \right] = \sum_{k=0}^{\infty} \frac{(a_1, \ldots, a_r; q)_k}{(b_1, \ldots, b_s; q)_k} (-1)^{(1+s-r)k} q^{(1+s-r)\binom{k}{2}} \frac{z^k}{(q;q)_k}. $$

The $q$-binomial coefficients have two less obvious formulas as follows:

$$ \begin{bmatrix} n \\ p \end{bmatrix}_q = {}_2\phi_1 \left[ \left. \begin{matrix} q^{-p}, q^{p-n} \\ q \end{matrix} \right| q; q^{n+1} \right] = q^{p(p-n)} {}_2\phi_1 \left[ \left. \begin{matrix} q^{-p}, q^{p-n} \\ q \end{matrix} \right| q; q \right]. \tag{1} $$

Actually these expressions are obtained by specializing $a = q^{p-n}, c = q$ in the celebrated $q$-Chu-Vandermonde formula [1, p. 236]:

$$ \frac{(c/a;q)_p}{(c;q)_p} = {}_2\phi_1 \left[ \left. \begin{matrix} q^{-p}, a \\ c \end{matrix} \right| q; cq^p/a \right] = a^{-p} {}_2\phi_1 \left[ \left. \begin{matrix} q^{-p}, a \\ c \end{matrix} \right| q; q \right]. $$

Of course using this relation as a definition of $q$-binomial coefficients would be rather tautological. However, quite surprisingly, it is possible to define two new families of *$q$-positive integers*, i.e., a polynomial of $q$ with *non negative integer coefficients*, by slightly modifying the $q$-Chu-Vandermonde formula. In fact there are two such $q$-analogues of Lassalle's generalized binomial coefficients.

In the next two sections we present the first $q$-analogue and its *raison d'être* in the context of linearization problem of $q$-binomial coefficients. In Section 4 we outline the second $q$-analogue. At the end this paper we give the first values of our generalized $q$-binomial coefficients, which seems to suggest some *unimodal* properties of the coefficients of these polynomials.

## 2 The first $q$-analogue

For any non negative integers $n, p$ and $k$, define the coefficient

$$ \binom{n}{p,k}_q = q^{(n-p)p} \begin{bmatrix} n \\ k \end{bmatrix}_q {}_3\phi_2 \left[ \left. \begin{matrix} q^{1-k}, q^{-p}, q^{p-n} \\ q, q^{1-n} \end{matrix} \right| q; q \right]. \tag{2} $$

Applying Sears' transformation [1, p. 61, (3.2.5)]

$$ {}_3\phi_2 \left[ \left. \begin{matrix} q^{-p}, a, b \\ c, d \end{matrix} \right| q; q \right] = \frac{(c/a;q)_p}{(c;q)_p} a^p \, {}_3\phi_2 \left[ \left. \begin{matrix} q^{-p}, a, d/b \\ d, q^{1-p}a/c \end{matrix} \right| q; bq/c \right] \tag{3} $$

with $a = q^{p-n}$, $b = q^{1-k}$, $c = q$ and $d = q^{1-n}$, we get

$$ \binom{n}{p,k}_q = \begin{bmatrix} n \\ k \end{bmatrix}_q \begin{bmatrix} n \\ p \end{bmatrix}_q {}_3\phi_2 \left[ \left. \begin{matrix} q^{-p}, q^{p-n}, q^{k-n} \\ q^{1-n}, q^{-n} \end{matrix} \right| q; q^{1-k} \right]. \tag{4} $$

It follows that

$$ \binom{n}{p,k}_q = 0 \quad \text{for} \quad k > n \quad \text{or} \quad p > n, $$

and for $p, k \le n$,

$$\binom{n}{p,k}_q = \binom{n}{n-p,k}_q, \qquad \binom{n}{p,n}_q = \begin{bmatrix} n \\ p \end{bmatrix}_q,$$

the last equations following directly from the $q$-Chu-Vandermonde formula.

Set $[n]_q = 1 + q + \cdots + q^{n-1} = (1-q^n)/(1-q)$ for $n \ge 0$. We can rewrite (2) as follows:

$$\binom{n}{p,k}_q = \frac{[n]_q}{[k]_q} \sum_{r\ge 0} \begin{bmatrix} p \\ r \end{bmatrix}_q \begin{bmatrix} n-p \\ r \end{bmatrix}_q \begin{bmatrix} n-r-1 \\ k-r-1 \end{bmatrix}_q q^{(n-p)p+r(r-k)}. \tag{5}$$

Therefore

$$\binom{n}{0,k}_q = \begin{bmatrix} n \\ k \end{bmatrix}_q, \qquad \binom{n}{1,k}_q = [k]_q q^{n-k} \begin{bmatrix} n \\ k \end{bmatrix}_q,$$

and

$$\binom{n}{2,k}_q = q^{2n-3-k}[k]_q \begin{bmatrix} n \\ k \end{bmatrix}_q + q^{2(n-k)} \frac{[n]_q[n-3]_q}{[2]_q} \begin{bmatrix} n-2 \\ k-2 \end{bmatrix}_q.$$

For $p > 0$, we have also

$$\binom{n}{p,0}_q = 0, \quad \binom{n}{p,1}_q = q^{(n-p)p}[n]_q,$$

$$\binom{n}{p,2}_q = q^{(n-p)p} \frac{[n]_q}{[2]_q} \left([n-1]_q + [p]_q[n-p]_q\right).$$

These results suggest that $\binom{n}{p,k}_q$ are polynomials of $q$ with *non negative integer coefficients.* Indeed, it is not hard to see that they are polynomials with integer coefficients.

