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Adjacency preserving mappings of symmetric and hermitian matrices

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Summary. Let D be a division ring with an involution $^{-}$ and $F = \{a \in D \mid \overline{a} = a\}$. When $^{-}$ is the identity map then $D = F$ is a field and we assume $\mathrm{char}(F) \neq 2$. When $^{-}$ is not the identity map we assume that F is a subfield of D and is contained in the center of D. Let n be an integer, $n \geq 2$, and $\mathcal{H}_n(D)$ be the space of hermitian matrices which includes the space $\mathcal{S}_n(F)$ of symmetric matrices as a particular case. If a bijective map φ of $\mathcal{H}_n(D)$ preserves the adjacency then also φ^{-1} preserves the adjacency.

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1. Introduction

Let D be a division ring which possesses an involution $^{-}$. By an *involution* $^{-}$ of D we mean a bijection $^{-} : D \to D$ with the properties $\overline{a+b} = \overline{a} + \overline{b}$, $\overline{ab} = \overline{b}\,\overline{a}$, and $\overline{\overline{a}} = a$ for all $a, b \in D$. Let $F = \{a \in D \mid \overline{a} = a\}$ be the set of fixed elements of $^{-}$. If $^{-}$ is the identity map, then $D = F$ is a field.

Let n be an integer, $n \geq 2$. An $n \times n$ matrix H over D is called *hermitian* if ${}^t\overline{H} = H$. When $^{-}$ is the identity and $D = F$ is a field, hermitian matrices are merely symmetric matrices. Denote by $\mathcal{H}_n(D)$ the space of $n \times n$ hermitian matrices over D. When $^{-}$ is the identity and $D = F$ is a field, $\mathcal{H}_n(D)$ is usually denoted by $\mathcal{S}_n(F)$, called the space of $n \times n$ symmetric matrices over F. Let $A, B \in \mathcal{H}_n(D)$. A, B are said to be *adjacent* and we write $A \sim B$ if $\mathrm{rank}(A - B) = 1$. The Fundamental Theorem of the geometry of hermitian matrices (and symmetric matrices) reads as follows.

Theorem 1.1. *Let D be a division ring which possesses an involution $^{-}$ and denote the set of fixed elements of $^{-}$ in D by F. If $^{-}$ is not the identity map, assume that F is a subfield of D and is contained in the center of D. Let n be an integer, $n \geq 2$. Then any bijective map φ from $\mathcal{H}_n(D)$ to itself for which both the map φ*

and its inverse φ^{-1} *preserve the adjacency in* $\mathcal{H}_n(D)$ *is of the form*

$$X^\varphi = \alpha P X^{\sigma\, t}\overline{P} + H_0 \qquad \text{for all } X \in \mathcal{H}_n(D), \tag{1}$$

where $\alpha \in F^* := F \setminus \{0\}$*,* $P \in \mathrm{GL}_n(D)$*,* $H_0 \in \mathcal{H}_n(D)$*, and* σ *is an automorphism of* D *which commutes with* $^-$*, i.e.,* $\overline{a}^\sigma = \overline{a^\sigma}$ *for all* $a \in D$*, unless* $n = 3$ *and* $D = \mathbb{F}_2$ *and* $^-$ *is the identity map of* $\mathbb{F}_2$*. In this latter case, there is an extra bijective map* ϵ *of* $\mathcal{S}_3(\mathbb{F}_2)$*, and* φ *might also be the product of a map of the form* (1) *and* ϵ*. Conversely, any map of the form* (1) *or* ϵ *is bijective, and both the map and its inverse preserve the adjacency.*

This theorem was proved by L. K. Hua, Z.-X. Wan et al., cf. [2, 3, 4, 5, 10, 11]. It should be remarked that in the statement of this theorem in [10, 11], when $^-$ is not the identity map it is further assumed that the trace map $x \mapsto x + \overline{x}$ is surjective. But this assumption was removed in [5].

In [14] the problem was posed whether for each type of geometry of matrices it is sufficient to demand that the map φ from the space of matrices of a certain type to itself is bijective and preserves the adjacency. In the present paper we solve this problem for $\mathcal{S}_n(F)$ under the assumption that $\mathrm{char}(F) \neq 2$ and also for $\mathcal{H}_n(D)$ under the assumption that $^-$ is not the identity map and that the set F of fixed elements of $^-$ in D is a subfield of D and is contained in the center of D.

