

$Sp(2\nu, q)$ is strongly regular and compute its parameters. We also prove that the chromatic number of $Sp(2\nu, q)$ is $q^\nu + 1$. Section 3 is devoted to discuss the group of automorphisms $\text{Aut}(Sp(2\nu, q))$ of the graph. The structure of this group depends on q and ν . When $q = 2$, $\text{Aut}(Sp(2\nu, 2))$ is isomorphic to the symplectic group of degree 2ν over \mathbb{F}_2 . When $q > 2$, $\text{Aut}(Sp(2\nu, q))$ is the product of two subgroups which are identified clearly (cf. Theorem 3.4).

2. STRONGLY REGULARITY AND CHROMATIC NUMBERS OF SYMPLECTIC GRAPHS

For any subspace V of $\mathbb{F}_q^{(2\nu)}$, we denote the subspace of $\mathbb{F}_q^{(2\nu)}$ formed by all $\beta \in \mathbb{F}_q^{(2\nu)}$ such that $\alpha K^t \beta = 0$ for all $\alpha \in V$ by V^\perp . Then $[\alpha] \sim [\beta]$ if and only if $\beta \notin [\alpha]^\perp$.

Denote the vertex set of the graph $Sp(2\nu, q)$ by $V(Sp(2\nu, q))$. We first show that $Sp(2\nu, q)$ is strongly regular.

Theorem 2.1. *$Sp(2\nu, q)$ is a strongly regular graph with parameters*

$$\left(\frac{q^{2\nu} - 1}{q - 1}, q^{2\nu-1}, q^{2\nu-2}(q - 1), q^{2\nu-2}(q - 1) \right)$$

and eigenvalues $q^{2\nu-1}$, $q^{\nu-1}$ and $-q^{\nu-1}$.

Proof. As $|\mathbb{F}_q^{(2\nu)}| = q^{2\nu}$, it follows that $|V(Sp(2\nu, q))| = \frac{q^{2\nu}-1}{q-1}$. For any $[\alpha] \in V(Sp(2\nu, q))$, since $\dim([\alpha]^\perp) = 2\nu - 1$, we see that the degree of $[\alpha]$ which is just the number of one dimensional subspaces $[\beta]$ such that $\beta \notin [\alpha]^\perp$, is $\frac{q^{2\nu}-q^{2\nu-1}}{q-1} = q^{2\nu-1}$.

Let $[\alpha], [\beta]$ be any two different vertices of $Sp(2\nu, q)$ which are adjacent with each other or not. Then $\dim([\alpha, \beta]^\perp) = 2\nu - 2$. Note that a vertex $[\gamma]$ is adjacent with both $[\alpha]$ and $[\beta]$ is equivalent to that $\gamma \notin [\alpha]^\perp \cup [\beta]^\perp$. But

$$|([\alpha]^\perp \cup [\beta]^\perp)| = |[\alpha]^\perp| + |[\beta]^\perp| - |[\alpha, \beta]^\perp|.$$

Hence the number of vertices which are adjacent with both $[\alpha]$ and $[\beta]$ is $\frac{q^{2\nu}-2q^{2\nu-1}+q^{2\nu-2}}{q-1} = q^{2\nu-2}(q - 1)$. Therefore $Sp(2\nu, q)$ is a strongly regular graph with parameter

$$\left(\frac{q^{2\nu} - 1}{q - 1}, q^{2\nu-1}, q^{2\nu-2}(q - 1), q^{2\nu-2}(q - 1) \right).$$

By the same arguments as in [3, Section 10.2], we get that the eigenvalues of $Sp(2\nu, q)$ are $q^{2\nu-1}$, $q^{\nu-1}$ and $-q^{\nu-1}$. \square

Let $n \geq 2$. We say that a graph X is n -partite if there are subsets X_1, \dots, X_n of the vertex set $V(X)$ of X such that $V(X) = X_1 \cup \dots \cup X_n$, where $X_i \cap X_j = \emptyset$ for all $i \neq j$, and that there is no edge of X joining two vertices of the same subset. We are going to show that $Sp(2\nu, q)$ is $(q^\nu + 1)$ -partite. We need some results about subspaces of $\mathbb{F}_q^{(2\nu)}$. A subspace V of $\mathbb{F}_q^{(2\nu)}$ is called *totally isotropic* if $V \subseteq V^\perp$. Then totally isotropic subspaces of $\mathbb{F}_q^{(2\nu)}$ are of dimension $\leq \nu$ and there exist totally isotropic subspaces of dimension ν which are called *maximal totally isotropic subspaces*, cf. [6, Corollary 3.8].

The following lemma is due to Dye[1].

Lemma 2.2. *There exist maximal totally isotropic subspaces V_i , $i = 1, \dots, q^\nu + 1$, of $\mathbb{F}_q^{(2\nu)}$ such that*

$$\mathbb{F}_q^{(2\nu)} = V_1 \cup \dots \cup V_{q^\nu+1},$$

where $V_i \cap V_j = \{0\}$ for all $i \neq j$.

Proposition 2.3. *$Sp(2\nu, q)$ is $(q^\nu + 1)$ -partite. That is, there exist subsets $X_1, \dots, X_{q^\nu+1}$ of $V(Sp(2\nu, q))$ such that*

$$V(Sp(2\nu, q)) = X_1 \cup \dots \cup X_{q^\nu+1},$$

where $X_i \cap X_j = \emptyset$ for all $i \neq j$, and there is no edge of $Sp(2\nu, q)$ joining two vertices of the same subset. Moreover, the subsets $X_1, \dots, X_{q^\nu+1}$ can be so chosen that for any two distinct indices i and j , every $\alpha \in X_i$ is adjacent with exactly $q^{\nu-1}$ vertices in X_j .

Proof. Let $\mathbb{F}_q^{(2\nu)} = V_1 \cup \dots \cup V_{q^\nu+1}$ as in 2.2. Set $X_i = \{[\alpha] : \alpha \neq 0 \in V_i\}$, $i = 1, \dots, q^\nu + 1$. Then

$$V(Sp(2\nu, q)) = X_1 \cup \dots \cup X_{q^\nu+1}, \quad X_i \cap X_j = \emptyset, \quad \text{for all } i \neq j.$$

As V_i is totally isotropic, we see that there is no edge joining any two vertices in X_i . Thus $Sp(2\nu, q)$ is $(q^\nu + 1)$ -partite. For any $i \neq j$, let $[\alpha] \in X_i$. Since V_j is maximal totally isotropic of dimension ν , it follows that $\alpha \notin V_j = V_j^\perp$ and $\dim([\alpha]^\perp \cap V_j) = \dim([\alpha, V_j]^\perp) = \nu - 1$. Note that, for any $[\beta] \in X_j$, $[\beta]$ is adjacent with $[\alpha]$ if and only if $\beta \in V_j \setminus ([\alpha]^\perp \cap V_j)$. Hence the number of vertices in X_j which is adjacent with $[\alpha]$ is $\frac{q^\nu-1}{q-1} - \frac{q^{\nu-1}-1}{q-1} = q^{\nu-1}$. \square

Now we can compute the chromatic number of $Sp(2\nu, q)$.

