Removable Edges in a Cycle of a 4-Connected Graph *

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Abstract

Let G be a 4-connected graph. For an edge e of G, we do the following operations on G: first, delete the edge e from G, resulting the graph $G-e$; second, for all the vertices x of degree 3 in $G-e$, delete x from $G-e$ and then completely connect the 3 neighbors of x by a triangle. If multiple edges occur, we use single edges to replace them. The final resultant graph is denoted by $G\ominus e$. If $G\ominus e$ is still 4-connected, then e is called a *removable edge* of G. In this paper, we investigate the problem on how many removable edges there are in a cycle of a 4-connected graph, and give examples to show that our results are in some sense best possible.

Key Words: 4-Connected graph, Removable edge, Edge-vertex-cut fragment, Edge-vertex-cut atom

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1 Introduction

All graphs considered here are simple and finite. For notations and terminology not given here, we refer the reader(s) to [1]. In this paper we shall study the removable edges in a cycle of a 4-connected graph. First of all, we give the definition of a removable edge for a 4-connected graph. Let G be a 4-connected graph and e an edge of G. Consider the graph $G-e$ obtained by deleting the edge e from G. If $G-e$ has vertices of degree 3, we do the following operations on $G-e$. For all

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vertices x of degree 3 in $G - e$, delete x from $G - e$ and then completely connect the three neighbors of x by a triangle. If multiple edges occur, we use single edges to replace them. The final resultant graph is denoted by $G \ominus e$. Note that if there is no vertex of degree 3 in $G - e$, then $G \ominus e$ is simply the graph $G - e$.

Definition 1.1. For a 4-connected graph G and an edge e of G, if $G \ominus e$ is still 4-connected, then the edge e is called *removable*; otherwise, it is called *unremovable*. The set of all removable edges of G is denoted by $E_R(G)$; whereas the set of unremovable edges of G is denoted by $E_N(G)$.

Definition 1.2. A 2-cyclic graph G of order n is defined to be the square of the cycle C_n, namely, G can be obtained from C_n by adding edges between all pairs of vertices of C_n which are at distance 2 in C_n.

The aim to introduce the concept of removable edges in 4-connected graphs is to find new method to construct 4-connected graphs and to prove some properties of 4-connected graphs inductively. In [2], Yin proved that there always exist removable edges in 4-connected graphs G unless G is a 2-cyclic graph of order 5 or 6. He showed that a 4-connected graph can be obtained from a 2-cyclic graph by the following four operations: (i) adding edges, (ii) splitting vertices, (iii) adding vertices and removing edges, and (iv) extending vertices. He also obtained a lower bound for the number of removable edges and contractible edges in a 4-connected graph G. In this paper we shall investigate how many removable edges there are in a cycle of a 4-connected graph G, and give examples to show that our results are best possible in some sense.

For convenience we introduce the following notations. Without specific statement, in the sequel G always denotes a 4-connected graph. The vertex set and edge set of G is denoted, respectively, by $V(G)$ and $E(G)$. The order and size of G is denoted, respectively, by $|G|$ and $|E(G)|$. For $x \in V(G)$, we simply write $x \in G$. The neighborhood of $x \in G$ is denoted by $\Gamma_G(x)$ and the degree of x is denoted by $d(x)$. If x and y are the two end-vertices of an edge e, we write $e = xy$. For a nonempty subset F of $E(G)$, or N of $V(G)$, the induced subgraph by F or N in G is denoted by $[F]$ or $[N]$. Let $A, B \subset V(G)$ such that $A \neq \emptyset \neq B$ and $A \bigcap B = \emptyset$, define $[A, B] = \{xy \in E(G) \mid x \in A, y \in B\}$. If H is a subgraph of G, we say that G contains H. For a subset S of $V(G)$, $G - S$ denotes the graph obtained by deleting all the vertices in S from G together with all the incident edges. If $G - S$ is disconnected, we say that S is a vertex-cut of G. If $|S| = s$ for such an S, we say that S is an s-vertex-cut. For $e = xy \in E(G)$ and $S \subset V(G)$ such that $|S| = 3$, if $G - e - S$ has exactly two (connected) components, say A and B, such that $|A| \geq 2$ and $|B| \geq 2$, then we say that (e, S) is a *separating pair* and $(e, S; A, B)$ is a *separating group*, in which A and B are called the *edge-vertex-cut fragments*. If, moreover, $|A| = 2$, then A is called an *edge-vertex-cut atom*. For an edge-vertex-cut atom A, let $A = \{x, z\}$ and $S = \{a, b, c\}$, if $ax, bx \in E(G), cx \notin E(G)$, then A is called a *1-edge-vertex-cut atom*; whereas if $ax, bx, cx \in E(G)$, then A is called a *2-edge-vertex-cut atom*. It is easy to see that if A is an edge-vertex-cut atom, then A is either a 1-edge-vertex-cut atom or a 2-edge-vertex-cut atom. Let $E_0 \subset E_N(G)$ such that $E_0 \neq \emptyset$ and let $(xy, S; A, B)$ be a separating group of G such that $x \in A$ and $y \in B$. If $xy \in E_0$, then A and B are called *E_0-edge-vertex-cut fragments*. An E_0-edge-vertex-cut fragment is called an *E_0-edge-vertex-cut end-fragment* of G if it does not contain any other E_0-edge-vertex-cut fragment of G as a proper subset. It is easy to see that any E_0-edge-vertex-cut fragment of G contains a such end-fragment. Similarly, if $|A| = 2$, then A is called an *E_0-edge-vertex-cut atom.*

2 Some Known Results

In the sequel we shall use the following results on the existence of removable edges in 4-connected graphs, which were obtained by Yin in [2].

Theorem 2.1. Let G be a 4-connected graph with $|G| \geq 7$. An edge e of G is unremovable if and only if there is a separating pair (e, S), or a separating group $(e, S; A, B)$ in G.