Theorem 1.2. *Let* D *be a division ring which possesses an involution* $^-$ *and denote the set of fixed elements of* $^-$ *by* F*. When* $^-$ *is the identity map, hence* $D = F$ *is a field, then assume that* $\mathrm{char}(F) \neq 2$*. When* $^-$ *is not the identity map, assume that* F *is a subfield of* D *and is contained in the center of* D*. Let* n *be an integer,* $n \geq 2$*. If a bijective map* φ *from* $\mathcal{H}_n(D)$ *to itself preserves the adjacency in* $\mathcal{H}_n(D)$ *then also* φ^{-1} *preserves the adjacency.*

There is a close relation between the projective space $P\mathcal{S}_n(F)$ of symmetric matrices and $\mathcal{S}_n(F)$ [1, 6, 12]. Theorem 1.2 is also true in the projective space $P\mathcal{S}_n(F)$ of symmetric matrices [7, 8], even under milder hypotheses. The result can be extended to the dual polar space [9].

2. Some lemmas

The basic notations and properties of the space of hermitian matrices and that of symmetric matrices are described in the book [12] of Z.-X. Wan, which we will follow.

In the following our discussion on hermitian matrices includes symmetric matrices over fields of characteristic other than two as a particular case.

We call $n \times n$ hermitian matrices over D the *points* of the space $\mathcal{H}_n(D)$. Let A, B be two points of $\mathcal{H}_n(D)$. The *distance* $d(A, B)$ between A and B is defined to be the smallest nonnegative integer k with the property that there exists a sequence of consecutively adjacent points $A = A_0, A_1, \dots, A_k = B$. The distance satisfies the triangle inequality

$$d(A,B) + d(B,C) \geq d(A,C) \quad \text{for all } A, B, C \in \mathcal{H}_n(D).$$

From now on when $^-$ is the identity map then $D = F$ is a field and we assume that $\mathrm{char}(F) \neq 2$, and when $^-$ is not the identity map we assume that the set $F = \{a \in D \mid \overline{a} = a\}$ is a subfield of D and is contained in the center of D.

For any two points $A, B \in \mathcal{H}_n(D)$, it was proved in [12] that

$$d(A,B) = \mathrm{rank}(A - B).$$

For any two adjacent points $A, B \in \mathcal{H}_n(D)$ the line $l = AB$ joining A and B is defined to be the set consisting of A, B, and all points X which are adjacent to both A and B. It was also proved in [12] that $l = \{A + \lambda(B - A) \mid \lambda \in F\}$.

Lemma 2.1. *Let $P \in \mathcal{H}_n(D)$ be a point and let l be a line of $\mathcal{H}_n(D)$. Then either the distance between P and any point of l is the same, or there is a point $Q \in l$ such that $d(P,X) = d(P,Q) + 1$ for all $X \in l \setminus \{Q\}$.*

Proof. Since the transformations of the form (1) operate transitively on the set of lines, we may assume that $l = \{\lambda\, {}^t\overline{e_1} e_1 \mid \lambda \in F\}$ where $e_1 = (1, 0, \dots, 0) \in D^n$. We can find a cogredient transformation which leaves ${}^t\overline{e_1} e_1$ fixed and takes P to a matrix of the form

$$P_1 = \begin{pmatrix} p_{11} & p_{12} & \cdots & p_{1r} & p_{1,r+1} & \cdots & p_{1n} \\ \overline{p_{12}} & \lambda_2 & & & & & \\ \vdots & & \ddots & & & & \\ \overline{p_{1r}} & & & \lambda_r & & & \\ \overline{p_{1,r+1}} & & & & 0 & & \\ \vdots & & & & & \ddots & \\ \overline{p_{1n}} & & & & & & 0 \end{pmatrix},$$

where $\lambda_2, \dots, \lambda_r \in F^*$ and $p_{11} \in F$, $p_{12}, \dots, p_{1n} \in D$.

Case 1. $p_{1,r+1} = \dots = p_{1n} = 0$. Then there is a point Q in l such that $d(P_1, X) = d(P_1, Q) + 1 = r$ for all $X \in l \setminus \{Q\}$.