Theorem 2.4. $\chi(Sp(2\nu, q)) = q^\nu + 1$.

Proof. By 2.3, we see that $\chi(Sp(2\nu, q)) \leq q^\nu + 1$. Note that $\chi(Sp(2\nu, q))$ is the minimal n such that $Sp(2\nu, q)$ is n -partite. Suppose that $Sp(2\nu, q)$ is n -partite. Then there exist subsets Y_1, \dots, Y_n of $V(Sp(2\nu, q))$ such that

$$V(Sp(2\nu, q)) = Y_1 \cup \dots \cup Y_n, \quad Y_i \cap Y_j = \emptyset, \quad \text{for all } i \neq j,$$

and there is no edge joining any two vertices in the same Y_i for $i = 1, \dots, n$. We want to show that $n \geq q^\nu + 1$. Suppose that $n < q^\nu + 1$. From the above equality, we have $\sum_{i=1}^n |Y_i| = \frac{q^{2\nu}-1}{q-1} = \left(\frac{q^\nu-1}{q-1}\right)(q^\nu + 1)$. Then there exists some i such that $|Y_i| > \frac{q^\nu-1}{q-1}$. Let W_i be the subspace of $\mathbb{F}_q^{(2\nu)}$ generated by all α such that $[\alpha] \in Y_i$. Then W_i is a totally isotropic subspace, hence $\dim W_i \leq \nu$. This turns out $|Y_i| \leq \frac{q^\nu-1}{q-1}$, a contradiction. Hence $\chi(Sp(2\nu, q)) = q^\nu + 1$. \square

3. AUTOMORPHISMS OF SYMPLECTIC GRAPHS

We recall that a $2\nu \times 2\nu$ matrix T is called a *symplectic matrix* (or *generalized symplectic matrix*) of order 2ν over \mathbb{F}_q if $TK^tT = K$ (or $TK^tT = kK$ for some $k \in \mathbb{F}_q^*$, respectively). The set of symplectic matrices (or generalized symplectic matrices) of order 2ν over \mathbb{F}_q forms a group with respect to the matrix multiplication, which is called the *symplectic group* (or *generalized symplectic group, respectively*), of degree 2ν over \mathbb{F}_q and denoted by $Sp_{2\nu}(\mathbb{F}_q)$ (or $GSp_{2\nu}(\mathbb{F}_q)$). The center of $Sp_{2\nu}(\mathbb{F}_q)$ consists of the identity matrix E and $-E$, and the factor group $Sp_{2\nu}(\mathbb{F}_q)/\{E, -E\}$ is called the *projective symplectic group* of degree 2ν over \mathbb{F}_q and denoted by $PSp_{2\nu}(\mathbb{F}_q)$. The center of $GSp_{2\nu}(\mathbb{F}_q)$ consists of all kE , where $k \in \mathbb{F}_q^*$, and the factor group of $GSp_{2\nu}(\mathbb{F}_q)$ with respect to its center is called the *projective generalized symplectic group* of degree 2ν over \mathbb{F}_q and denoted by $PGSp_{2\nu}(\mathbb{F}_q)$. Clearly, $PGSp_{2\nu}(\mathbb{F}_q) \cong PSp_{2\nu}(\mathbb{F}_q)$, and when $q = 2$, $GSp_{2\nu}(\mathbb{F}_2) = Sp_{2\nu}(\mathbb{F}_2)$.

Proposition 3.1. *Let T be a $2\nu \times 2\nu$ nonsingular matrix over \mathbb{F}_q and*

$$\begin{aligned} \sigma_T : V(Sp(2\nu, q)) &\rightarrow V(Sp(2\nu, q)) \\ [\alpha] &\mapsto [\alpha T]. \end{aligned}$$

Then

- (1) $T \in GSp_{2\nu}(\mathbb{F}_q)$ if and only if $\sigma_T \in \text{Aut}(Sp(2\nu, q))$. In particular, when $q = 2$, $T \in Sp_{2\nu}(\mathbb{F}_2)$ if and only if $\sigma_T \in \text{Aut}(Sp(2\nu, 2))$
- (2) For any $T_1, T_2 \in GSp_{2\nu}(\mathbb{F}_q)$, $\sigma_{T_1} = \sigma_{T_2}$ if and only if $T_1 = kT_2$ for some $k \in \mathbb{F}_q^*$;

Proof. It is clear that σ_T is an one-one correspondence from $V(Sp(2\nu, q))$ to itself.

(1) First assume $T \in GSp_{2\nu}(\mathbb{F}_q)$. Then $TK^tT = kK$ for some $k \in \mathbb{F}_q^*$. For any $[\alpha], [\beta] \in V(Sp(2\nu, q))$, since $\alpha K^t\beta = k^{-1}(\alpha T)K^t(\beta T)$, $[\alpha] \sim [\beta]$ if and only if $\sigma_T([\alpha]) \sim \sigma_T([\beta])$, hence $\sigma_T \in \text{Aut}(Sp(2\nu, q))$.

Conversely, assume $\sigma_T \in \text{Aut}(Sp(2\nu, q))$. Then, for any $\alpha, \beta \neq 0 \in \mathbb{F}_q^{(2\nu)}$, $\alpha K^t\beta = 0$ if and only if $\alpha(TK^tT)^t\beta = 0$. Hence, for any $\alpha \neq 0 \in \mathbb{F}_q^{(2\nu)}$, the two systems of linear equations $(\alpha K)^tX = 0$, $(\alpha TK^tT)^tX = 0$ have the same solutions. But $\text{rank}(\alpha K) = \text{rank}(\alpha TK^tT) = 1$, we see that $\alpha K = k(\alpha TK^tT)$ for some $k \in \mathbb{F}_q^*$, which depends on α . Take $\alpha = (1, 0, \dots, 0), (0, 1, \dots, 0), \dots, (0, 0, \dots, 1)$, we get that $K = \text{diag}(k_1, k_2, \dots, k_{2\nu})TK^tT$, for some $k_1, k_2, \dots, k_{2\nu} \in \mathbb{F}_q^*$. Take $\alpha = (1, 1, \dots, 1)$, we see that $k_1 = k_2 = \dots = k_{2\nu}$, hence $K = k_1TK^tT$.