We can rewrite (4) as follows

$$\binom{n}{p,k}_q = \sum_{i\ge 0} (-1)^i \begin{bmatrix} n-i \\ k \end{bmatrix}_q \frac{[n]}{[n-i]} \begin{bmatrix} n-i \\ i \end{bmatrix}_q \begin{bmatrix} n-2i \\ p-i \end{bmatrix}_q q^{i(i-1)/2}. \tag{6}$$

Since

$$\frac{[n]}{[n-i]} \begin{bmatrix} n-i \\ i \end{bmatrix}_q = \frac{1-q^{n-i}+q^{n-i}-q^n}{1-q^{n-i}} \begin{bmatrix} n-i \\ i \end{bmatrix}_q = \begin{bmatrix} n-i \\ i \end{bmatrix}_q + q^{n-i} \begin{bmatrix} n-i-1 \\ i-1 \end{bmatrix}_q,$$

we see that $\binom{n}{p,k}_q$ is a $q$-integer, i.e., a polynomial of $q$ with integer coefficients.

Two generating functions for $\binom{n}{p,k}_1$ were published in [3]. We consider their $q$-analogues below.

**Theorem 1.** *There holds*

$$\sum_{0\le k,p\le n} \binom{n}{p,k}_q x^p q^{\binom{p}{2}} y^k q^{\binom{k}{2}}$$

$$= (-y;q)_n(-x;q)_n \sum_{i\ge 0} \frac{1-q^n}{1-q^{n-i}} \begin{bmatrix} n-i \\ i \end{bmatrix}_q \frac{(-x)^i q^{i^2-i}}{(-yq^{n-i};q)_i(-x;q)_i(-xq^{n-i};q)_i}. \tag{7}$$

*Proof.* Using (6) one has

$$\begin{aligned}
&\sum_{0\le k,p\le n} \binom{n}{p,k}_q x^p q^{\binom{p}{2}} y^k q^{\binom{k}{2}}\\
=&\sum_{0\le k,p\le n}\sum_{i\ge 0}^{n}(-1)^i\frac{1-q^n}{1-q^{n-i}}\begin{bmatrix}n-i\\k\end{bmatrix}_q\begin{bmatrix}n-i\\i\end{bmatrix}_q\begin{bmatrix}n-2i\\p-i\end{bmatrix}_q q^{\binom{i}{2}}y^k q^{\binom{k}{2}}x^p q^{\binom{p}{2}}\\
=&\sum_{i\ge 0}(-1)^i\frac{1-q^n}{1-q^{n-i}}x^i\sum_{k\ge 0}\begin{bmatrix}n-i\\k\end{bmatrix}_q y^k q^{\binom{k}{2}}\sum_{p\ge 0}\begin{bmatrix}n-2i\\p-i\end{bmatrix}_q (xq^i)^{p-i}q^{\binom{p-i}{2}}q^{i^2-i}\\
=&\sum_{i\ge 0}(-1)^i\frac{1-q^n}{1-q^{n-i}}x^i(-y;q)_{n-i}(-xq^i;q)_{n-2i}q^{i^2-i},
\end{aligned}$$

which yields (7). □

**Remark.** When $q=1$, since

$$\sum_{k<n}\binom{n-k}{k}\frac{n}{n-k}z^k=\left(\frac{1+\sqrt{1+4z}}{2}\right)^n+\left(\frac{1-\sqrt{1+4z}}{2}\right)^n,$$

setting $z=\frac{-x}{(1+x)^2(1+y)}$, the above generating functions can be written as follows:

$$\sum_{0\le k,p\le n}\begin{bmatrix}n\\p,k\end{bmatrix}_1 x^p y^k = 2^{-n}[(1+y)(1+x)]^n\left((1+\sqrt{1+4z})^n+(1-\sqrt{1+4z})^n\right).$$

The following is the $q$-analogue of Lassalle's recurrence relation in [3, Lemma 3.2].

**Proposition 1.** *For $k\ne 0$ and $0\le p\le n$, we have*

$$(1-q^{n-p+1})\binom{n}{p-1,k}_q-(1-q^p)\binom{n}{p,k}_q=\frac{[n]_q}{[n-1]_q}q^{p-1}(1-q^{n-2p+1})\binom{n-1}{p-1,k}_q. \tag{8}$$

*Proof.* Indeed, up to the factor $[n]_q/[k]_q$ and using (5), the left-hand side can be written as