Case 2. There is some s, $r + 1 \leq s \leq n$ with $p_{1s} \neq 0$. Then $d(P_1, X) = r + 1$ for all $X \in l$. □

Corollary 2.1. *Let $P \in \mathcal{H}_n(D)$ be a point with $\mathrm{rank}(P) = k$. Then we can find a cogredient transformation which leaves ${}^t\overline{e_1} e_1$ fixed and takes P to a matrix of one of the following forms*

$$\begin{pmatrix} \mu_1 & \mu_2 & & & & & \\ \overline{\mu_2} & 0 & & & & & \\ & & \lambda_3 & & & & \\ & & & \ddots & & & \\ & & & & \lambda_k & & \\ & & & & & 0 & \\ & & & & & & \ddots \\ & & & & & & & 0 \end{pmatrix}, \quad \begin{pmatrix} \lambda_1 & & & & & \\ & \lambda_2 & & & & \\ & & \ddots & & & \\ & & & \lambda_k & & \\ & & & & 0 & \\ & & & & & \ddots \\ & & & & & & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & & & & & & \\ & \lambda_1 & & & & & \\ & & \lambda_2 & & & & \\ & & & \ddots & & & \\ & & & & \lambda_k & & \\ & & & & & 0 & \\ & & & & & & \ddots \\ & & & & & & & 0 \end{pmatrix},$$

where $\lambda_1, \ldots, \lambda_k \in F^*$ *and* $\mu_1 \in F$, $\mu_2 \in D^*$. *Let* $l := \{\lambda\, {}^t\overline{e_1} e_1 \mid \lambda \in F\}$. *In the first case,* $d(P, X) = k$ *for all* $X \in l$. *In the second case there exists* $Q \in l$ *such that* $d(P, Q) = k - 1$ *and* $d(P, X) = k$ *for all* $X \in l \setminus \{Q\}$. *In the third case there exists* $Q \in l$ *such that* $d(P, Q) = k$ *and* $d(P, X) = k + 1$ *for all* $X \in l \setminus \{Q\}$.

Lemma 2.2. *Let* $A \in \mathcal{H}_n(D)$ *be a matrix with* $\operatorname{rank}(A) = k + 1$. *A matrix* $B \in \mathcal{H}_n(D)$ *has rank* k *and* $A \sim B$ *if and only if there exists an* $x \in D^n$ *with* $xA\,{}^t\overline{x} \neq 0$ *and*

$$B = A - (xA\,{}^t\overline{x})^{-1}\, {}^t\overline{(xA)}(xA).$$

Proof. Let there exist $x \in D^n$ with $xA\,{}^t\overline{x} \neq 0$. Let $B = A - (xA\,{}^t\overline{x})^{-1}\, {}^t\overline{(xA)}(xA)$. Then $A \sim B$. For $y \in D^n$, $yA = 0$ we have $yB = 0$ thus $\ker(A) \subset \ker(B)$. $xA\,{}^t\overline{x} \neq 0$ implies $xA \neq 0$. But $xB = 0$, thus $\ker(A) \subsetneq \ker(B)$, and $\operatorname{rank}(B) = \operatorname{rank}(A) - 1 = k$.

Now let $B \in \mathcal{H}_n(D)$ satisfy $\operatorname{rank}(B) = k$ and $A \sim B$. Then $B = A - \lambda\, {}^t\overline{y} y$ where $\lambda \in F^*$ and $y \in D^n \setminus \{0\}$. There exists $T \in \mathrm{GL}_n(D)$ such that $yT = e_1 = (1, 0, \ldots, 0)$. Let $B_1 = {}^t\overline{T} B T$, $A_1 = {}^t\overline{T} A T$, then $B_1 = A_1 - \lambda\, {}^t\overline{e_1} e_1$. Since $\operatorname{rank}(A) = k + 1$ and $\operatorname{rank}(B) = k$, by Corollary 2.1, under a cogredient transformation which leaves ${}^t\overline{e_1} e_1$ fixed, we can assume

$$A_1 = \begin{pmatrix} a_{11} & 0 & \cdots & 0 & 0 \\ 0 & \lambda_2 & & & \\ \vdots & & \ddots & & \vdots \\ 0 & & & \lambda_{k+1} & \\ 0 & & \cdots & & 0 \end{pmatrix}, \qquad a_{11}, \lambda_2, \ldots, \lambda_{k+1} \in F^*.$$

Then $a_{11} = \lambda$. Let $x = e_1\, {}^t\overline{T}$, then $B = A - (xA\,{}^t\overline{x})^{-1}\, {}^t\overline{(xA)}(xA)$. □

Lemma 2.3. *Let* $A, B \in \mathrm{GL}_n(D)$ *satisfy* $A \neq B$. *Then* $(B - A)B^{-1}(B - A) \neq B - A$.