(2) It is clear that $\sigma_{T_1} = \sigma_{T_2}$ if $T_1 = kT_2$ for some $k \in \mathbb{F}_q^*$. Conversely, suppose that $\sigma_{T_1} = \sigma_{T_2}$. Then, for any $\alpha \neq 0 \in \mathbb{F}_q^{(2\nu)}$, $\alpha T_1 = k\alpha T_2$ for some $k \in \mathbb{F}_q^*$. Take $\alpha = (1, 0, \dots, 0), (0, 1, \dots, 0)$, and so on as above, we see that $T_1 = kT_2$ for some $k \in \mathbb{F}_q^*$. \square

By 3.1, every generalized symplectic matrix in $GSp_{2\nu}(\mathbb{F}_q)$ induces an automorphism of $Sp(2\nu, q)$ and two generalized symplectic matrices T_1 and T_2 induce the same automorphism of $Sp(2\nu, q)$ if and only if $T_1 = kT_2$ for some $k \in \mathbb{F}_q^*$. Thus $PSp_{2\nu}(\mathbb{F}_q)$ can be regarded as a subgroup of $\text{Aut}(Sp(2\nu, q))$.

Proposition 3.2. $Sp(2\nu, q)$ is vertex transitive and edge transitive.

Proof. For any $[\alpha], [\beta] \in V(Sp(2\nu, q))$, there exists $T \in Sp_{2\nu}(\mathbb{F}_q)$ such that $\alpha T = \beta$ by [6, Lemma 3.11]. Then $\sigma_T \in \text{Aut}(Sp(2\nu, q))$ such that $\sigma_T([\alpha]) = [\beta]$. Hence $Sp(2\nu, q)$ is vertex transitive.

Let $[\alpha_1], [\alpha_2], [\beta_1], [\beta_2] \in V(Sp(2\nu, q))$ such that $[\alpha_1] \sim [\alpha_2]$ and $[\beta_1] \sim [\beta_2]$. We may assume that $\alpha_1 K^t \alpha_2 = \beta_1 K^t \beta_2$. Then, by [6, Lemma 3.11] again, there exists $T \in Sp_{2\nu}(\mathbb{F}_q)$ such that $\alpha_1 T = \beta_1$ and $\alpha_2 T = \beta_2$. Then $\sigma_T \in \text{Aut}(Sp(2\nu, q))$ such that $\sigma_T([\alpha_1]) = [\beta_1]$ and $\sigma_T([\alpha_2]) = [\beta_2]$. Hence $Sp(2\nu, q)$ is edge transitive. \square

When $q = 2$, we have the following

Proposition 3.3. $\text{Aut}(Sp(2\nu, 2)) \cong Sp_{2\nu}(\mathbb{F}_2)$.

Proof. Let

$$\begin{aligned} \sigma : Sp_{2\nu}(\mathbb{F}_2) &\rightarrow \text{Aut}(Sp(2\nu, 2)) \\ T &\mapsto \sigma_T. \end{aligned}$$

Then, by 3.1, σ is an injection. Clearly, σ preserves the operation. It remains to show that, for any $\tau \in \text{Aut}(Sp(2\nu, 2))$, there exists a $T \in Sp_{2\nu}(\mathbb{F}_2)$ such that $\tau = \sigma_T$.

Note that, for any $\alpha \neq 0 \in \mathbb{F}_2^{(2\nu)}$, we have that $[\alpha] = \{0, \alpha\}$. We will denote the uniquely defined element $\tau([\alpha]) \setminus \{0\}$ by $\tau(\alpha)$ and set $\tau(0) = 0$. Then from $\tau \in \text{Aut}(Sp(2\nu, 2))$ we see that $\alpha K^t \beta = \tau(\alpha) K^t (\tau(\beta))$ for any $\alpha, \beta \in \mathbb{F}_2^{(2\nu)}$ (not necessarily non-zero). Fix any $\alpha \in \mathbb{F}_2^{(2\nu)}$. Let $\beta_1, \beta_2 \in \mathbb{F}_2^{(2\nu)}$. Then

$$\begin{aligned} \alpha K^t \beta_1 &= \tau(\alpha) K^t (\tau(\beta_1)), \\ \alpha K^t \beta_2 &= \tau(\alpha) K^t (\tau(\beta_2)). \end{aligned}$$

Thus

$$\alpha K^t (\beta_1 + \beta_2) = \tau(\alpha) K^t (\tau(\beta_1) + \tau(\beta_2)).$$

But

$$\alpha K^t (\beta_1 + \beta_2) = \tau(\alpha) K^t (\tau(\beta_1 + \beta_2)),$$

hence

$$\tau(\alpha) K^t (\tau(\beta_1 + \beta_2) + \tau(\beta_1) + \tau(\beta_2)) = 0.$$

This is true for any $\alpha \in \mathbb{F}_2^{(2\nu)}$, it follows that $\tau(\beta_1 + \beta_2) + \tau(\beta_1) + \tau(\beta_2) = 0$, i.e., $\tau(\beta_1 + \beta_2) = \tau(\beta_1) + \tau(\beta_2)$. Set

$$T = \begin{pmatrix} \tau(1, 0, \dots, 0) \\ \tau(0, 1, \dots, 0) \\ \vdots \\ \tau(0, 0, \dots, 1) \end{pmatrix}.$$

Then $\tau(\alpha) = \alpha T$ for any $\alpha \in \mathbb{F}_2^{(2\nu)}$. Thus T is nonsingular. By 3.1 $T \in Sp_{2\nu}(\mathbb{F}_2)$ and $\tau = \sigma_T$ as required. \square

From now on, we assume that $q > 2$. In $\mathbb{F}_q^{(2\nu)}$, let us set

$$\begin{aligned} e_1 &= (1, 0, 0, 0, \dots, 0, 0), \\ f_1 &= (0, 1, 0, 0, \dots, 0, 0), \\ e_2 &= (0, 0, 1, 0, \dots, 0, 0), \\ f_2 &= (0, 0, 0, 1, \dots, 0, 0), \\ &\dots\dots \\ e_\nu &= (0, 0, 0, 0, \dots, 1, 0), \\ f_\nu &= (0, 0, 0, 0, \dots, 0, 1). \end{aligned}$$

Then $e_i, f_i, i = 1, \dots, \nu$, form a basis of $\mathbb{F}_q^{(2\nu)}$ and $e_i K^t f_i = 1, e_i K^t e_j = 0, f_i K^t f_j = 0, i, j = 1, \dots, \nu$, and $e_i K^t f_j = 0, i \neq j, i, j = 1, \dots, \nu$.