Theorem 2.2. Let G be a 4-connected graph with $|G| \geq 8$ and let $(xy, S; A, B)$ be a separating group of G such that $x \in A$, $y \in B$ and $|A| \geq 3$. Then, every edge in $[\{x\}, S]$ is removable.

Corollary 2.3. Let G be a 4-connected graph with $|G| \geq 8$. Then, every 3-cycle of G contains at least one removable edge.

Theorem 2.4. Let G be a 4-connected graph with $|G| \geq 7$. If for an unremovable edge xy, i.e., $xy \in E_N(G)$, there is a separating group $(xy, S; A, B)$, then all the edge in $E([S])$ are removable, i.e., $E([S]) \subset E_R(G)$.

3 Notations and Terminology for Subgraphs With Special Structures

For convenience we introduce the following definitions for subgraphs of G with special structures.

Definition 3.1. Let G be a 4-connected graph and H a subgraph of G such that $V(H) = \{a, x_1, x_2, x_3, x_4, v_1, v_2, v_3, v_4\}$ and $E(H) = \{ax_1, ax_2, ax_3, ax_4, x_1x_2, x_2x_3, x_3x_4, x_4x_1, x_1v_1, x_2v_2, x_3v_3, x_4v_4\}$. If H satisfies the following conditions

(i) $d(a) = d(x_i) = 4$ for $i = 1, 2, 3, 4$,

(ii) $ax_1, ax_2, ax_3, ax_4 \in E_N(G)$ and $x_1x_2, x_2x_3, x_3x_4, x_4x_1 \in E_R(G)$,

then H is called a *helm*. The vertices a, x_i for $i = 1, 2, 3, 4$ of a helm H are called *inner vertices* of H.

Definition 3.2. Let G be a 4-connected graph and H a subgraph of G such that $V(H) = \{a, b, x_1, x_2, \cdots, x_{l+3}\}$ and $E(H) = \{x_1x_2, x_2x_3, \cdots, x_{l+2}x_{l+3}, ax_2, ax_3, \cdots, ax_{l+2}, bx_2, bx_3, \cdots, bx_{l+2}$, where $l \geq 1$. If H satisfies the following conditions

(i) $x_ix_{i+1} \in E_N(G)$ for $i = 1, 2, \cdots, l+2$,

(ii) $ax_j, bx_j \in E_R(G)$ for $j = 2, 3, \cdots, l+2$,

(iii) $d(x_j) = 4$ for $j = 2, 3, \cdots, l+2$, then H is called an *l-bi-fan*.

An l-bi-fan H is said to be *maximal* if $\Gamma_G(x_1) \neq \{a, b, x_2, u\}$ and $\Gamma_G(x_{l+3}) \neq \{a, b, x_{l+2}, v\}$ for any $u, v \in G$. The vertices of an l-bi-fan or a maximal l-bi-fan H satisfying the condition (iii) are called *inner vertices* of H.

Definition 3.3. Let G be a 4-connected graph and H a subgraph of G such that $V(H) = \{x_1, x_2, \cdots, x_{l+2}, y_1, y_2, \cdots, y_{l+2}\}$ and $E(H) = E_1(H) \cup E_2(H)$ where $E_1(H) = \{x_1x_2, x_2x_3, \cdots, x_{l+1}x_{l+2}, y_1y_2, y_2y_3, \cdots, y_{l+1}y_{l+2}\}$ and $E_2(H) = \{y_1x_2, x_2y_2, y_2x_3, \cdots, y_lx_{l+1}, x_{l+1}y_{l+1}, y_{l+1}x_{l+2}\}$. Then, H is called an *l-belt* if the following conditions are satisfied

(i) $E_1(H) \subset E_N(H)$ and $E_2(H) \subset E_R(H)$,

(ii) $d(x_i) = d(y_j) = 4$ for $i = 2, 3, \cdots, l+1; j = 2, 3, \cdots l+1$.

An l-belt H is said to be *maximal* if $\Gamma_G(y_1) \neq \{x_1, x_2, y_2, u\}$ and $\Gamma_G(x_{l+2}) \neq \{x_{l+1}, y_{l+1}, y_{l+2}, v\}$ for any $u, v \in G$. The vertices of an l-belt or a maximal l-belt H satisfying the condition (ii) are called *inner vertices* of H.

Definition 3.4. Let G be a 4-connected graph and H a subgraph of G such that $V(H) = \{x_1, x_2, \cdots, x_{l+2}, x_{l+3}, y_1, y_2, \cdots, y_{l+2}\}$ and $E(H) = E_1(H) \cup E_2(H)$ where $E_1(H) = \{x_1x_2, x_2x_3, \cdots, x_{l+1}x_{l+2}, x_{l+2}x_{l+3}, y_1y_2, y_2y_3, \cdots, y_{l+1}y_{l+2}\}$ and $E_2(H) = \{y_1x_2, x_2y_2, y_2x_3, \cdots, y_lx_{l+1}, x_{l+1}y_{l+1}, y_{l+1}x_{l+2}, x_{l+2}y_{l+2}\}$. Then, H is called an *l-co-belt* if the following conditions are satisfied

(i) $E_1(H) \subset E_N(H)$ and $E_2(H) \subset E_R(H)$,

(ii) $d(x_i) = d(y_j) = 4$ for $i = 2, 3, \cdots, l+1, l+2; j = 2, 3, \cdots l+1$.

An l-co-belt H is said to be *maximal* if $\Gamma_G(y_1) \neq \{x_1, x_2, y_2, u\}$ and $\Gamma_G(y_{l+2}) \neq \{x_{l+2}, y_{l+1}, x_{l+3}, v\}$ for any $u, v \in G$. The vertices of an l-co-belt or a maximal l-co-belt H satisfying the condition (ii) are called *inner vertices* of H.