$$\begin{aligned}
&\sum_{r\ge 0}q^A\begin{bmatrix}n-r-1\\k-r-1\end{bmatrix}_q\left((1-q^{n-p+1})q^{p-1}\begin{bmatrix}p-1\\r\end{bmatrix}_q\begin{bmatrix}n-p+1\\r\end{bmatrix}_q-(1-q^p)q^{n-p}\begin{bmatrix}p\\r\end{bmatrix}_q\begin{bmatrix}n-p\\r\end{bmatrix}_q\right)\\
=&(1-q^{n-2p+1})\sum_{r\ge 0}q^A\begin{bmatrix}n-r-1\\k-r-1\end{bmatrix}_q\left(q^{p-1}\begin{bmatrix}p-1\\r\end{bmatrix}_q\begin{bmatrix}n-p\\r\end{bmatrix}_q-q^{n-2r+1}\begin{bmatrix}p-1\\r-1\end{bmatrix}_q\begin{bmatrix}n-p\\r-1\end{bmatrix}_q\right)\\
=&(1-q^{n-2p+1})\sum_{r\ge 0}q^A\begin{bmatrix}p-1\\r\end{bmatrix}_q\begin{bmatrix}n-p\\r\end{bmatrix}_q q^{p-1}\left(\begin{bmatrix}n-r-1\\k-r-1\end{bmatrix}_q-q^{n-k}\begin{bmatrix}n-r-2\\k-r-2\end{bmatrix}_q\right)\\
=&(1-q^{n-2p+1})\sum_{r\ge 0}q^A\begin{bmatrix}p-1\\r\end{bmatrix}_q\begin{bmatrix}n-p\\r\end{bmatrix}_q\begin{bmatrix}n-r-2\\k-r-1\end{bmatrix}_q q^{p-1},
\end{aligned}$$

where $A=r(r-k)+(n-p)(p-1)$. The result follows then from Equation (5). □

For the second generating function, we will further need the $q$-binomial formula [1, 236]:

$$\sum_{n\geq 0}\frac{(a;q)_n}{(q;q)_n}x^n=\frac{(ax;q)_\infty}{(x;q)_\infty}, \tag{9}$$

Jackson's $q$-Pfaff-Kummer transformation [1, p.241]:

$${}_2\phi_1\left[\begin{matrix}a,b\\ c\end{matrix}\middle| q;z\right]=\frac{(az;q)_\infty}{(z;q)_\infty}{}_2\phi_2\left[\begin{matrix}a,c/b\\ c,az\end{matrix}\middle| q;bz\right], \tag{10}$$

and another transformation of Sears [1, p. 61, (3.2.2)]:

$${}_3\phi_2\left[\begin{matrix}q^{-p},a,b\\ c,d\end{matrix}\middle| q;cdq^p/ab\right]=\frac{(c/a;q)_p}{(c;q)_p}\,{}_3\phi_2\left[\begin{matrix}q^{-p},a,d/b\\ d,q^{1-p}a/c\end{matrix}\middle| q;q\right]. \tag{11}$$

**Theorem 2.** *For $0\leq p\leq n$, we have*

$$\sum_{k\geq 1}\binom{n}{p,k}_q y^k q^{\binom{k}{2}} \;=\; yq^{p(n-p)}(-y;q)_p[n]_q\,{}_2\phi_1\left[\begin{matrix}q^{p+1},q^{p-n+1}\\ q^2\end{matrix}\middle| q;-yq^{n-p}\right]. \tag{12}$$

*Proof.* By applying (3) to (2) and then applying (11) we get:

$$\begin{aligned}\binom{n}{p,k}_q &= q^{(n-p)p}\begin{bmatrix}n\\ k\end{bmatrix}\frac{(q^{1-p};q)_{k-1}}{(q^{1-n};q)_{k-1}}q^{(k-1)(p-n)}{}_3\phi_2\left[\begin{matrix}q^{1-k},q^{p-n},q^{1+p}\\ q^{p-k+1};q\end{matrix}\middle| q;q^{n-p}\right]\\ &= q^{(n-p)p}\begin{bmatrix}n\\ k\end{bmatrix}\frac{(q^{1-p};q)_{k-1}}{(q^{1-n};q)_{k-1}}q^{(p-n)(k-1)}\frac{(q^{-k};q)_{k-1}}{(q^{p-k+1};q)_{k-1}}{}_3\phi_2\left[\begin{matrix}q^{1-k},q^{1+p},q^{n-p+1}\\ q,q^2\end{matrix}\middle| q;q\right]\\ &= (-1)^{k-1}q^{(n-p)p-\frac{(k-1)(k+2)}{2}}[n]_q{}_3\phi_2\left[\begin{matrix}q^{1-k},q^{1+p},q^{n-p+1}\\ q,q^2\end{matrix}\middle| q;q\right].\end{aligned} \tag{13}$$

Hence

$$\sum_{k\geq 1}\binom{n}{p,k}_q y^k q^{\binom{k}{2}} \;=\; yq^{p(n-p)}[n]_q\sum_{l\geq 0}\frac{(q^{p+1};q)_l(q^{n-p+1};q)_l}{(q^2;q)_l(q;q)_l(q;q)_l}q^l\sum_{k\geq l}(q^{-k};q)_l(-yq^{-1})^k.$$

But the $q$-binomial formula (9) implies that

$$\sum_{k\geq l}(q^{-k};q)_l(-yq^{-1})^k=(-1)^l q^{\binom{l}{2}}\sum_{k\geq l}(q^{k-l+1};q)_l(-yq^{-l-1})^k=\frac{(q;q)_l}{(1+yq^{-1})(-y^{-1}q^2;q)_l}.$$

Hence

$$\sum_{k\geq 1}\binom{n}{p,k}_q y^k q^{\binom{k}{2}}=\frac{yq^{p(n-p)}}{1+yq^{-1}}[n]_q\sum_{l\geq 0}\frac{(q^{p+1};q)_l(q^{n-p+1};q)_l q^l}{(-y^{-1}q^2;q)_l(q^2;q)_l(q;q)_l}. \tag{14}$$