In order to describe $\text{Aut}(Sp(2\nu, q))$ for any prime power q , we need some definition from group theory. Let φ be the natural action of $\text{Aut}(\mathbb{F}_q)$ on the group $\mathbb{F}_q^* \times \dots \times \mathbb{F}_q^*$ (ν in number) defined by

$$\varphi(\pi)((k_1, \dots, k_\nu)) = (\pi(k_1), \dots, \pi(k_\nu)), \text{ for all } \pi \in \text{Aut}(\mathbb{F}_q) \text{ and } k_1, \dots, k_\nu \in \mathbb{F}_q^*,$$

then the semi-direct product of $\mathbb{F}_q^* \times \dots \times \mathbb{F}_q^*$ by $\text{Aut}(\mathbb{F}_q)$ corresponding to φ , denoted by $(\mathbb{F}_q^* \times \dots \times \mathbb{F}_q^*) \rtimes_\varphi \text{Aut}(\mathbb{F}_q)$, is the group consisting of all elements of the form (k_1, \dots, k_ν, π) , where $k_1, \dots, k_\nu \in \mathbb{F}_q^*$ and $\pi \in \text{Aut}(\mathbb{F}_q)$, with multiplication defined by

$$(k_1, \dots, k_\nu, \pi)(k'_1, \dots, k'_\nu, \pi') = (k_1 \pi(k'_1), \dots, k_\nu \pi(k'_\nu), \pi \pi').$$

Then the main result about $\text{Aut}(Sp(2\nu, q))$ is as follows.

Theorem 3.4. *Regard $PSp_{2\nu}(\mathbb{F}_q)$ as a subgroup of $\text{Aut}(Sp(2\nu, q))$ and let E be the subgroup of $\text{Aut}(Sp(2\nu, q))$ defined as follows*

$$E = \{\sigma \in \text{Aut}(Sp(2\nu, q)) : \sigma([e_i]) = [e_i], \sigma([f_i]) = [f_i], i = 1, \dots, \nu\}.$$

Then

- (1) $\text{Aut}(Sp(2\nu, q)) = PSp_{2\nu}(\mathbb{F}_q) \cdot E$;
- (2) If $\nu = 1$, then E is isomorphic to the symmetric group on $q - 1$ elements;
- (3) If $\nu > 1$, then

$$E \cong \underbrace{(\mathbb{F}_q^* \times \dots \times \mathbb{F}_q^*)}_\nu \rtimes_\varphi \text{Aut}(\mathbb{F}_q).$$

Proof. (1) Let $\tau \in \text{Aut}(Sp(2\nu, q))$. Suppose that $\tau([e_i]) = [e'_i], \tau([f_i]) = [f'_i], i = 1, \dots, \nu$. Then $e'_i K^t f'_i \neq 0, e'_i K^t e'_j = 0, f'_i K^t f'_j = 0, i, j = 1, \dots, \nu$ and $e'_i K^t f'_j = 0, i \neq j, i, j = 1, \dots, \nu$. We may choose $e'_i, f'_i, i = 1, \dots, \nu$, such that $e'_i K^t f'_i = 1, i =$

$1, \dots, \nu$. Let

$$A = \begin{pmatrix} e_1 \\ f_1 \\ e_2 \\ f_2 \\ \vdots \\ e_\nu \\ f_\nu \end{pmatrix}, \quad A' = \begin{pmatrix} e'_1 \\ f'_1 \\ e'_2 \\ f'_2 \\ \vdots \\ e'_\nu \\ f'_\nu \end{pmatrix}.$$

Then $AK^tA = K = A'K^tA'$. Thus, by [6, Lemma 3.11], there exists $T \in Sp_{2\nu}(\mathbb{F}_q)$ such that $A = A'T$, i.e., $e'_iT = e_i$, $f'_iT = f_i$, $i = 1, \dots, \nu$. Set $\tau_1 = \sigma_T\tau$. Then $\tau_1([e_i]) = [e_i]$, $\tau_1([f_i]) = [f_i]$, $i = 1, \dots, \nu$, hence $\tau_1 \in E$. Thus $\tau \in PSp_{2\nu}(\mathbb{F}_q) \cdot E$. It follows that $\text{Aut}(Sp(2\nu, q)) = PSp_{2\nu}(\mathbb{F}_q) \cdot E$.

(2) When $\nu = 1$, it is clear that E is isomorphic to the symmetric group on the $q - 1$ vertices of $Sp(2, q)$ since $Sp(2, q)$ is a complete graph.

(3) Suppose that $\nu > 1$. Firstly, let us write out some elements of E . Let $k_1, \dots, k_\nu \in \mathbb{F}_q^*$ and $\pi \in \text{Aut}(\mathbb{F}_q)$. Let $\sigma_{(k_1, \dots, k_\nu, \pi)}$ be the map which takes any vertex $[a_1, a_2, a_3, a_4, \dots, a_{2\nu-1}, a_{2\nu}]$ of $Sp(2\nu, q)$ to the vertex

$$[\pi(a_1), k_1\pi(a_2), k_2\pi(a_3), k_1k_2^{-1}\pi(a_4), \dots, k_\nu\pi(a_{2\nu-1}), k_1k_\nu^{-1}\pi(a_{2\nu})].$$