Definition 3.5. Let G be a 4-connected graph and H a subgraph of G such that $V(H) = \{x_1, x_2, x_3, y_1, y_2, y_3, y_4\}$ and $E(H) = \{x_1x_2, x_2x_3, y_1y_2, y_2y_3, y_3y_4, x_1y_2, x_2y_2, x_2y_3, x_3y_3\}$. Then, H is called a *W-framework* if the following conditions are satisfied

(i) $x_ix_{i+1} \in E_N(G)$ for $i = 1, 2$,

(ii) $d(x_2) = d(y_2) = d(y_3) = 4$,

(iii) $y_2y_3, x_1y_2, x_2y_2, x_2y_3, x_3y_3 \in E_R(G)$.

The vertex x_2 of a W-framework H is called the *inner vertex* of H.

Definition 3.6. Let G be a 4-connected graph and H a subgraph of G such that $V(H) = \{x_1, x_2, x_3, y_1, y_2, y_3, y_4\}$ and $E(H) = \{x_1x_2, x_2x_3, x_1x_3, y_1y_2, y_2y_3, y_3y_4, x_1y_2, x_2y_2, x_2y_3, x_3y_3\}$. Then, H is called a *W'-framework* if the following conditions are satisfied

(i) $x_ix_{i+1} \in E_N(G)$ for $i = 1, 2$,

(ii) $d(x_2) = d(x_3) = d(y_2) = d(y_3) = 4$ and $d(x_1) \geq 5$,

(iii) $y_2y_3, x_1y_2, x_2y_3, x_3y_3, x_1x_3 \in E_R(G), x_2y_2 \in E_N(G)$.

The vertices x_2, x_3 of a W'-framework H are called *inner vertices* of H.

After we have done the above preparations, we can state and prove our main results in the next section.

4 The Main Results

In this section we shall consider the problem on how many removable edges there are in a cycle of a 4-connected graph G. Before we give our main results, we need to show some lemmas.

Lemma 4.1. Let G be a 4-connected graph, $(xy, S; A, B)$ be a separating group of G such that $x \in A, y \in B, S = \{a, b, c\}$ and A be a 1-edge-vertex atom, say, $A = \{x, z\}$. Then, one of the following conclusions holds:

(i) $ax, bx, zx \in E_R(G)$.

(ii) $ax \in E_N(G), d(x) = d(z) = 4, bx, zx, az \in E_R(G), zc \in E_N(G)$.

(iii) $ax \in E_N(G), ay \in E_R(G)$. And, if $d(a) = 4, d(y) \geq 5$, then $az, zb, zx, by, ay \in E_R(G), bx \in E_N(G)$. If $d(a) \geq 5, d(y) = 4$, then $by, bx, bz, az \in E_R(G), zx \in E_N(G)$. If $d(a) = d(y) = 4$, then $az, bz, by \in E_R(G), bx, zx \in E_N(G)$. If $d(a) \geq 5, d(y) \geq 5$, then $az, zx, bx, by \in E_R(G)$.

(iv) $ax, bx, ac, bc \in E_R(G), zx, zc \in E_N(G), \{za, zb\} \cap E_N(G) \neq \emptyset, d(x) = d(c) = d(z) = 4$. If $za \in E_N(G)$, then the following conclusion holds: $d(b) = 4$, and if $d(a) = 4$, then $bz \in E_N(G)$; if $d(a) \geq 5$, then $bz \in E_R(G)$ holds. If $bz \in E_N(G)$, then the following conclusion holds: $d(a) = 4$, and if $d(b) = 4$, then $az \in E_N(G)$; if $d(b) \geq 5$, then $az \in E_R(G)$.

(v) $ax, bx, az, bz \in E_R(G), xz \in E_N(G), d(x) = d(z) = 4$.

(vi) $bx \in E_N(G), by \in E_R(G)$. And, if $d(a) = 4, d(y) \geq 5$, then $bz, za, zx, ay, by \in E_R(G), ax \in E_N(G)$. If $d(b) \geq 5, d(y) = 4$, then $ay, ax, az, bz \in E_R(G), zx \in E_N(G)$. If $d(b) = d(y) = 4$, then $bz, az, ay \in E_R(G), ax, zx \in E_N(G)$. If $d(b) \geq 5, d(y) \geq 5$, then $bz, zx, ax, ay \in E_R(G)$.

Proof. If $ax, bx, zx \in E_R(G)$, then the conclusion (i) holds. So, we may assume that $\{ax, bx, zx\} \cap E_N(G) \neq \emptyset$. Next we will distinguish the following cases to proceed the proof.

Case 1. $ax \in E_N(G)$.

Then, we take the corresponding separating group $(ax, T; C, D)$ such that $x \in C, a \in D$, and so, $x \in A \cap C, y \in B \cap (C \cup T)$. Let

$$\begin{aligned} X_1 &= (C \cap S) \cup (S \cap T) \cup (A \cap T), \\ X_2 &= (A \cap T) \cup (S \cap T) \cup (S \cap D), \\ X_3 &= (D \cap S) \cup (S \cap T) \cup (B \cap T), \\ X_4 &= (B \cap T) \cup (S \cap T) \cup (C \cap S). \end{aligned}$$

Subcase 1.1. $y \in B \cap C$.

