A note on the degree monotonicity of cages

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Abstract

A $(k; g)$-graph is a k-regular graph with girth g. A $(k; g)$-cage is a $(k; g)$-graph with the *least* number of vertices. The order of a $(k; g)$-cage is denoted by $f(k; g)$. In this paper we show that $f(k + 2; g) \geq f(k; g)$ for $k \geq 2$ and present some partial results to support the conjecture that $f(k_1; g) < f(k_2; g)$ if $k_1 < k_2$.

1 Introduction

In this paper, we consider only finite simple graphs, and refer to them as graphs.

Suppose that V' (or E') is a nonempty subset of V (or E). The induced subgraph (or the edge-induced subgraph) of G by V' is denoted by $G[V']$ (or $G[E']$). The subgraph obtained from G by deleting the vertices in V' together with their incident edges is denoted by $G-V'$. The graph obtained from G by adding a set of edges E' is denoted by $G \cup E'$. For a vertex v of G and a set of vertices $S \subseteq V(G)$, we use $N_S(v)$ to denote the set of vertices in S that are adjacent to v. A component in a graph is odd if it has an odd number of vertices. We denote by $o(G)$ the number of odd components of G. The number of edges between subgraphs H_1 and H_2 in a graph G is denoted by $e_G(H_1, H_2)$.

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The length of a shortest cycle in a graph G is called the *girth* of G. Clearly, adding edges to a graph G might decrease the girth of G. If G' is obtained from G by adding edges, we use the term *smaller cycle of* G' to denote any cycle of G' having length less than the girth.

A k-regular graph with girth g is called a $(k; g)$-graph and a $(k; g)$-*cage* is a $(k; g)$-graph with the least number of vertices. We use $f(k; g)$ to denote the number of vertices in any $(k; g)$-cage.

Cages were introduced by Tutte [11] in 1947, and since then have been widely studied. The problem of finding cages has a prominent place in both extremal graph theory and algebraic graph theory. A survey paper by Wong [12] in 1982 refers to 70 publications. The study of cages has led to interesting applications of algebra to graph theory. Recently, it has also attracted some attention from researchers in computer science (see [2], [8]). In these papers, new computer search algorithms are used to find new cages or provide better bounds for $f(k; g)$. However, most of the work so far is on the existence problem, i.e. finding cages, or estimating $f(k; g)$. Very little is known on the structural properties of cages.

The first fundamental properties of cages, girth monotonicity, were established by Erdős and Sachs [4], Holton and Sheeham [6], Fu, Huang and Rodger [5], independently. They proved the following monotonicity result with respect of girth of the cages which turns out to be the foundation in exploring the connectivity of cages.

Girth Monotonicity Theorem. If $k \geq 3$ and $3 \leq g_1 < g_2$, then $f(k; g_1) < f(k; g_2)$.

Fu, Huang and Rodger [5] proved that all cages are 2-connected, and then subsequently showed that all cubic cages are 3-edge-connected. It follows from this theorem that all cubic cages are 3-connected. They then conjectured that $(k; g)$-cages are k-connected. Daven and Rodger [3], and independently Jiang and Mubayi [9], proved that all $(k; g)$-cages are 3-connected for $k \geq 3$. It was proven in [14] that $(4; g)$-cages are 4-connected and in [13] that $(k; g)$-cages are k-edge-connected when g is odd. Recently, Lin, Miller and Rodger [7] prove that $(k; g)$-cages are k-edge-connected when g is even.

Jiang and Mubayi also provided some structural properties of cages. They showed that $diam(G[S]) \geq \lfloor g/2 \rfloor$ where S is a cut-set of a $(k; g)$-cage G and every $(k; g)$-cage contains a non-separating g-cycle for $g \geq 5$. Moreover, they showed that every g-cycle in a $(g; k)$-cage is non-separating for $k \geq 3$ and $g \geq 4$ even. Related to this, it is easy to show that every vertex in a $(k; g)$-cage is contained in a g-cycle if g is even. The case of g odd is still open.

Similar to the Girth Monotonicity Theorem, we consider the following conjecture.

Degree Monotonicity Conjecture. If $k_1 < k_2$, then $f(k_1; g) < f(k_2; g)$.

We shall give some partial results to support this conjecture in the next section.

2 Results

It is well-known that a $(k;3)$-cage is K_{k+1}, a complete graph on $k+1$ vertices and a $(k;4)$-cage is $K_{k,k}$, a complete bipartite graph with k vertices in each partite set. Thus the Degree Monotonicity Conjecture is true when $g = 3$ or 4. It is also true for the known cages. For example, $f(3;5) = 10 < f(4;5) = 19 < f(5;5) = 30 < f(6;5) = 40 < f(7;5) = 50$ and $f(3;6) = 14 < f(4;6) = 26 < f(5;6) = 42 < f(6;6) = 62 < f(7;6) = 90 < f(8,6) = 104$.