Then it is clear that $\sigma_{(k_1, \dots, k_\nu, \pi)}$ is well-defined. Furthermore, it is easy to see that $\sigma_{(k_1, \dots, k_\nu, \pi)}$ is injective, but the vertex set of $Sp(2\nu, q)$ is finite, $\sigma_{(k_1, \dots, k_\nu, \pi)}$ is a bijection from $V(Sp(2\nu, q))$ to itself. Let $\alpha = [a_1, a_2, a_3, a_4, \dots, a_{2\nu-1}, a_{2\nu}]$, $\beta = [a'_1, a'_2, a'_3, a'_4, \dots, a'_{2\nu-1}, a'_{2\nu}]$ be two vertices of $Sp(2\nu, q)$. If $\alpha \not\sim \beta$, then, by definition,

$$(a_1a'_2 - a_2a'_1) + (a_3a'_4 - a_4a'_3) + \dots + (a_{2\nu-1}a'_{2\nu} - a_{2\nu}a'_{2\nu-1}) = 0,$$

which implies that

$$\begin{aligned} & (\pi(a_1)k_1\pi(a'_2) - \pi(a_2)k_1\pi(a'_1)) + (k_2\pi(a_3)k_1k_2^{-1}\pi(a'_4) - k_1k_2^{-1}\pi(a_4)k_2\pi(a'_3)) \\ & + \dots + (k_\nu\pi(a_{2\nu-1})k_1k_\nu^{-1}\pi(a'_{2\nu}) - k_1k_\nu^{-1}\pi(a_{2\nu})k_\nu\pi(a'_{2\nu-1})) = 0, \end{aligned}$$

i.e., $\sigma_{(k_1, \dots, k_\nu, \pi)}(\alpha) \not\sim \sigma_{(k_1, \dots, k_\nu, \pi)}(\beta)$. Since the edges set of $Sp(2\nu, q)$ is finite, $\alpha \not\sim \beta$ if and only if $\sigma_{(k_1, \dots, k_\nu, \pi)}(\alpha) \not\sim \sigma_{(k_1, \dots, k_\nu, \pi)}(\beta)$. Hence $\sigma_{(k_1, \dots, k_\nu, \pi)} \in \text{Aut}(Sp(2\nu, q))$. Note that $\sigma_{(k_1, \dots, k_\nu, \pi)}([e_i]) = [e_i]$, $\sigma_{(k_1, \dots, k_\nu, \pi)}([f_i]) = [f_i]$, $i = 1, \dots, \nu$, hence, $\sigma_{(k_1, \dots, k_\nu, \pi)} \in E$.

If we define a map h as $(k_1, \dots, k_\nu, \pi) \mapsto \sigma_{(k_1, \dots, k_\nu, \pi)}$, then it is easy to verify that h is a group homomorphism from $(\mathbb{F}_q^* \times \dots \times \mathbb{F}_q^*) \times_\varphi \text{Aut}(\mathbb{F}_q)$ to E . It is also easy to see that if $(k_1, \dots, k_\nu, \pi) \neq (k'_1, \dots, k'_\nu, \pi')$ then $\sigma_{(k_1, \dots, k_\nu, \pi)} \neq \sigma_{(k'_1, \dots, k'_\nu, \pi')}$. Thus, to show that h is a group isomorphism, it remains to show that every element of E is of the form $\sigma_{(k_1, \dots, k_\nu, \pi)}$.

Suppose that $\sigma \in E$. Note that if $\sigma([a_1, a_2, \dots, a_{2\nu}]) = [b_1, b_2, \dots, b_{2\nu}]$, then $a_{2i-1} \neq 0$ if and only if $[a_1, a_2, \dots, a_{2\nu}] \sim [f_i]$ and $a_{2i} \neq 0$ if and only if $[a_1, a_2, \dots, a_{2\nu}] \sim [e_i]$, and similar results are also true for b_i . But $\sigma([e_i]) = [e_i]$ and $\sigma([f_i]) = [f_i]$, it follows that $a_i = 0$ if and only if $b_i = 0$. For any vertex $[a_1, a_2, \dots, a_{2\nu}]$, if $a_1 = \dots = a_{i-1} = 0$ and $a_i \neq 0$ then $[a_1, a_2, \dots, a_{2\nu}]$ can be uniquely written as $[0, \dots, 0, 1, a'_{i+1}, \dots, a'_{2\nu}]$ and $\sigma([a_1, a_2, \dots, a_{2\nu}])$ can be uniquely written as

$[0, \dots, 0, 1, b'_{i+1}, \dots, b'_{2\nu}]$. Let us show how to determine $b'_{i+1}, \dots, b'_{2\nu}$ from $a'_{i+1}, \dots, a'_{2\nu}$. We will use frequently the fact that, for any vertices $[\alpha], [\beta]$, if $[\alpha] \not\sim [\beta]$ then $\sigma([\alpha]) \not\sim \sigma([\beta])$.

In the following, we will denote $[a_1, a'_1, a_2, a'_2, \dots, a_\nu, a'_\nu]$ by $\sum_{i=1}^\nu a_i[e_i] + \sum_{i=1}^\nu a'_i[f_i]$, for example, $[a, b, 0, \dots, 0]$ is denoted by $a[e_1] + b[f_1]$. Since σ is a bijection from $V(Sp(2\nu, q))$ to itself, we have permutations $\pi_i, i = 2, \dots, 2\nu$, of \mathbb{F}_q with $\pi(0) = 0$ such that

$$\begin{aligned}\sigma([e_1] + a_{2i-1}[e_i]) &= [e_1] + \pi_{2i-1}(a_{2i-1})[e_i], \\ \sigma([e_1] + a_{2i}[f_i]) &= [e_1] + \pi_{2i}(a_{2i})[f_i].\end{aligned}$$

We firstly consider the cases $\sigma([0, 1, a_3, \dots, a_{2\nu}])$ and $\sigma([1, a_2, a_3, \dots, a_{2\nu}])$. Let $\sigma([0, 1, a_3, \dots, a_{2\nu}]) = [0, 1, a'_3, \dots, a'_{2\nu}]$ and $j \geq 1$. If $a_{2j+1} \neq 0$, then, from $[0, 1, a_3, \dots, a_{2\nu}] \not\sim [e_1] + a_{2j+1}^{-1}[f_{j+1}]$ we have $[0, 1, a'_3, \dots, a'_{2\nu}] \not\sim [e_1] + \pi_{2j+2}(a_{2j+1}^{-1})[f_{j+1}]$, hence, $a'_{2j+1} = \pi_{2j+2}(a_{2j+1}^{-1})^{-1}$. If $a_{2j+2} \neq 0$, then from $[0, 1, a_3, \dots, a_{2\nu}] \not\sim [e_1] - a_{2j+2}^{-1}[e_{j+1}]$ we have $[0, 1, a'_3, \dots, a'_{2\nu}] \not\sim [e_1] + \pi_{2j+1}(-a_{2j+2}^{-1})[e_{j+1}]$, hence, $a'_{2j+2} = -\pi_{2j+1}(-a_{2j+2}^{-1})^{-1}$. Thus

$$(1) \quad \sigma([0, 1, a_3, \dots, a_{2\nu}]) = [0, 1, a'_3, \dots, a'_{2\nu}],$$

where $a'_{2j+1} = \pi_{2j+2}(a_{2j+1}^{-1})^{-1}$ if $a_{2j+1} \neq 0$ and $a'_{2j+2} = -\pi_{2j+1}(-a_{2j+2}^{-1})^{-1}$ if $a_{2j+2} \neq 0$.