Since $|A| = 2$ and A is a connected subgraph of G, we have that $A \cap D = \emptyset$. First, we claim that $A \cap T \neq \emptyset$. Otherwise, $A \cap T = \emptyset$, and so $|A \cap C| = 2$. Since $a \in S \cap D$, we have that $|X_1| \leq 2$. Then, $X_1 \cup \{x\}$ is a vertex-cut of G with cardinality less than 4, a contradiction. Hence, $A \cap T = \{z\}$. Second, we claim that $S \cap T = \emptyset$. Otherwise, $S \cap T \neq \emptyset$, and a contradiction will be deduced as follows. If $B \cap T = \emptyset$, since B is a connected subgraph of G, then we have that $B \cap D = \emptyset$. Then, $B = B \cap C$, and so $|S \cap T| = 2$. Noticing that $a \in S \cap D$ and $|S| = 3$, we have that $S \cap C = \emptyset$. From $|B| \geq 2$ we know that $|B \cap C| \geq 2$. Then, it is easy to see that $\{y\} \cup (S \cap T)$ is a vertex-cut of G with cardinality less than 4, a contradiction. So, $B \cap T \neq \emptyset$, and so $|S \cap T| = 1$. Noticing that $|T| = 3$, we have that $|B \cap T| = 1$. Since X_4 is a vertex-cut of $G - xy$, we have that $|X_4| \geq 3$, and so, $|S \cap C| \geq 1$. Since $S \cap D \neq \emptyset$, by noticing that $|S| = 3$, we have that $|S \cap D| = 1$, i.e., $S \cap D = \{a\}$. Note that $|X_3| = 3$. Since G is 4-connected, we have that $B \cap D = \emptyset$. Hence, $D = \{a\}$, which contradicts to that $|D| \geq 2$. Therefore, $S \cap T = \emptyset$. Note that $|B \cap T| = 2$. If $|S \cap D| = 1$, by a similar argument we can get that $D = \{a\}$, a contradiction. So, $|S \cap D| \geq 2$. Since $|X_4| \geq 3$, we have that $|S \cap C| \geq 1$. Therefore, $|S \cap C| = 1$ and $|S \cap D| = 2$. Since $bx \in E(G)$, obviously we have $b \in X_1$, and so $S \cap C = \{b\}$. Then, $S \cap D = \{a, c\}, \Gamma_G(x) = \{a, b, y, z\}, \Gamma_G(z) = \{x, a, b, c\}$. We claim that $xz \in E_R(G)$. Otherwise, $xz \in E_N(G)$, and we take the corresponding separating group $(xz, S'; A', B')$ such that $x \in A', z \in B'$. Since $xzax$ is a 3-cycle of G, we have that $a \in S'$ and $ax \in E_N(G)$. From Theorem 2.2 we know that $|A'| = 2$, say $A' = \{x, v_1\}$. Then, we have that axv_1a is a 3-cycle of G and $v_1 \neq z$, which is impossible to hold in G, and so, $xz \in E_R(G)$. We claim that $az \in E_R(G)$. Otherwise, $az \in E_N(G)$, and we take the corresponding separating group $(az, S'; A', B')$ such that $a \in A', z \in B'$. Obviously, $x \in S'$. Since $ax \in E_N(G)$, from Theorem 2.2 we have that $|A'| = 2$, say $A' = \{a, v_1\}$. Then, axv_1a is a 3-cycle of G and $v_1 \neq z$, which is impossible to hold in G, and so, $az \in E_R(G)$. Let $S' = \{x\} \cup (B \cap T), A' = C \cap (B \cup S), B' = G - bz - S' - A'$, then $(bz, S'; A', B')$ is a separating group of G, and so $bz \in E_N(G)$. We claim that $bx \in E_R(G)$. Otherwise, $bx \in E_N(G)$, and we take the corresponding separating group $(bx, S'; A', B')$ such that $b \in A', x \in B'$. Since $bxzb$ is a 3-cycle of G, we have that $z \in S'$. Since $bz \in E_N(G)$, we have that $|A'| = 2$, say $A' = \{b, v_1\}$. Then, bv_1zb is a 3-cycle of G, and $v_1 \neq x$, which is impossible to hold in G, and hence $bx \in E_R(G)$. Let $S_1 = \{a, b, y\}$, then (zc, S_1) is a separating pair of G, and so, $zc \in E_N(G)$. Obviously, $d(x) = d(z) = 4$. Hence, the conclusion (ii) holds.

Subcase 1.2. $y \in B \cap T$.

Since $xy \in E_N(G)$, from Theorem 2.2 we have that $|C| = 2$. If $|A \cap C| = 2$, then we have that $A = A \cap C = C$. Since $B \cap T \neq \emptyset \neq S \cap D$, we have that $|S \cap T| \leq 2$. It is easy to see that $\{x\} \cup X_1$ is a vertex-cut of G with cardinality less than 4, a contradiction. So, $A \cap C = \{x\}$. Since A and C are connected subgraphs of G, we have that $|S \cap C| = |A \cap T| = 1$ and $B \cap C = \emptyset = A \cap D$. We claim that $S \cap T = \emptyset$. Otherwise, $|S \cap T| = 1$, and so $|B \cap T| = 1$. Note that $|X_3| = 3$. Since G is 4-connected, we have that $B \cap D = \emptyset$, and so $B = B \cap T = \{y\}$, which contradicts to that $|B| \geq 2$. Therefore, $S \cap T = \emptyset$, and so $|B \cap T| = |S \cap D| = 2$. From $\Gamma_G(x) = \{z, b, a, y\}$ we know that $S \cap C = \{b\}$, and so $S \cap D = \{a, c\}, A \cap T = \{z\}$.

Let $B \cap T = \{u, y\}$. Next we will discuss the following subsubcases.