Proof. Assume $(B - A)B^{-1}(B - A) = B - A$. Then $(B - A)(I - B^{-1}A) = B - A$ and $(B - A)B^{-1}A = 0$, a contradiction to $A \neq B$. □

Lemma 2.4. *Let* $|F| = \infty$ *and* $A, B \in \mathcal{H}_n(D)$ *with* $A \neq B$, $\operatorname{rank}(A) = \operatorname{rank}(B) = n$, $\operatorname{rank}(B - A) \geq 2$. *Then there exists* $x \in D^n$ *such that*

$$x(B - A)\,{}^t\overline{x} \neq 0 \quad \text{and} \quad x(B - A)\,{}^t\overline{x} \neq x(B - A)B^{-1}(B - A)\,{}^t\overline{x}.$$

Proof. There exists $T \in \mathrm{GL}_n(D)$ with ${}^t\overline{T}(B-A)T = \mathrm{diag}(\lambda_1, \dots, \lambda_k, 0, \dots, 0)$, $\lambda_i \in F^*$, $k \geq 2$. Let $B_1 = {}^t\overline{T}BT$, $A_1 = {}^t\overline{T}AT$. Then $B_1^{-1} = T^{-1}B^{-1}\,{}^t\overline{T^{-1}}$, $(B_1 - A_1)B_1^{-1}(B_1 - A_1) \neq B_1 - A_1$. It is sufficient to show that there exists $x \in D^n$ such that

$$x(B_1 - A_1)\,{}^t\overline{x} \neq 0 \quad \text{and} \quad x(B_1 - A_1)B_1^{-1}(B_1 - A_1)\,{}^t\overline{x} \neq x(B_1 - A_1)\,{}^t\overline{x},$$

where $B_1 - A_1 = \mathrm{diag}(\lambda_1, \dots, \lambda_k, 0, \dots, 0)$. Let $B_1^{-1} = (\beta_{ij})$.

Case 1. $\beta_{ii} \neq \lambda_i^{-1}$ for some i, $1 \leq i \leq k$. Then

$$e_i(B_1 - A_1)\,{}^t\overline{e_i} = \lambda_i \neq 0 \quad \text{and} \quad e_i(B_1 - A_1)B_1^{-1}(B_1 - A_1)\,{}^t\overline{e_i} = \lambda_i\beta_{ii}\lambda_i \neq \lambda_i.$$

Case 2. $\beta_{ii} = \lambda_i^{-1}$ for all i, $1 \leq i \leq k$. Since $(B_1 - A_1)B_1^{-1}(B_1 - A_1) \neq B_1 - A_1$, there exist i, j, $1 \leq i, j \leq k$, $i \neq j$ such that $\beta_{ij} \neq 0$. Without loss of generality, we assume $\beta_{12} \neq 0$. It is enough to find $x_1, x_2 \in D$ such that

$$\lambda_1 x_1 \overline{x_1} + \lambda_2 x_2 \overline{x_2} \neq 0, \qquad x_1\lambda_1\beta_{12}\lambda_2\overline{x_2} + x_2\lambda_2\overline{\beta_{12}}\lambda_1\overline{x_1} \neq 0.$$

Case 2.1. $\overline{}$ is the identity, $D = F$ and $\mathrm{char}(F) \neq 2$. If $\lambda_1 + \lambda_2 \neq 0$, then choose $x_1 = x_2 = 1$. If $\lambda_1 + \lambda_2 = 0$, then choose $x_1 = 1$ and $x_2 \in F^*$ with $x_2^2 \neq 1$.

Case 2.2. $\overline{}$ is not the identity, $D \neq F$:

Case 2.2.1. When $\beta_{12} + \overline{\beta_{12}} \neq 0$, proceed as in Case 2.1.

Case 2.2.2. When $\beta_{12} + \overline{\beta_{12}} = 0$, choose $x_1 = 1$ and $x_2 \in D \setminus F$ with $\lambda_1 + \lambda_2 x_2 \overline{x_2} \neq 0$, $\beta_{12}\overline{x_2} + x_2\overline{\beta_{12}} \neq 0$. □

Lemma 2.5. *Let* $|F| = \infty$. *For all* $A, B \in \mathcal{H}_n(D)$ *with* $A \neq B$ *and* $\mathrm{rank}(A) = \mathrm{rank}(B) = n$ *there exists* $C \in \mathcal{H}_n(D)$ *with* $\mathrm{rank}(C) = n$, $B \sim C$ *and* $d(A, C) = d(A, B) - 1$.