For the case $\sigma([1, a_2, a_3, \dots, a_{2\nu}])$. Let $\sigma([1, a_2, a_3, \dots, a_{2\nu}]) = [1, a''_2, a''_3, \dots, a''_{2\nu}]$. From $[1, a_2, a_3, \dots, a_{2\nu}] \not\sim [e_1] + a_2[f_1]$ we get $[1, a''_2, a''_3, \dots, a''_{2\nu}] \not\sim [e_1] + \pi_2(a_2)[f_1]$, hence, $a''_2 = \pi_2(a_2)$. Let $j \geq 1$. If $a_{2j+1} \neq 0$, then, from $[1, a_2, a_3, \dots, a_{2\nu}] \not\sim [f_1] - a_{2j+1}^{-1}[f_{j+1}]$ and $\sigma([f_1] - a_{2j+1}^{-1}[f_{j+1}]) = [f_1] - \pi_{2j+1}(a_{2j+1})^{-1}[f_{j+1}]$ as been shown above, we have $[1, a''_2, a''_3, \dots, a''_{2\nu}] \not\sim [f_1] - \pi_{2j+1}(a_{2j+1})^{-1}[f_{j+1}]$, hence, $a''_{2j+1} = \pi_{2j+1}(a_{2j+1})$. Similarly, if $a_{2j+2} \neq 0$, then from $[1, a_2, a_3, \dots, a_{2\nu}] \not\sim [f_1] + a_{2j+2}^{-1}[e_{j+1}]$ we have $[1, a''_2, a''_3, \dots, a''_{2\nu}] \not\sim [f_1] + \pi_{2j+2}(a_{2j+2})^{-1}[e_{j+1}]$, hence, $a''_{2j+2} = \pi_{2j+2}(a_{2j+2})$. Thus, for any $a_2, a_3, \dots, a_{2\nu} \in \mathbb{F}_q$,

$$(2) \quad \sigma([1, a_2, a_3, \dots, a_{2\nu}]) = [1, \pi_2(a_2), \pi_3(a_3), \dots, \pi_{2\nu}(a_{2\nu})].$$

Then, let $i \geq 2$, we discuss the general cases $\sigma([0, \dots, 0, 1, a_{2i+1}, \dots, a_{2\nu}])$ and $\sigma([0, \dots, 0, 1, a_{2i}, \dots, a_{2\nu}])$. The above results of case $i = 1$ will be used. Let $\sigma([0, \dots, 0, 1, a_{2i+1}, \dots, a_{2\nu}]) = [0, \dots, 0, 1, a'_{2i+1}, \dots, a'_{2\nu}]$ and $j \geq i$. If $a_{2j+1} \neq 0$, then, from

$$[0, \dots, 0, 1, a_{2i+1}, \dots, a_{2\nu}] \not\sim [e_1] + [e_i] + a_{2j+1}^{-1}[f_{j+1}]$$

and $\sigma([e_1] + [e_i] + a_{2j+1}^{-1}[f_{j+1}]) = [e_1] + \pi_{2i-1}(1)[e_i] + \pi_{2j+2}(a_{2j+1}^{-1})[f_{j+1}]$ as been shown above, we have

$$[0, \dots, 0, 1, a'_{2i+1}, \dots, a'_{2\nu}] \not\sim [e_1] + \pi_{2i-1}(1)[e_i] + \pi_{2j+2}(a_{2j+1}^{-1})[f_{j+1}],$$

hence, $a'_{2j+1} = \pi_{2i-1}(1)\pi_{2j+2}(a_{2j+1}^{-1})^{-1}$. Similarly, if $a_{2j+2} \neq 0$, then from

$$[0, \dots, 0, 1, a_{2i+1}, \dots, a_{2\nu}] \not\sim [e_1] + [e_i] - a_{2j+2}^{-1}[e_{j+1}]$$

we have

$$[0, \dots, 0, 1, a'_{2i+1}, \dots, a'_{2\nu}] \not\sim [e_1] + \pi_{2i-1}(1)[e_i] + \pi_{2j+1}(-a_{2j+2}^{-1})[e_{j+1}],$$

hence, $a'_{2j+2} = -\pi_{2i-1}(1)\pi_{2j+1}(-a_{2j+2}^{-1})^{-1}$. Thus,

$$(3) \quad \sigma([0, \dots, 0, 1, a_{2i+1}, \dots, a_{2\nu}]) = [0, \dots, 0, 1, a'_{2i+1}, \dots, a'_{2\nu}],$$

where $a'_{2j+1} = \pi_{2i-1}(1)\pi_{2j+2}(a_{2j+1}^{-1})^{-1}$ if $a_{2j+1} \neq 0$ and $a'_{2j+2} = -\pi_{2i-1}(1)\pi_{2j+1}(-a_{2j+2}^{-1})^{-1}$ if $a_{2j+2} \neq 0$.

Finally, for the case $\sigma([0, \dots, 0, 1, a_{2i}, \dots, a_{2\nu}])$. Let $\sigma([0, \dots, 0, 1, a_{2i}, \dots, a_{2\nu}]) = [0, \dots, 0, 1, a''_{2i}, \dots, a''_{2\nu}]$. From

$$[0, \dots, 0, 1, a_{2i}, \dots, a_{2\nu}] \not\sim [e_1] + [e_i] + a_{2i}[f_i]$$

we get

$$[0, \dots, 0, 1, a''_{2i}, \dots, a''_{2\nu}] \not\sim [e_1] + \pi_{2i-1}(1)[e_i] + \pi_{2i}(a_{2i})[f_i],$$

hence, $a''_{2i} = \pi_{2i-1}(1)^{-1}\pi_{2i}(a_{2i})$. Let $j \geq i$. If $a_{2j+1} \neq 0$, then from

$$[0, \dots, 0, 1, a_{2i}, \dots, a_{2\nu}] \not\sim [f_i] - a_{2j+1}^{-1}[f_{j+1}]$$

we have

$$[0, \dots, 0, 1, a''_{2i}, \dots, a''_{2\nu}] \not\sim [f_i] - \pi_{2i-1}(1)^{-1}\pi_{2j+1}(a_{2j+1})^{-1}[f_{j+1}],$$