Using the formula

$$(a;q)_n=(a^{-1};q^{-1})_n(-a)^n q^{\binom{n}{2}} \tag{15}$$

and Jackson's transformation, we can rewrite the above sum as follows:

$$\begin{aligned}\sum_{k\geq 1}\binom{n}{p,k}_q y^k q^{\binom{k}{2}} &= \frac{yq^{p(n-p)}}{1+yq^{-1}}[n]_{q^2}\phi_2\left[\begin{matrix}q^{-1-p},q^{p-n-1}\\ q^{-2},-yq^{-2}\end{matrix}\middle| q^{-1};-yq^{n-2}\right]\\ &= yq^{p(n-p)}(-y;q)_p[n]_{q^2}\phi_1\left[\begin{matrix}q^{-1-p},q^{p-n-1}\\ q^{-2}\end{matrix}\middle| q^{-1};-yq^{p-1}\right].\end{aligned}$$

We then recover (12) by applying (15) again. □

**Remark**. When $q=1$, the above theorem reduces to Lassalle's generating function [3], which was proved by using induction and contiguous relation. One could give another proof of Theorem 2 by using Proposition 1.

The following result is crucial to prove that $\binom{n}{p,k}_q$ is a $q$-positive integer.

**Corollary 1.** *We have*

$$\binom{n}{p,k}_q = \frac{[n]_q}{[p]_q}\sum_{i=0}^{k-1}\begin{bmatrix}n-p\\ k-1-i\end{bmatrix}_q\begin{bmatrix}n-p+i\\ i\end{bmatrix}_q\begin{bmatrix}p\\ i+1\end{bmatrix}_q q^{(i+1)(i+1-k)+(n-p)p}.$$

*Proof.* Using the $q$-binomial formula we have

$$(-y;q)_p = \sum_{j=0}^{p}\begin{bmatrix}p\\ j\end{bmatrix} y^j q^{\binom{j}{2}}.$$

Extracting the coefficient of $y^k$ in (12) we obtain

$$\binom{n}{p,k}_q = \frac{[n]_q}{[n-p]_q}\sum_{i=0}^{k-1}\begin{bmatrix}p\\ k-1-i\end{bmatrix}_q\begin{bmatrix}p+i\\ i\end{bmatrix}_q\begin{bmatrix}n-p\\ i+1\end{bmatrix}_q q^{(n-p)p+(i+1)(i+1-k)}.$$

As $\binom{n}{p,k}_q=\binom{n}{n-p,k}_q$, substituting $p$ by $n-p$ yields the desired identity. □

By applying

$$\binom{n}{p,k}_q = \frac{[p]_q}{[n]_q}\binom{n}{p,k}_q + q^p\frac{[n-p]_q}{[n]_q}\binom{n}{n-p,k}_q,$$

we can write

$$\begin{aligned}\binom{n}{p,k}_q &= \sum_{i=0}^{k-1}\begin{bmatrix}n-p\\ k-1-i\end{bmatrix}_q\begin{bmatrix}n-p+i\\ i\end{bmatrix}_q\begin{bmatrix}p\\ i+1\end{bmatrix}_q q^{(n-p)p+(i+1)(i+1-k)}\\ &+ \sum_{i=0}^{k-1}\begin{bmatrix}p\\ k-1-i\end{bmatrix}_q\begin{bmatrix}p+i\\ i\end{bmatrix}_q\begin{bmatrix}n-p\\ i+1\end{bmatrix}_q q^{(n-p)(p+1)+(i+1)(i+1-k)},\end{aligned}$$

which implies the following

**Theorem 3.** *The polynomials* $\binom{n}{p,k}_q$ *is a q-positive integer.*

Since $q$-binomial coefficients have various nice combinatorial interpretations, it would be possible to derive a combinatorial interpretation for $\binom{n}{p,k}_q$ from the above expression.

## 3 Further extensions

It is surprising that the general numbers $c_k^{(\mathbf{r})}$ of Lassalle [3] have also a $q$-analogue, which are also $q$-positive integers. We shall explain such a $q$-analog in this section. Note that the $q$-Chu-Vandermonde formula:

$$\begin{bmatrix} n+m \\ k \end{bmatrix}_q = \sum_{i\geq 0} \begin{bmatrix} n \\ i \end{bmatrix}_q \begin{bmatrix} m \\ k-i \end{bmatrix}_q q^{(n-i)(k-i)} \tag{16}$$

implies that

$$\begin{aligned} \begin{bmatrix} x \\ r_1 \end{bmatrix}_q \begin{bmatrix} x \\ r_2 \end{bmatrix}_q &= \begin{bmatrix} x \\ r_1 \end{bmatrix}_q \sum_{k\geq 0} \begin{bmatrix} r_1 \\ k \end{bmatrix}_q \begin{bmatrix} x-r_1 \\ r_2-k \end{bmatrix}_q q^{(r_1-k)(r_2-k)} \\ &= \sum_{k\geq 0} q^{(r_1-k)(r_2-k)} \begin{bmatrix} r_1+r_2-k \\ k, r_1-k, r_2-k \end{bmatrix}_q \begin{bmatrix} x \\ r_1+r_2-k \end{bmatrix}_q. \end{aligned} \tag{17}$$