Proof. If $A \sim B$ then choose $C = A$. Assume $d(A, B) = k \geq 2$. By Lemma 2.4, there exists $x \in D^n$ such that

$$x(B - A)\,{}^t\overline{x} \neq 0 \quad \text{and} \quad x(B - A)\,{}^t\overline{x} \neq x(B - A)B^{-1}(B - A)\,{}^t\overline{x}.$$

Let

$$C = B - (x(B - A)\,{}^t\overline{x})^{-1}\,{}^t\overline{(x(B - A))}\big(x(B - A)\big).$$

By Lemma 2.2 we have $C \sim B$ and $d(A, C) = d(A, B) - 1$. Assume $\mathrm{rank}(C) \neq n$. Then by Lemma 2.2 there is $y \in D^n$ with

$$C = B - (yB\,{}^t\overline{y})^{-1}\,{}^t\overline{yB}(yB).$$

Then $yB = \nu x(B - A)$ for some $\nu \in D^*$ and

$$C = B - \Big(x(B - A)B^{-1}(B - A)\,{}^t\overline{x}\Big)^{-1}\,{}^t\overline{(x(B - A))}\big(x(B - A)\big).$$

Thus

$$x(B - A)\,{}^t\overline{x} = x(B - A)B^{-1}(B - A)\,{}^t\overline{x},$$

a contradiction. □

Lemma 2.6. *Let $|F| = \infty$. Let $A, B \in \mathcal{H}_n(D)$, $A \sim B$, $\operatorname{rank}(A) = \operatorname{rank}(B) = n$. Let $A - B = \lambda_0\, {}^t\overline{x}x$, $\lambda_0 \in F^*$, and $l = \{A - \lambda\, {}^t\overline{x}x \mid \lambda \in F\}$ be the line containing both A and B. Suppose all points in l are of rank n. Then there are two points $C, D \in \mathcal{H}_n(D)$ with $\operatorname{rank}(C) = \operatorname{rank}(D) = n$, $A \sim C$, $C \sim D$, $D \sim B$, and the line containing A, C contains a point of rank $n-1$, so do the line containing C, D and the line containing D, B.*

Proof. There exists $T \in \mathrm{GL}_n(D)$ with $xT = (1, 0, \ldots, 0) = e_1$. Let $A_1 = {}^t\overline{T}AT$, $B_1 = {}^t\overline{T}BT$, $l_1 = \{A_1 - \lambda\, {}^t\overline{e_1}e_1 \mid \lambda \in F\}$. It is sufficient to prove the lemma for A_1, B_1 and l_1. We drop the subscript, i.e., let $A, B \in l = \{A - \lambda\, {}^t\overline{e_1}e_1 \mid \lambda \in F\}$, $\operatorname{rank}(A) = \operatorname{rank}(B) = n$. Since $\operatorname{rank}(A) = n$, by Corollary 2.1, under a cogredient transformation which leaves ${}^t\overline{e_1}e_1$ fixed we can assume

$$A = \begin{pmatrix} a_{11} & a_{12} & & & \\ \overline{a_{12}} & 0 & & & \\ & & \lambda_3 & & \\ & & & \ddots & \\ & & & & \lambda_n \end{pmatrix}, \qquad B = A - \lambda_0\, {}^t\overline{e_1}e_1,$$

where $a_{11} \in F$, $a_{12} \in D^*$, $\lambda_3, \ldots, \lambda_n \in F^*$, because in the case $A = \operatorname{diag}(a_{11}, \lambda_2, \ldots, \lambda_n)$ there would exist one point in l which is of rank $n-1$. Choose $\mu \in F^*$ such that $a_{11} - \mu \neq 0$ and $a_{11} - \lambda_0 - \mu \neq 0$. Let $\mu_1 = -\overline{a_{12}}(a_{11} - \mu)^{-1}a_{12}$, $\mu_2 = -\overline{a_{12}}(a_{11} - \lambda_0 - \mu)^{-1}a_{12}$, then $\mu_1, \mu_2 \in F^*$, $\mu_1 \neq \mu_2$. Let $C = \operatorname{diag}(\mu, \mu_1, \lambda_3, \ldots, \lambda_n)$ and $D = \operatorname{diag}(\mu, \mu_2, \lambda_3, \ldots, \lambda_n)$. It is easy to verify that C, D satisfy the requirements of Lemma 2.6. □