hence, $a''_{2j+1} = \pi_{2i-1}(1)^{-1}\pi_{2j+1}(a_{2j+1})$. If $a_{2j+2} \neq 0$, then from

$$[0, \dots, 0, 1, a_{2i}, \dots, a_{2\nu}] \not\sim [f_i] + a_{2j+2}^{-1}[e_{j+1}]$$

we have

$$[0, \dots, 0, 1, a''_{2i}, \dots, a''_{2\nu}] \not\sim [f_i] + \pi_{2i-1}(1)\pi_{2j+2}(a_{2j+2})^{-1}[e_{j+1}],$$

hence, $a''_{2j+2} = \pi_{2i-1}(1)^{-1}\pi_{2j+2}(a_{2j+2})$. Thus, for any $a_{2i}, a_{2i+1}, \dots, a_{2\nu} \in \mathbb{F}_q$,

$$(4) \quad \begin{aligned} & \sigma([0, \dots, 0, 1, a_{2i}, a_{2i+1}, \dots, a_{2\nu}]) \\ &= [0, \dots, 0, 1, \pi_{2i-1}(1)^{-1}\pi_{2i}(a_{2i}), \pi_{2i-1}(1)^{-1}\pi_{2i+1}(a_{2i+1}), \dots, \pi_{2i-1}(1)^{-1}\pi_{2\nu}(a_{2\nu})]. \end{aligned}$$

Having represented σ by π_i , $i = 2, \dots, 2\nu$, let us discuss some properties of π_i .

Lemma 3.5. (1) For any $i \geq 1$ and $a \in \mathbb{F}_q$,

$$\pi_{2i+1}(1)\pi_{2i+2}(a) = \pi_{2i+2}(1)\pi_{2i+1}(a) = \pi_2(a);$$

(2) For any $i \geq 2$ and $a, b \in \mathbb{F}_q$,

$$\begin{aligned} \pi_i(a+b) &= \pi_i(a) + \pi_i(b); \\ \pi_i(-a) &= -\pi_i(a); \\ \pi_i(ab) &= \pi_i(a)\pi_i(b)\pi_i(1)^{-1}; \\ \pi_i(a^{-1}) &= \pi_i(a)^{-1}\pi_i(1)^2 \text{ if } a \neq 0. \end{aligned}$$

Proof. (1) We may assume that $a \neq 0$. Since $[e_1] + a[e_{i+1}] + a[f_{i+1}] \not\sim [e_{i+1}] + [f_{i+1}]$, it follows that $\sigma([e_1] + a[e_{i+1}] + a[f_{i+1}]) \not\sim \sigma([e_{i+1}] + [f_{i+1}])$, but

$$\begin{aligned} \sigma([e_1] + a[e_{i+1}] + a[f_{i+1}]) &= [e_1] + \pi_{2i+1}(a)[e_{i+1}] + \pi_{2i+2}(a)[f_{i+1}], \\ \sigma([e_{i+1}] + [f_{i+1}]) &= [e_{i+1}] + \pi_{2i+1}(1)^{-1}\pi_{2i+2}(1)[f_{i+1}], \end{aligned}$$

we have that

$$\pi_{2i+1}(1)^{-1}\pi_{2i+2}(1)\pi_{2i+1}(a) - \pi_{2i+2}(a) = 0,$$

i.e.,

$$\pi_{2i+1}(1)\pi_{2i+2}(a) = \pi_{2i+2}(1)\pi_{2i+1}(a).$$

Similarly, since $[e_1] + a[f_1] + [e_{i+1}] \not\sim [e_1] + a[f_{i+1}]$, we have that $[e_1] + \pi_2(a)[f_1] + \pi_{2i+1}(1)[e_{i+1}] \not\sim [e_1] + \pi_{2i+2}(a)[f_{i+1}]$, hence, $\pi_{2i+1}(1)\pi_{2i+2}(a) = \pi_2(a)$.

(2) From $[e_1] + (a+b)[f_1] + [e_2] \not\sim [e_1] + a[f_1] + b[f_2]$ we have that

$$[e_1] + \pi_2(a+b)[f_1] + \pi_3(1)[e_2] \not\sim [e_1] + \pi_2(a)[f_1] + \pi_4(b)[f_2].$$

Then $\pi_2(a) - \pi_2(a+b) + \pi_3(1)\pi_4(b) = 0$, but $\pi_3(1)\pi_4(b) = \pi_2(b)$, hence, $\pi_2(a+b) = \pi_2(a) + \pi_2(b)$. It turns out from (1) that this equality holds for all $i \geq 2$. Thus $\pi_i(-a) = -\pi_i(a)$ as $\pi_i(0) = 0$.

For multiplication, let $i \geq 1$, from $[e_1] + b[e_{i+1}] + ab[f_{i+1}] \not\sim [e_{i+1}] + a[f_{i+1}]$ we get that

$$[e_1] + \pi_{2i+1}(b)[e_{i+1}] + \pi_{2i+2}(ab)[f_{i+1}] \not\sim [e_{i+1}] + \pi_{2i+1}(1)^{-1}\pi_{2i+2}(a)[f_{i+1}],$$

hence, $\pi_{2i+1}(b)\pi_{2i+1}(1)^{-1}\pi_{2i+2}(a) - \pi_{2i+2}(ab) = 0$, but $\pi_{2i+1}(b)\pi_{2i+1}(1)^{-1} = \pi_{2i+2}(b)\pi_{2i+2}(1)^{-1}$. Thus

$$\pi_{2i+2}(ab) = \pi_{2i+2}(a)\pi_{2i+2}(b)\pi_{2i+2}(1)^{-1}.$$

It follows from $\pi_{2i+1}(1)\pi_{2i+2}(a) = \pi_{2i+2}(1)\pi_{2i+1}(a)$ and $\pi_{2i+1}(1)\pi_{2i+2}(1) = \pi_{2i+2}(1)\pi_{2i+1}(1)$ that the above equality also holds for $2i+1$. It remains to consider π_2 . We have

$$\begin{aligned} \pi_2(ab) &= \pi_3(1)\pi_4(ab) \\ &= \pi_3(1)\pi_4(1)^{-1}\pi_4(a)\pi_4(b) \\ &= \pi_3(1)^{-1}\pi_4(1)^{-1}\pi_2(a)\pi_2(b) \\ &= \pi_2(a)\pi_2(b)\pi_2(1)^{-1}. \end{aligned}$$