Set $\mathbf{r} = (r_1, \ldots, r_m)$ and $|\mathbf{r}| = r_1 + \cdots + r_m$. Iterating (17) yields:

$$\begin{bmatrix} x \\ r_1 \end{bmatrix}_q \cdots \begin{bmatrix} x \\ r_m \end{bmatrix}_q = \sum_{l\geq 0} d_l(\mathbf{r}; q) \begin{bmatrix} x \\ l \end{bmatrix}_q, \tag{18}$$

where $d_l(\mathbf{r}; q)$ are $q$-positive integers given by

$$\begin{aligned} d_l(\mathbf{r}; q) &= \sum_{k_1,\ldots,k_{m-2}\geq 0} \begin{bmatrix} r_1+r_2-k_1 \\ k_1, r_1-k_1, r_2-k_1 \end{bmatrix}_q \begin{bmatrix} r_1+r_2+r_3-k_1-k_2 \\ k_2, r_1+r_2-k_1-k_2, r_3-k_2 \end{bmatrix}_q \\ &\quad \times \cdots \times \begin{bmatrix} r_1+\ldots+r_{m-2}-k_1-\ldots-k_{m-3} \\ k_{m-3}, r_1+\ldots+r_{m-3}-k_1-\ldots-k_{m-3}, r_{m-2}-k_{m-3} \end{bmatrix}_q \\ &\quad \times \begin{bmatrix} l \\ r_m \end{bmatrix}_q \begin{bmatrix} r_m \\ |\mathbf{r}|-k_1-\ldots-k_{m-2}-l \end{bmatrix}_q q^B, \end{aligned} \tag{19}$$

where

$$\begin{aligned} B &= (r_1-k_1)(r_2-k_1) + (r_1+r_2-k_1-k_2)(r_3-k_2) + \cdots \\ &\quad + (r_1+r_2+\cdots+r_{m-2}-k_1-\cdots-k_{m-2})(r_{m-1}-k_{m-2}) \\ &\quad + (l-r_m)(l-r_1-\cdots-r_{m-1}+k_1+\cdots+k_{m-2}). \end{aligned}$$

In particular, for $m = 2$ we have

$$d_l(r_1, r_2; q) = q^{(l-r_1)(l-r_2)} \begin{bmatrix} l \\ r_1 \end{bmatrix}_q \begin{bmatrix} r_1 \\ l-r_2 \end{bmatrix}_q. \tag{20}$$

On the other hand, identity (16) implies also

$$\begin{bmatrix} x+r_1-1 \\ r_1 \end{bmatrix}_q = \sum_{k\geq 0} \begin{bmatrix} r_1-1 \\ r_1-k \end{bmatrix}_q \begin{bmatrix} x \\ k \end{bmatrix}_q q^{k(k-1)}. \tag{21}$$

From (18) and (21) we derive the following result.

$$\begin{bmatrix} x+r_1-1 \\ r_1 \end{bmatrix}_q \cdots \begin{bmatrix} x+r_m-1 \\ r_m \end{bmatrix}_q = \sum_{l\geq 0} \tilde{c}_l(\mathbf{r};q) \begin{bmatrix} x \\ l \end{bmatrix}_q.$$

where

$$\tilde{c}_l(\mathbf{r};q) = \sum_{\mathbf{k}} d_l(\mathbf{k};q) \prod_{i=1}^{m} \begin{bmatrix} r_i-1 \\ k_i-1 \end{bmatrix}_q q^{k_i(k_i-1)}. \tag{22}$$

**Theorem 4.** *The polynomial*

$$c_l(\mathbf{r};q) = \frac{[r_1+\ldots+r_m]_q}{[l]_q} \tilde{c}_l(\mathbf{r};q)$$

*is a $q$-positive integer.*

*Proof.* By (19) there is a polynomial $P_m(\mathbf{k};q) \in \mathbb{N}[q]$ such that

$$d_l(\mathbf{k};q) = \frac{[l]_q}{[k_m]_q} P_m(k;q).$$

Since $d_l(\mathbf{k};q)$ is symmetric with respect to $\mathbf{k} = (k_1,\ldots,k_m)$, the above formula infers that there is a polynomial $P_i(\mathbf{k};q) \in \mathbb{N}[q]$ for each $j \in \{1,\ldots,m\}$ such that

$$d_l(\mathbf{k};q) = \frac{[l]_q}{[k_j]_q} P_j(\mathbf{k};q).$$

Therefore, using (3) we have

$$\begin{aligned} c_l(\mathbf{r};q) &= \sum_{j=1}^{m} \frac{[r_j]q^{r_1+\cdots+r_{j-1}}}{[r_1+\cdots+r_m]} c_l(\mathbf{r};q) \\ &= \sum_{j=1}^{m} q^{r_1+\cdots+r_{j-1}} \sum_{\mathbf{k}} P_i(\mathbf{k};q) \begin{bmatrix} r_j \\ k_j \end{bmatrix}_q q^{k_j(k_j-1)} \prod_{i\neq j} \begin{bmatrix} r_i-1 \\ k_i-1 \end{bmatrix}_q q^{k_i(k_i-1)}, \end{aligned}$$

which is clearly a $q$-positive integer. □

We can also derive a simpler formula for $\tilde{c}_j(r_1,\ldots,r_m;q)$ using the *$q$-difference operator*. Set $[x;q] = (q^x-1)/(q-1)$ and

$$[x;q]_n = [x;q][x-1;q]\cdots[x-n+1;q] = \frac{(q^{x-n+1};q)_n}{(1-q)^n}.$$

We define the $q$-difference operator $\Delta_q$ by

$$\Delta_q^0 f(x) = f(x),\ \Delta_q^{n+1} f(x) = \Delta_q^n (E - q^n I) f(x),$$

where $If(x) = f(x)$ and $Ef(x) = f(x+1)$. Note that

$$\Delta_q^n f(x) = (E - q^{n-1}I)(E - q^{n-2}I)\cdots(E-I)f(x).$$

By the $q$-Chu-Vandermonde formula, we have

$$\Delta_q^n f(x) = \sum_{k=0}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\binom{k}{2}} f(x+n-k). \tag{23}$$