3. Proof of Theorem 1.2

Let φ be a bijective map from $\mathcal{H}_n(D)$ to itself which preserves adjacency, i.e. $A \sim B$ implies $A^\varphi \sim B^\varphi$ for all $A, B \in \mathcal{H}_n(D)$. Clearly, for all $A, B \in \mathcal{H}_n(D)$, $d(A^\varphi, B^\varphi) \leq d(A, B)$, and l^φ is contained in a line for all lines l. If $^-$ is the identity map then $D = F$. If $^-$ is not the identity map, then D is either a separable quadratic extension of F or a division ring of generalized quaternions over F (cf. Theorem 1.1 in [5]). Thus if F is finite, D is finite and the geometry of $\mathcal{H}_n(D)$ contains only finitely many points and lines. Then l^φ is a line for all lines l, and $A^\varphi \sim B^\varphi$ implies $A \sim B$ for all $A, B \in \mathcal{H}_n(D)$.

Now let F be infinite.

Lemma 3.1. *Let φ be a bijective map which preserves adjacency and assume that $0^\varphi = 0$. Then for any $B \in \mathcal{H}_n(D)$ with $d(0, B) = n$ we have $d(0, B^\varphi) = n$.*

Proof. Suppose $d(0, B^\varphi) \neq n$, then $d(0, B^\varphi) \leq n-1$. Let $C \in \mathcal{H}_n(D)$, $d(0, C) = n$. Then $\operatorname{rank}(B) = \operatorname{rank}(C) = n$. By Lemma 2.5 and Lemma 2.6 there is a sequence of points $B_0 = B, B_1, \ldots, B_k = C$ such that $\mathrm{rank}(B_i) = n \; \forall i = 1, \ldots, k$, $B_i \sim B_{i+1}$ $\forall i = 0, \ldots, k-1$, and each line $l_i = B_i B_{i+1}$ contains a point Q_i of rank $n-1$. Then $d(0, Q_i) = n-1$. It follows that $d(0, Q_i^{\varphi}) \leq d(0, Q_i) = n-1$. But $d(0, B^{\varphi}) \leq n-1$, and by Lemma 2.1, $d(0, B_1^{\varphi}) \leq n-1$. Analogously, $d(0, B_2^{\varphi}) \leq n-1$, ..., $d(0, B_k^{\varphi}) \leq n-1$, i.e. $d(0, C^{\varphi}) \leq n-1$. This contradicts the surjectivity of φ. □

Proof of Theorem 1.2. Let φ be a bijective map from $\mathcal{H}_n(D)$ to itself which preserves adjacency. First we prove that for $A, B \in \mathcal{H}_n(D)$, $d(A, B) = n$ implies $d(A^{\varphi}, B^{\varphi}) = n$. Let σ be the map $X \mapsto X^{\sigma} = X + A$ for all $X \in \mathcal{H}_n(D)$ and let σ' be the map $X \mapsto X^{\sigma'} = X - A^{\varphi}$ for all $X \in \mathcal{H}_n(D)$. Let $\varphi' = \sigma' \circ \varphi \circ \sigma$, then φ' is bijective and preserves adjacency, $0^{\varphi'} = 0$. $d(0, B - A) = d(A, B) = n$, by Lemma 3.1 we have $n = d(0, (B-A)^{\varphi'}) = d(A^{\varphi}, B^{\varphi})$.

Then we prove that $d(A, B) = d(A^{\varphi}, B^{\varphi})$ for all $A, B \in \mathcal{H}_n(D)$. If $d(A, B) = n$, then $d(A^{\varphi}, B^{\varphi}) = n$ from above. Suppose $d(A, B) < n$. Then there is a point C such that $d(A, B) + d(B, C) = d(A, C) = n$. This implies $n = d(A, C) = d(A, B) + d(B, C) \geq d(A^{\varphi}, B^{\varphi}) + d(B^{\varphi}, C^{\varphi}) \geq d(A^{\varphi}, C^{\varphi}) = n$. Hence $d(A, B) = d(A^{\varphi}, B^{\varphi})$. In particular, $d(A, B) = 1$ if, and only if, $d(A^{\varphi}, B^{\varphi}) = 1$. Therefore also φ^{-1} preserves adjacency. □

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