Subsubcase 1.2.1. If $ay \notin E(G)$, we claim that $xz \in E_R(G)$. Otherwise, $xz \in E_N(G)$, and we take the corresponding separating group $(xz, S'; A', B')$ such that $z \in A', x \in B'$. Since $azxa$ is a 3-cycle of G, we have that $a \in S'$. Since $ax \in E_N(G)$, from Theorem 2.2 we have that $|B'| = 2$, say $B' = \{x, v_1\}$. Then, axv_1a is a 3-cycle of G. However, $ay \notin E(G)$ and $v_1 \neq z$, which is impossible to hold in G. Hence, $xz \in E_R(G)$. By symmetry, we can show that $bx \in E_R(G)$. We claim that $az \in E_R(G)$. Otherwise, $az \in E_N(G)$, and we take the corresponding separating group $(az, S'; A', B')$ such that $a \in A', z \in B'$. Since $azxa$ is a 3-cycle of G, we have that $x \in S'$. Since $ax \in E_N(G)$, we have that $|A'| = 2$, say $A' = \{a, v_1\}$. Then, axv_1a is a 3-cycle of G, an analogous argument can lead to a contradiction. So, $az \in E_R(G)$. By symmetry, we have that $by \in E_R(G)$. Let $S' = \{a, b, y\}$. Obviously, (zc, S') is a separating pair of G, and so $zc \in E_N(G)$. Hence, the conclusion (ii) holds.

Subsubcase 1.2.2. If $ay \in E(G)$, then from Corollary 2.3 we know that $ay \in E_R(G)$. Then, we consider the following cases.

(1.) If $d(a) \geq 5$ and $d(y) \geq 5$, we claim that $xz \in E_R(G)$. Otherwise, $xz \in E_N(G)$, and we take the corresponding separating group $(xz, S'; A', B')$ such that $x \in A', z \in B'$. Since $azxa$ is a 3-cycle of G, we have that $a \in S'$. Since $ax \in E_N(G)$, from Theorem 2.2 we know that $|A'| = 2$, say $A' = \{x, v_1\}$. Then, axv_1a is a 3-cycle of G. Noticing that $d(v_1) = 4$ and $d(y) \geq 5$, we have that $v_1 \neq y$, which is impossible to hold in G. Hence, $xz \in E_R(G)$. By symmetry, we can show that $bx \in E_R(G)$. We claim that $az \in E_R(G)$. Otherwise, $az \in E_N(G)$, and we take the corresponding separating group $(az, S'; A', B')$. Obviously, $x \in S'$, and an analogous argument can lead to a contradiction. So, $az \in E_R(G)$. By symmetry, we have that $by \in E_R(G)$. Hence, the conclusion (iii) holds.

(2.) If $d(a) = 4$ and $d(y) \geq 5$, we let $\Gamma_G(a) = \{x, y, z, v\}$. Let $A' = \{a, x\}, S' = \{v, z, y\}, B' = G - bx - S' - A'$, then $(bx, S'; A', B')$ is a separating group of G, and so $bx \in E_N(G)$. We claim that $bz \in E_R(G)$. Otherwise, $bz \in E_N(G)$, and we take the corresponding separating group $(bz, S'; A', B')$ such that $b \in A', z \in B'$. Noticing that $bzxb$ is a 3-cycle of G, we have $x \in S'$. Since $bx \in E_N(G)$, from Theorem 2.2 we have that $|A'| = 2$, say, $A' = \{b, v_1\}$. Then, bxv_1b is a 3-cycle of G. Noticing that $d(y) \geq 5$ and $d(v_1) = 4$, we have that $v_1 \neq y$, which is impossible to hold in G. Therefore, $bz \in E_R(G)$. We claim that $az \in E_R(G)$. Otherwise, $az \in E_N(G)$, and we take the separating group $(az, S'; A', B')$ such that $a \in A', z \in B'$. Obviously, $x \in S'$. Since $ax \in E_N(G)$, from Theorem 2.2 we have that $|A'| = 2$, say $A' = \{a, v_1\}$. Then, axv_1a is a 3-cycle of G and $v_1 \neq z$. Note that $d(v_1) = 4, d(y) \geq 5$, and so, $v_1 \neq y$, which is impossible to hold in G. So, $az \in E_R(G)$. By an analogous argument we can show that $zx \in E_R(G)$. We claim that $by \in E_R(G)$. Otherwise, $by \in E_N(G)$, and we take the separating group $(by, S'; A', B')$ such that $b \in A', y \in B'$. Obviously, $x \in S'$. Since $xy \in E_N(G)$, from Theorem 2.2 we have that $|B'| = 2$, say $B' = \{y, v_1\}$. Then, xyv_1x is a 3-cycle of G. It is easy to see that this is true only if $v_1 = a$. From $\Gamma_G(a) = \{x, y, z, v\}$ we know that $S' = \{x, z, v\}$. Since $d(y) \geq 5$, we have $yz \in E(G)$, which is impossible to hold in G. So, $by \in E_R(G)$. Hence, the conclusion (iii) holds.

(3.) If $d(a) \geq 5$ and $d(y) = 4$. By an analogous argument used in (2.) we can show that the conclusion (iii) holds.

(4.) If $d(a) = d(y) = 4$, we let $\Gamma_G(a) = \{x, y, z, v\}$, $A_1 = \{a, x\}, S_1 = \{z, y, v\}, B_1 = G - bx - S_1 - A_1$. Then, $(bx, S_1; A_1, B_1)$ is a separating group of G, and so, $bx \in E_N(G)$. By symmetry, we have that $ax, xy, zx \in E_N(G)$. From Corollary 2.3 we have that $az, by, bz \in E_R(G)$. Hence, the conclusion (iii) holds.

If $bx \in E_N(G)$, we may employ a similar argument to show that the conclusion (iv) holds. So, next we may assume that $ax, bx \in E_R(G)$.

Case 2. $xz \in E_N(G)$.

We take the corresponding separating group $(xz, T; C, D)$ such that $x \in C, z \in D$. Then, we have that $x \in A \cap C, z \in A \cap D$. Since $xzax, xzbx$ are two 3-cycles of G, we have that $a, b \in S \cap T$. Since $A \cap D = \{z\}$ and D is a connected subgraph of G as well as $|D| \geq 2$, we can get that $S \cap D \neq \emptyset$. Since $S = \{a, b, c\}$, we have that $S \cap D = \{c\}$. Obviously, $|B \cap T| = 1$.