It's easy to see that

$$\Delta_q^n [x;q]_m = [m;q]_n [x;q]_{m-n} q^{n(x+n-m)}.$$

We have also

$$p(x) = \sum_{n\geq 0} \frac{\Delta_q^n p(0)}{[n]!} [x;q]_n. \tag{24}$$

It follows from (24), (23) and (3) that

$$\tilde{c}_k(r_1,\ldots,r_m;q) = \sum_{j=1}^{k} (-1)^{k-j} \begin{bmatrix} k \\ j \end{bmatrix}_q q^{\binom{k-j}{2}} \prod_{l=1}^{m} \begin{bmatrix} j+r_l-1 \\ r_l \end{bmatrix}_q. \tag{25}$$

Set $\mathbf{r} = (r_1,\ldots,r_m)$ and

$$\begin{aligned} c_k(\mathbf{r};q) &= \frac{[r_1+\ldots+r_m]_q}{[k]_q} \tilde{c}_k(r_1,\ldots,r_m;q) \\ &= \sum_{i=1}^{m} q^{r_1+\cdots+r_{i-1}} \frac{[r_i]_q}{[k]_q} \sum_{j=1}^{k} (-1)^{k-j} \begin{bmatrix} k \\ j \end{bmatrix}_q q^{\binom{k-j}{2}} \prod_{l=1}^{m} \begin{bmatrix} j+r_l-1 \\ r_l \end{bmatrix}_q \\ &= \sum_{i=1}^{m} \sum_{j=1}^{k} (-1)^{k-j} q^{r_1+\cdots+r_{i-1}+\binom{k-j}{2}} \\ &\qquad \times \begin{bmatrix} k-1 \\ j-1 \end{bmatrix}_q \begin{bmatrix} j+r_i-1 \\ r_i \end{bmatrix}_q \prod_{l=1,\, l\neq j}^{m} \begin{bmatrix} r_l+i-1 \\ r_l \end{bmatrix}_q. \end{aligned} \tag{26}$$

Thus we have obtained another proof of the $q$-integrality of $c_k(\mathbf{r};q)$.

Finally, when $\mathbf{r} = (r_1, r_2)$, we have the following result.

**Theorem 5.** *The coefficients* $\binom{r_1+r_2}{r_1,k}_q$ *satisfy*

$$\begin{bmatrix} x+r_1-1 \\ r_1 \end{bmatrix}_q \begin{bmatrix} x+r_2-1 \\ r_2 \end{bmatrix}_q = \sum_{k\geq 1} \frac{[k]_q q^{k(k-1)-r_1 r_2}}{[r_1+r_2]_q} \binom{r_1+r_2}{r_1,k}_q \begin{bmatrix} x \\ k \end{bmatrix}_q. \tag{27}$$

*Proof.* By (25) we have

$$\begin{aligned} c_k(r_1,r_2;q) &= \frac{[r_1+r_2]_q}{[k]_q} \sum_{j=0}^{k} (-1)^{k-j} \begin{bmatrix} k \\ j \end{bmatrix}_q q^{\binom{k-j}{2}} \begin{bmatrix} j+r_1-1 \\ r_1 \end{bmatrix}_q \begin{bmatrix} j+r_2-1 \\ r_2 \end{bmatrix}_q \\ &= (-1)^{k-1} [r_1+r_2]_q q^{\binom{k-1}{2}} {}_3\phi_2 \left[ \begin{matrix} q^{1-k}, q^{1+r_1}, q^{1+r_2} \\ q, q^2 \end{matrix} \middle| q; q \right]. \end{aligned}$$

Comparing with (13) we see that $c_k(r_1,r_2;q) = q^{k(k-1)-r_1 r_2} \binom{r_1+r_2}{r_1,k}_q$. □

## 4 The Second $q$-analogue

In this section, we give another new family of q-positive integers $\begin{bmatrix} n \\ p,k \end{bmatrix}_q$, which have the similar properties as $\binom{n}{p,k}_q$. Since the proofs are similar we omit the details.

**Definition 1.** *For any positive integers $n, p, k$, define*

$$\begin{bmatrix} n \\ p,k \end{bmatrix}_q = \begin{bmatrix} n \\ k \end{bmatrix}_q {}_3\phi_2 \left[ \begin{matrix} q^{1-k}, q^{-p}, q^{p-n} \\ q, q^{1-n} \end{matrix} \middle| q; q^{k+1} \right]. \tag{28}$$

Obviously we have

$$\begin{bmatrix} n \\ p,k \end{bmatrix}_q = \begin{bmatrix} n \\ n-p,k \end{bmatrix}_q, \qquad \begin{bmatrix} n \\ p,n \end{bmatrix}_q = \begin{bmatrix} n \\ p \end{bmatrix}_q.$$

The definition (28) could be written as follows:

$$\begin{bmatrix} n \\ p,k \end{bmatrix}_q = \frac{[n]_q}{[k]_q} \sum_{r\geq 0} \begin{bmatrix} p \\ r \end{bmatrix}_q \begin{bmatrix} n-p \\ r \end{bmatrix}_q \begin{bmatrix} n-r-1 \\ k-r-1 \end{bmatrix}_q q^{r^2}.$$