Finally, if $a \neq 0$, then from $\pi_i(1) = \pi_i(aa^{-1}) = \pi_i(a)\pi_i(a^{-1})\pi_i(1)^{-1}$ we obtain that $\pi_i(a^{-1}) = \pi_i(a)^{-1}\pi_i(1)^2$, then the proof of lemma is complete. \square

We continue the proof of the theorem. Let us denote the identity automorphism on \mathbb{F}_q by π_1 . Then when $i = 1$, (3) reduces to (1) and (4) reduces to (2). Therefore (3) and (4) hold for all i , where $1 \leq i \leq \nu$. By the above lemma, for any $i \geq 1$, we can rewrite (3) in the form of (4) as follows. In (3), for any $j \geq i$, we have

$$\begin{aligned} a'_{2j+1} &= \pi_{2i-1}(1)\pi_{2j+2}(a_{2j+1}^{-1})^{-1} \\ &= \pi_{2i-1}(1)\pi_{2j+2}(a_{2j+1})\pi_{2j+2}(1)^{-2} \\ &= \pi_{2i-1}(1)\pi_{2j+1}(1)^{-1}\pi_{2j+2}(1)^{-1}\pi_{2j+1}(a_{2j+1}) \\ &= \pi_{2i-1}(1)\pi_2(1)^{-1}\pi_{2j+1}(a_{2j+1}) \\ &= \pi_{2i}(1)^{-1}\pi_{2j+1}(a_{2j+1}), \end{aligned}$$

and

$$\begin{aligned}
a'_{2j+2} &= -\pi_{2i-1}(1)\pi_{2j+1}(-a_{2j+2}^{-1})^{-1} \\
&= \pi_{2i-1}(1)\pi_{2j+1}(a_{2j+2}^{-1})^{-1} \\
&= \pi_{2i-1}(1)\pi_{2j+1}(a_{2j+2})\pi_{2j+1}(1)^{-2} \\
&= \pi_{2i-1}(1)\pi_{2j+1}(1)^{-1}\pi_{2j+2}(1)^{-1}\pi_{2j+2}(a_{2j+2}) \\
&= \pi_{2i-1}(1)\pi_2(1)^{-1}\pi_{2j+2}(a_{2j+2}) \\
&= \pi_{2i}(1)^{-1}\pi_{2j+2}(a_{2j+2}).
\end{aligned}$$

Hence, for any $a_{2i+1}, \dots, a_{2\nu} \in \mathbb{F}_q$,

$$(5) \quad \sigma([0, \dots, 0, 1, a_{2i+1}, \dots, a_{2\nu}]) = [0, \dots, 0, 1, \pi_{2i}(1)^{-1}\pi_{2i+1}(a_{2i+1}), \dots, \pi_{2i}(1)^{-1}\pi_{2\nu}(a_{2\nu})],$$

which is of the same form as (4).

Now let $k_1 = \pi_2(1)$, $\pi = k_1^{-1}\pi_2$, $k_2 = \pi_3(1)$, $k_3 = \pi_5(1), \dots, k_\nu = \pi_{2\nu-1}(1)$. Then $\pi \in \text{Aut}(\mathbb{F}_q)$, $\pi_2 = k_1\pi$, $\pi_3 = k_2\pi$, $\pi_4 = k_1k_2^{-1}\pi, \dots, \pi_{2\nu-1} = k_\nu\pi$, $\pi_{2\nu} = k_1k_\nu^{-1}\pi$. Assembling (4) and (5), we obtain

$$\begin{aligned}
&\sigma([a_1, a_2, a_3, a_4, \dots, a_{2\nu-1}, a_{2\nu}]) \\
&= [\pi(a_1), k_1\pi(a_2), k_2\pi(a_3), k_1k_2^{-1}\pi(a_4), \dots, k_\nu\pi(a_{2\nu-1}), k_1k_\nu^{-1}\pi(a_{2\nu})].
\end{aligned}$$

Hence $\sigma = h(k_1, \dots, k_\nu, \pi)$, as required. \square

Corollary 3.6. *When $\nu = 1$,*

$$|\text{Aut}(Sp(2, q))| = q(q^2 - 1) \cdot (q - 2)!,$$

and when $\nu \geq 2$,

$$|\text{Aut}(Sp(2\nu, q))| = q^{\nu^2} \prod_{i=1}^{\nu} (q^{2i} - 1) \cdot [\mathbb{F}_q : \mathbb{F}_p].$$

Proof. Note that $PSp_{2\nu}(\mathbb{F}_q) \cap E$ consists of σ which is reduced from some matrix of the form $\text{diag}(k_1, l_1, k_2, l_2, \dots, k_\nu, l_\nu)$, with $k_i l_i = 1$, $i = 1, \dots, \nu$. Thus $|PSp_{2\nu}(\mathbb{F}_q) \cap E| = \frac{1}{2}(q - 1)^\nu$. Hence

$$\begin{aligned}
|\text{Aut}(Sp(2\nu, q))| &= \frac{|PSp_{2\nu}(\mathbb{F}_q)||E|}{|PSp_{2\nu}(\mathbb{F}_q) \cap E|} \\
&= \frac{\frac{1}{2}q^{\nu^2} \prod_{i=1}^{\nu} (q^{2i} - 1) \cdot |E|}{\frac{1}{2}(q - 1)^\nu}.
\end{aligned}$$

Thus, when $\nu = 1$, $|\text{Aut}(Sp(2, q))| = q(q^2 - 1) \cdot (q - 2)!$, and when $\nu \geq 2$,

$$\begin{aligned} & |\text{Aut}(Sp(2\nu, q))| \\ &= \frac{\frac{1}{2}q^{\nu^2} \prod_{i=1}^{\nu} (q^{2i} - 1) \cdot (q - 1)^{\nu} \cdot |\text{Aut}(\mathbb{F}_q)|}{\frac{1}{2}(q - 1)^{\nu}} \\ &= q^{\nu^2} \prod_{i=1}^{\nu} (q^{2i} - 1) \cdot |\text{Aut}(\mathbb{F}_q)| \\ &= q^{\nu^2} \prod_{i=1}^{\nu} (q^{2i} - 1) \cdot [\mathbb{F}_q : \mathbb{F}_p], \end{aligned}$$

as is well-known that $|\text{Aut}(\mathbb{F}_q)| = [\mathbb{F}_q : \mathbb{F}_p]$ where $p = \text{char}(\mathbb{F}_q)$. \square

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