Subcase 2.1. If $az \in E_N(G)$, from Theorem 2.2 we have that $|D| = 2$, and so $D = \{z, c\}$. It is easy to see that $ac, bc \in E(G)$. From Theorem 2.4 we have that $ac, bc \in E_R(G)$. Obviously, $d(x) = d(c) = d(z) = 4$ and $\Gamma_G(x) = \{z, b, a, y\}$. Let $A_1 = \{x, z\}, S_1 = \{y, a, b\}, B_1 = G - zc - S_1 - A_1$, then $(zc, S_1; A_1, B_1)$ is a separating group of G, and so $zc \in E_N(G)$. We take the separating group $(az, S'; A', B')$ such that $a \in A', z \in B'$. Obviously, $x \in S'$. Since $xz \in E_N(G)$, we have that $|B'| = 2$, say $B' = \{z, v_1\}$. Then, xzv_1x is a 3-cycle of G, which is true only if $v_1 = b$, and so $d(b) = 4$. Here, if $d(a) = 4$, let $\Gamma_G(a) = \{x, z, c, v\}$, $A_1 = \{a, z\}, S_1 = \{c, x, v\}$ and $B_1 = G - bz - S_1 - B_1$. Then $(bz, S_1; A_1, B_1)$ is a separating group of G, and so $bz \in E_N(G)$. If $d(a) \geq 5$, we claim that $bz \in E_R(G)$. Otherwise, $bz \in E_N(G)$, then we take the corresponding separating group $(bz, S_1; A_1, B_1)$ such that $b \in A_1, z \in B_1$. Obviously, $x \in S_1$. Since $xz \in E_N(G)$, from Theorem 2.2 we have $|B_1| = 2$, say $B_1 = \{z, v_1\}$. Then xv_1zx is a 3-cycle of G. Note that $d(a) \geq 5, d(v_1) = 4$, and so $v_1 \neq a$. Which is impossible to hold in G. So, $bz \in E_R(G)$. Hence, the conclusion (iv) holds.

Subcase 2.2. If $bz \in E_N(G)$, we may employ a similar argument used in Subcase 2.1 to show that the conclusion (iv) holds.

Therefore, we may assume that $az, bz \in E_R(G)$. Obviously, $d(x) = d(z) = 4$, and so the conclusion (v) holds. The proof is now complete.□

Corollary 4.2. Let G be a 4-connected graph and $(xy, S; A, B)$ be a separating group of G such that $x \in A, y \in B, S = \{a, b, c\}$. Let A be a 1-edge-vertex-cut atom, say $A = \{x, z\}$, If $\{xa, xb, xz\} \cap E_N(G) \neq \emptyset$, then we have that x is an inner vertex of one of the following subgraphs in G: helm, co-belt, belt, W'-framework, W-framework or bi-fan.

Lemma 4.3. Let G be a 4-connected graph, $(xy, S; A, B)$ be a separating group of G, and A be a 2-edge-vertex-cut atom, say $A = \{x, z\}$ and $S = \{a, b, c\}$. Then, $ax, bx, cx, xz \in E_R(G)$.

Proof. By contradiction. We consider the following cases.

(1.) If $ax \in E_N(G)$, we take the corresponding separating group $(ax, T; C, D)$ such that $x \in C, a \in D$. Then, $x \in A \cap C, a \in S \cap D$. Let $X = (D \cap S) \cup (S \cap T) \cup (B \cap T)$. Since $bx, cx \in E(G)$, we can get that $b, c \in S \cap (C \cup T)$, and so $|S \cap D| = 1$. We claim that $A \cap T \neq \emptyset$. Otherwise, $A \cap T = \emptyset$. Since $|A| = 2$ and A is a connected subgraph of G, we have that $A \cap C = \{x, z\}$. It is easy to see that $\{b, c, x\}$ would be a 3-vertex-cut of G, a contradiction. Therefore, $A \cap T = \{z\}, A \cap D = \emptyset$. Obviously, $|X| \geq 3$. Since $|S \cap D| = 1$ and $|D| \geq 2$, we have that $B \cap D \neq \emptyset$, and so $|X| \geq 4$. However, by noticing that $|A \cap T| = 1$, we have that $|(S \cup B) \cap T| = 2$, and so $|X| = 3$, a contradiction.

If $bx \in E_N(G)$ or $cx \in E_N(G)$, we may employ a similar argument. So, next we may assume that $bx, cx \in E_R(G)$.

(2.) If $xz \in E_N(G)$, we take the corresponding separating group $(xz, T; C, D)$ such that $x \in C, z \in D$. Then, we have that $x \in A \cap C, z \in A \cap D$. It is easy to see that $a, b, c \in S \cap T$. Since $|T| = 3$, we have that $y \in B \cap C$. Let $X = (D \cap S) \cup (S \cap T) \cup (B \cap T)$, and so $|X| = 3$. Then, we have that $B \cap D = \emptyset$. Noticing that $D \cap S = \emptyset$, we have that $D = A \cap D = \{z\}$, which contradicts to that $|D| \geq 2$. Therefore, $xz \in E_R(G)$.

From the above arguments, we know that the lemma holds. □

Now we present our main results. For convenience we denote by $\Re$ the set of all helms, maximal l-bi-fans, maximal l-belts, maximal l-co-belts, W-frameworks and W'-frameworks of a graph G.

Definition 4.4. Let C be a cycle of a 4-connected graph G and H a subgraph of G belonging to $\Re$. If C contains an inner vertex of H, then we say that C *passes through* H.