Therefore

$$\begin{bmatrix} n \\ 0,k \end{bmatrix}_q = \begin{bmatrix} n \\ k \end{bmatrix}_q, \quad \begin{bmatrix} n, \\ 1,k \end{bmatrix}_q = [k]_q \begin{bmatrix} n \\ k \end{bmatrix}_q,$$

and

$$\begin{bmatrix} n \\ 2,k \end{bmatrix}_q = [k]_q \begin{bmatrix} n \\ k \end{bmatrix}_q + q^2 \frac{[n]_q [n-3]_q}{[2]_q} \begin{bmatrix} n-2 \\ k-2 \end{bmatrix}_q.$$

For $p \geq 0$, we have also

$$\begin{bmatrix} n \\ p,0 \end{bmatrix}_q = 0, \quad \begin{bmatrix} n \\ p,1 \end{bmatrix}_q = [n]_q, \quad \begin{bmatrix} n \\ p,2 \end{bmatrix}_q = \frac{[n]_q}{[2]_q}([n-1]_q + [p]_q[n-p]_q).$$

Applying (11) to the definition (28), we get

$$\begin{bmatrix} n \\ p,k \end{bmatrix}_q = \sum_{r\geq 0} (-1)^i \begin{bmatrix} n-i \\ k \end{bmatrix}_q \frac{[n]}{[n-i]} \begin{bmatrix} n-i \\ i \end{bmatrix}_q \begin{bmatrix} n-2i \\ p-i \end{bmatrix}_q q^{i(i-1)/2+ki}.$$

The following two generating functions are obtained in the same way as for $\binom{n}{p,k}_q$.

**Theorem 6.** *There holds*

$$\begin{aligned} &\sum_{0\leq k,p\leq n} \begin{bmatrix} n \\ p,k \end{bmatrix}_q x^p q^{\binom{p}{2}} y^k q^{\binom{k}{2}} \\ = \quad &(-y;q)_n(-x;q)_n \sum_{i\geq 0} \frac{1-q^n}{1-q^{n-i}} \begin{bmatrix} n-i \\ i \end{bmatrix}_q \frac{(-x)^i q^{i^2-i}}{(-y;q)_i(-x;q)_i(-xq^{n-i};q)_i}. \end{aligned} \tag{29}$$

**Theorem 7.** *For $0 \leq p \leq n$, there holds*

$$\sum_{k\geq 1} \begin{bmatrix} n \\ p,k \end{bmatrix}_q y^k q^{\binom{k}{2}} = y(-yq^{n-p};q)_p[n]_q\, {}_2\phi_1 \left[ \begin{matrix} q^{p+1}, q^{p-n+1} \\ q^2 \end{matrix} \middle| q; -yq^{n-p} \right]. \tag{30}$$

As (8) we get the recurrence relation

**Proposition 2.** *For $k \neq 0$ and $0 \leq p \leq n$, there holds*

$$(1-q^{n-p+1}) \begin{bmatrix} n \\ p-1,k \end{bmatrix}_q - (1-q^p) \begin{bmatrix} n \\ p,k \end{bmatrix}_q = \frac{[n]_q}{[n-1]_q}(1-q^{n-2p+1})q^{k+p-1} \begin{bmatrix} n-1 \\ p-1,k \end{bmatrix}_q. \tag{31}$$

**Corollary 2.** *We have*

$$\begin{bmatrix} n \\ p,k \end{bmatrix}_q = \frac{[n]_q}{[p]_q} \sum_{i=0}^{k-1} \begin{bmatrix} n-p \\ k-1-i \end{bmatrix}_q \begin{bmatrix} n-p+i \\ i \end{bmatrix}_q \begin{bmatrix} p \\ i+1 \end{bmatrix}_q q^{(i+1+p-n)(i+1-k)}.$$

As in the proof of Theorem 3, we can prove that $\begin{bmatrix} n \\ p,k \end{bmatrix}_q$ is also a q-positive integer by using the above Corollary.

## 5 Tables of the generalized $q$-binomial coefficients

- $\begin{bmatrix} n \\ 0,k \end{bmatrix}_q = \binom{n}{0,k}_q = \begin{bmatrix} n \\ k \end{bmatrix}_q$ for $k \geq 0$,
- $\begin{bmatrix} n \\ p,0 \end{bmatrix}_q = \binom{n}{p,0}_q = 0$, for $p \geq 0$.
- $n = 1$, $\begin{bmatrix} n \\ 1,1 \end{bmatrix}_q = \binom{n}{1,1}_q = 1$.

**Tables of $\binom{n}{p,k}_q$**

$n = 2$

| $p\backslash k$ | 1 | 2 |
|---|---|---|
| 1 | $q[2]_q$ | $[2]_q$ |
| 2 | $[2]_q$ | 1 |