Proposition 2.1. $f(2;g) < f(3;g)$

Proof. Since any $(3;g)$-cage contains a cycle of length $\geq g$, $f(3;g) \geq f(2;g) = g$. If $f(3;g) = f(2;g) = g$, any vertex on the g-cycle in the $(3;g)$-cage has to be adjacent to another vertex on a g-cycle. This leads to a cycle of length less than g in the $(3;g)$-cage, a contradiction. □

Let v be any vertex of a $(k;g)$-cage G, where g is even, V_i be the set of vertices of distance i from v and $X = V(G) - v - \cup_{i=1}^{\frac{1}{2}g-1} V_i$. It follows that $|V_i| = k(k-1)^{i-1}$ for $1 \leq i \leq \frac{1}{2}g - 1$. Since G has girth at least g, $|E(G[V_{\frac{1}{2}g-1}])| = 0$. Hence, there are $k(k-1)^{\frac{1}{2}g-1}$ edges between $V_{\frac{1}{2}g-1}$ and X, and, in turn, $|X| \geq (k-1)^{\frac{1}{2}g-1}$. Thus, $f(k;g) \geq 1 + k + k(k-1) + ... + k(k-1)^{\frac{1}{2}g-2} + (k-1)^{\frac{1}{2}g-1}$ for g is even. For $g = 6, 8$ or 12 and $k-1$ is a prime power, $(k;g)$-cages exist and they are the generalized triangles, the generalized quadrangles and the generalized hexagons of order $k-1$. Thus $f(k;g) = 1 + k + k(k-1) + ... + k(k-1)^{\frac{1}{2}g-2} + (k-1)^{\frac{1}{2}g-1}$ for $g = 6, 8$ or 12 and $k-1$ is a prime power. This leads to the following theorem.

Proposition 2.2. *$f(k_1; g) < f(k_2; g)$ where $k_1 < k_2$ and $k_1 - 1$ is a prime power, and $g = 6, 8$ or 12.*

Proof. $f(k_1; g) = 1 + k_1 + k_1(k_1-1) + ... + k_1(k_1-1)^{\frac{1}{2}g-2} + (k_1-1)^{\frac{1}{2}g-1} < 1 + k_2 + k_2(k_2-1) + ... + k_2(k_2-1)^{\frac{1}{2}g-2} + (k_2-1)^{\frac{1}{2}g-1} \leq f(k_2; g)$ for $k_1 < k_2$ and $k_1 - 1$ is a prime power, and $g = 6, 8$ or 12. □

In 1973, Berge (see [1]) conjectured that every 4-regular graph contains a 3-regular graph. Tashkinov [10] proved this conjecture in 1984. We restate this result in terms of cages as follows.

Proposition 2.3. *$f(4;g) \geq f(3;g)$ and $f(4;g) > f(3;g)$ if $f(4;g)$ is odd.*

In order to extend the above result to general k, we study the perfect matching (i.e., 1-factor) in cages. We need the following recent result as a lemma.

Lemma 2.1. *([7] and [13]) $(k;g)$-cages are k-edge connected.*

Now we are ready to prove the existence of perfect matching in cages. In fact, we show that there are many perfect matchings in a cage with even order.

Theorem 2.1. *If the order of $(k;g)$-cage G is even, then for any edge $e \in E(G)$ there exists a 1-factor containing e.*

Proof. We proceed the proof with a contradiction. If there exists an edge $e \in E(G)$ so that there does not exists a 1-factor containing it, then $G-V(e)$ has no 1-factor. By Tutte's 1-factor Theorem, there exists a set $S' \subseteq V(G) - V(e)$ so that $o(G - V(e) - S') > |S'|$. By the parity, we can see that $o(G - V(e) - S') \geq |S'| + 2$.

Let $S = S' \cup V(e)$, we have $o(G - S) \geq |S|$. Let C_1, C_2, ..., C_w be the odd components of $G - S$. From Lemma 2.1, we have $e_G(C_i, S) \geq k$. Counting the edges between S and $\cup_i C_i$. We conclude that

$$kw \leq e(\cup_i C_i, S) \leq k|S| - 2$$

This implies that $o(G - S) = w < |S|$, a contradiction. □

Theorem 2.2. *If $f(k;g)$ is even, then $f(k-1;g) \leq f(k;g)$. In particular, for any $r \geq 1$ and $g \geq 3$, $f(2r;g) \leq f(2r+1;g)$.*

Proof. From Theorem 2.1, G has a 1-factor, says M. Then $G - M$ is a $(k-1)$-regular graph and its girth, g', is greater than or equal to g. Therefore $f(k;g) = |V(G-M)| \geq f(k-1;g') \geq f(k-1;g)$, from the Girth Monotonicity Theorem.

Since a regular graph of odd degree has an even number of vertices. Let $k = 2r+1$. It follows that $f(2r;g) \leq f(2r+1;g)$. □

Theorem 2.1 shows that there exists a 1-factor in $(k;g)$-cages if $f(k;g)$ is even. However, because the number of vertices in a $(k;g)$-cage may be odd, some $(k;g)$-cages may not contain a 1-factor. For example, the $(4;5)$-cage has 19 vertices. However, the $(4;5)$-cage is the only known cage with an odd number of vertices.

To provide further support for the Degree Monotonicity Conjecture, we prove a weaker form of this conjecture.

Theorem 2.3. *For $k \geq 2$, $f(k;g) \leq f(k+2;g)$.*

This result follows immediately from the next theorem, which is of independent interest. However, we need the next classic result from Petersen first.

Lemma 2.2. *(Petersen, see [1]). A $2r$-regular graph is 2-factorable.*

Theorem 2.4. *Every (k,g)-cage has a 2-factor.*

Proof. Let G be a (k,g)-cage. Then G is k-regular. If k is even, by Lemma 2.2, G has a 2-factor. If k is odd, then the order of G must be even. From the proof of Theorem 2.1, we see that in this case G has a 1-factor F. Thus $G - F$ is a $(k-1)$-regular graph (note that $k-1$ is even) and thus has a 2-factor T by Lemma 2.2 again. Of course, T is a 2-factor of G as well. $\square$

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