Theorem 4.5. Let G be a 4-connected graph and C a cycle of G. If C does not pass through any subgraph of G belonging to $\Re$, then there are least two removable edges of G in C.

Proof. By contradiction. Assume that C does not pass through any subgraph of G belonging to $\Re$, and there is at most one removable edge of G in C. Let $F = E(C) \cap E_R(G)$, then $|F| \leq 1$. Denote $E(C) - F$ by E_0. We take the separating group $(uw, S'; A', B')$ such that $u \in A', w \in B'$ and $uw \in E_0$. From $|F| \leq 1$ we know that $(E(A') \cup ([A', S']) \cap F = \emptyset$ or $(E(B') \cup ([S', B']) \cap F = \emptyset$. Without loss of generality, we may assume that $(E(A') \cup ([A', S']) \cap F = \emptyset$. Since A' is an E_0-edge-vertex-cut fragment, A' must contain an E_0-edge-vertex-cut end-fragment as its subgraph, say A. Then, we have that $(E(A) \cup ([A, S]) \cap F = \emptyset$, and we take a separating group $(xy, S; A, B)$ such that $x \in A, y \in B$ with $xy \in E_0$. Next, we will consider $|A|$ by cases.

Case 1. $|A| = 2$. Then, A is a 1-edge-vertex-cut atom or a 2-edge-vertex-cut atom, say, $A = \{x, z\}$. Let $S = \{a, b, c\}$.

Subcase 1.1. If A is a 2-edge-vertex-cut atom, since $xy \in E(C)$ and C is a cycle of G, we have that $\{xa, xb, xc, xz\} \cap E(C) \neq \emptyset$. From Lemma 4.3 we know that $\{xa, xb, xc, xz\} \subset E_R(G)$, which contradicts to that $(E(A) \cup [A, S]) \cap F = \emptyset$.

Subcase 1.2. If A is a 1-edge-vertex-cut atom, by noticing that C is a cycle of G and $([E(A) \cup [A, S]) \cap F = \emptyset$, then obviously $\{xa, xb, xz\} \cap E_N(G) \neq \emptyset$. From Corollary 4.2 we know that x is an inner vertex of one of the subgraphs of G belonging to $\Re$. Since $xy \in E(C)$, this contradicts to that C does not pass through any subgraph of

G belonging to $\Re$.

Case 2. $|A| \geq 3$. Then, we will discuss the following subcases.

Subcase 2.1. If there exists an $xz \in E_0 \cap E(A \cup [A, S])$, then obviously $z \notin S$; otherwise, we would have $|A| = 2$, a contradiction to that $|A| \geq 3$. We take the separating group $(xz, S_1; A_1, B_1)$ such that $x \in A_1, z \in B_1$. Then, we have that $x \in A \cap A_1, z \in A \cap B_1$. Let

$$\begin{aligned}
X_1 &= (A_1 \cap S) \cup (S \cap S_1) \cup (A \cap S_1), \\
X_2 &= (A \cap S_1) \cup (S \cap S_1) \cup (B_1 \cap S), \\
X_3 &= (B_1 \cap S) \cup (S \cap S_1) \cup (B \cap S_1), \\
X_4 &= (B \cap S_1) \cup (S \cap S_1) \cup (A_1 \cap S).
\end{aligned}$$

If $y \in B \cap S_1$, from Theorem 2.2 we have that $|A_1| = 2$, say $A_1 = \{x, v_1\}$. We claim that A_1 is a 1-edge-vertex-cut atom; otherwise, A_1 is a 2-edge-vertex-cut atom, and then, from Lemma 4.3 we have $xy \in E_R(G)$, a contradiction. From Corollary 4.2 we know that x is an inner vertex of some subgraph of G belonging to $\Re$, a contradiction to the assumption. Therefore, $y \notin B \cap S_1$, and so $y \in A_1 \cap B$. Since $A \cap B_1 \neq \emptyset$, we have that X_2 is a vertex-cut of $G - xz$, and so $|X_2| \geq 3$. By an analogous argument, we can deduce that $|X_4| \geq 3$. Since $|X_2| + |X_4| = |S| + |S_1| = 6$, we can get that $|X_2| = |X_4| = 3$, and so $|A_1 \cap S| = |A \cap S_1|, |B \cap S_1| = |B_1 \cap S|$. We claim that $A \cap B_1 = \{z\}$. Otherwise, $|A \cap B_1| \geq 2$. Then, $(xz, X_2; A \cap B_1, A_1 \cup B)$ is a separating group of G and $xz \in E_0$. It is easy to see that $A \cap B_1$ is an E_0-edge-vertex-cut fragment contained in A, which contradicts to that A is an E_0-edge-vertex-cut end-fragment of G. Therefore, $A \cap B_1 = \{z\}$. Since $|B_1| \geq 2$ and B_1 is a connected subgraph of G, we have that $B_1 \cap S \neq \emptyset$.

Subsubcase 2.1.1. If $|B_1 \cap S| = |B \cap S_1| = 3$, then $|X_1| = 0$, and so $\{z, y\}$ would be 2-vertex-cut of G, a contradiction.