$n = 3$,

| $p\backslash k$ | 1 | 2 | 3 |
|---|---|---|---|
| 1 | $q^2[3]_q$ | $q[2]_q[3]_q$ | $[3]_q$ |
| 2 | $q^2[3]_q$ | $q[2]_q[3]_q$ | $[3]_q$ |
| 3 | $[3]_q$ | $[3]_q$ | 1 |

n=4,

| $p\backslash k$ | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| 1 | $q^3[4]_q$ | $q^2[3]_q[4]_q$ | $q[2]_q[3]_q(1+q^2)$ | $[4]_q$ |
| 2 | $q^4[4]_q$ | $(1+q^2)(1+3q+2q^2+q^3)$ | $[2]_q(1+q^2)(2+q+q^2)$ | $[3]_q(1+q^2)$ |
| 3 | $q^3[4]_q$ | $q^2[3]_q[4]_q$ | $q[2]_q[3]_q(1+q^2)$ | $[4]_q$ |
| 4 | $[4]_q$ | $[2]_q[3]_q$ | $[4]_q$ | 1 |

n=5,

| $p \backslash k$ | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| 1 | $[5]_q$ | $[2]_q[5]_q(1+q^2)$ | $[3]_q[5]_q(1+q^2)$ | $[2]_q[5]_q(1+q^2)$ | $[5]_q$ |
| 2 | $[5]_q$ | $[5]_q(1+q+2q^2+q^3)$ | $[3]_q[5]_q(1+2q^2)$ | $[5]_q(1+q+2q^2+2q^3+q^4)$ | $[5]_q(1+q^2)$ |
| 3 | $[5]_q$ | $[5]_q(1+q+2q^2+q^3)$ | $[3]_q[5]_q(1+2q^2)$ | $[5]_q(1+q+2q^2+2q^3+q^4)$ | $[5]_q(1+q^2)$ |
| 4 | $[5]_q$ | $[2]_q[5]_q(1+q^2)$ | $[3]_q[5]_q(1+q^2)$ | $[2]_q[5]_q(1+q^2)$ | $[5]_q$ |
| 5 | $[5]_q$ | $[5]_q(1+q^2)$ | $[5]_q(1+q^2)$ | $[5]_q$ | 1 |

**Tables of $\begin{bmatrix} n \\ p,k \end{bmatrix}_q$**

$n=2$,

| $p\backslash k$ | 1 | 2 |
|---|---|---|
| 1 | $[2]_q$ | $[2]_q$ |
| 2 | $[2]_q$ | 1 |

$n=3$,

| $p\backslash k$ | 1 | 2 | 3 |
|---|---|---|---|
| 1 | $[3]_q$ | $[2]_q[3]_q$ | $[3]_q$ |
| 2 | $[3]_q$ | $[2]_q[3]_q$ | $[3]_q$ |
| 3 | $[3]_q$ | $[3]_q$ | 1 |

$n=4$,

| $p\backslash k$ | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| 1 | $[4]_q$ | $[3]_q[4]_q$ | $[2]_q[3]_q(1+q^2)$ | $[4]_q$ |
| 2 | $[4]_q$ | $(1+q^2)(1+2q+3q^2+q^3)$ | $[2]_q(1+q^2)(1+q+2q^2)$ | $(1+q^2)[3]_q$ |
| 3 | $[4]_q$ | $[3]_q[4]_q$ | $[2]_q[3]_q(1+q^2)$ | $[4]_q$ |
| 4 | $[4]_q$ | $[3]_q(1+q^2)$ | $[4]_q$ | 1 |

$n=5$,

| $p\backslash k$ | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| 1 | $q^4[5]_q$ | $[2]_q[5]_q(1+q^2)q^3$ | $[3]_q[5]_q(1+q^2)q^2$ | $[2]_q[5]_q(1+q^2)q$ | $[5]_q$ |
| 2 | $q^6[5]_q$ | $[5]_q(1+2q+q^2+q^3)q^5$ | $[3]_q[5]_q(2+q^2)q^4$ | $[5]_q(1+2q+2q^2+q^3+q^4)q^2$ | $[5]_q(1+q^2)$ |
| 3 | $q^6[5]_q$ | $[5]_q(1+2q+q^2+q^3)q^5$ | $[3]_q[5]_q(2+q^2)q^4$ | $[5]_q(1+2q+2q^2+q^3+q^4)q^2$ | $[5]_q(1+q^2)$ |
| 4 | $q^4[5]_q$ | $[2]_q[5]_q(1+q^2)q^3$ | $[3]_q[5]_q(1+q^2)q^2$ | $[2]_q[5]_q(1+q^2)q$ | $[5]_q$ |
| 5 | $[5]_q$ | $[5]_q(1+q^2)$ | $[5]_q(1+q^2)$ | $[5]_q$ | 1 |

The above tables seem to suggest that the sequences of the coefficients in $\binom{n}{p,k}_q$ and $\begin{bmatrix} n \\ p,k \end{bmatrix}_q$ are *unimodal* for each fixed pair $(n,p)$ or $(n,k)$.

**Remark** H. Rosengren noticed that the algebraic expressions of our coefficients $\binom{n}{p,k}_q$ and $\begin{bmatrix} n \\ p,k \end{bmatrix}_q$ are limit cases of his coefficients $R_k^l$ (see "An elementary approach to $6j$-symbols" available at http://www.arxiv.org/abs/math.CA/0312310). It would be interesting to see whether it is possible to obtain further (interesting) families of $q$-positive integers as degenerate cases of his coefficients.

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## References

[1] G. Gasper & M. Rahman, *Basic hypergeometric series*, Encyclopedia of Math. and its Applications, vol. **35**, 1990.

[2] F. Jouhet, B. Lass and J. Zeng, *Sur une généralisation des coefficients binomiaux*, Electronic J. Combin., vol.11(1), R47, 2004.

[3] M. Lassalle, *A new family of positive integers*, Ann. Combin. 6(2002), no. 3-4, 399-405.

[4] M. Lassalle, *Jack polynomials and some identities for partitions*, to appear in Trans. of Amer. Math. Soc., 2004.