Subsubcase 2.1.2. If $|B_1 \cap S| = |B \cap S_1| = 2$, since X_1 is a vertex-cut of $G - xy - xz$, then $|X_1| \geq 2$. Noticing that $|S| = |S_1| = 3$, we have that $|A \cap S_1| = |A_1 \cap S| = 1, S \cap S_1 = \emptyset$. We claim that $A \cap A_1 = \{x\}$. Otherwise, $|A \cap A_1| \geq 2$. Then, $\{x\} \cup X_1$ would be a 3-vertex-cut of G, a contradiction. Let $A \cap S_1 = \{a\}, A_1 \cap S = \{b\}, S \cap B_1 = \{v_1, v_2\}$. From $A \cap B_1 = \{z\}$ we can get that $\Gamma_G(z) = \{x, a, v_1, v_2\}$. We claim that $ab \in E(G)$. Otherwise, $\{x, v_1, v_2\}$ would be a 3-vertex-cut of G, a contradiction. We claim that $av_1, av_2 \in E(G)$. Otherwise, without loss of generality, we may assume that $av_1 \notin E(G)$. Let $A' = \{x, a\}, S' = \{b, z, v_2\}, B' = G - xy - S' - A'$, then $(xy, S'; A', B')$ is a separating group of G. Since $xy \in E_0$, A' is an E_0-edge-vertex-cut fragment contained in A, which contradicts to that A is an E_0-edge-vertex-cut end-fragment. So, $av_1, av_2 \in E(G)$, and hence $\Gamma_G(a) = \{x, z, b, v_1, v_2\}$. Let $S_0 = \{x, v_1, v_2\}, A_0 = \{a, z\}, B_0 = G - ab - S_0 - A_0$, then $(ab, S_0; A_0, B_0)$ is a separating group of G, and so $ab \in E_N(G)$.

We claim that $az \in E_R(G)$. Otherwise, $az \in E_N(G)$, and we take the corresponding separating group $(az, S'; A', B')$ such that $a \in A', z \in B'$. Since $axza, av_1za, av_2za$ are 3-cycles of G, we have that $x, v_1, v_2 \in S'$. Since $xz \in E_N(G)$, from Theorem 2.2 we have that $|B'| = 2$, say $B' = \{z, u\}$. Then, $uzxu$ is a 3-cycle of G, which is impossible to hold in G, and so $az \in E_R(G)$.

Since $(E(A) \cup ([A, S])) \cap F = \emptyset$ and C is a cycle of G, we can get that $\{zv_1, zv_2\} \cap E_N(G) \neq \emptyset$. Without loss of generality, we may assume that $zv_1 \in E_N(G)$. We take the separating group $(zv_1, T; C', D')$ such that $z \in C', v_1 \in D'$. Then, we have that $z \in C' \cap B_1, v_1 \in B_1 \cap D'$. Obviously, $a \in S_1 \cap T$. Let

$$\begin{aligned} Y_1 &= (A_1 \cap T) \cup (S_1 \cap T) \cup (C' \cap S_1), \\ Y_2 &= (C' \cap S_1) \cup (S_1 \cap T) \cup (B_1 \cap T), \\ Y_3 &= (B_1 \cap T) \cup (S_1 \cap T) \cup (S_1 \cap D'), \\ Y_4 &= (D' \cap S_1) \cup (S_1 \cap T) \cup (A_1 \cap T). \end{aligned}$$

(1.) If $x \in A_1 \cap C'$, then Y_1 is a vertex-cut of $G - xz$, and so $|Y_1| \geq 3$. By a similar argument, we have that $|Y_3| \geq 3$. Since $|Y_1| + |Y_3| = |S_1| + |T| = 6$, we can conclude that $|Y_1| = |Y_3| = 3$ and $|A_1 \cap T| = |S_1 \cap D'|, |S_1 \cap C'| = |B_1 \cap T|$. Since $a \in S_1$, from Theorem 2.4 we know that $b \notin T \cup S_1$. Since $bx, zv_2 \in E(G)$, we have that $b \in A_1 \cap C'$ and $v_2 \notin D' \cap B_1$. From $\Gamma_G(a) = \{v_1, v_2, z, x, b\}$, we know that $\Gamma_G(a) \cap (B_1 \cap D') = \{v_1\}$. Then, we have that $|A_1 \cap T| = |S_1 \cap D'| = 0, 1$ or 2.

(1.1.) If $|A_1 \cap T| = |D' \cap S_1| = 2$, then $|S_1 \cap C'| = |B_1 \cap T| = 0$. Since $zv_2 \in E(G)$, we have $v_2 \in B_1 \cap C'$, and hence $\{a, z\}$ would be 2-vertex-cut of G, a contradiction.

(1.2.) If $|A_1 \cap T| = |D' \cap S_1| = 1$, then $|S_1 \cap T| \leq 2$. First, we claim that $B_1 \cap D' = \{v_1\}$. Otherwise, $|B_1 \cap D'| \geq 2$. Then, from $\Gamma_G(a) \cap (B_1 \cap D') = \{v_1\}$, we can conclude that $\{v_1\} \cup (Y_3 - \{a\})$ would be a 3-vertex-cut of G, a contradiction. So, $B_1 \cap D' = \{v_1\}$. Let $D' \cap S_1 = \{u_1\}$. If $A_1 \cap D' \neq \emptyset$, from $\Gamma_G(a) = \{x, z, b, v_1, v_2\}$ we can get that $A_1 \cap D' \cap \Gamma_G(a) = \emptyset$, and so $Y_4 - \{a\}$ would be a vertex-cut of G with cardinality less than 4, a contradiction. Therefore, $A_1 \cap D' = \emptyset$. Then, $au_1 \in E(G)$. However, it is easy to see that $u_1 \notin \{x, z, b, v_1, v_2\}$, a contradiction.

(1.3.) If $|D' \cap S_1| = |A_1 \cap T| = 0$, since D' is a connected subgraph of G, we have that $A_1 \cap D' = \emptyset$. Then, $|D'| = |D' \cap B_1| \geq 2$. Since $\Gamma_G(a) \cap (B_1 \cap D') = \{v_1\}$, by noticing that $|Y_3| = 3$, we have that $\{v_1\} \cup (Y_3 - \{a\})$ would be a 3-vertex-cut of G, a contradiction.

(2.) If $x \in A_1 \cap T$, from Theorem 2.2 we have that $|C'| = 2$. Since C' is a connected subgraph of G, we have that $A_1 \cap C' = \emptyset$. If $S_1 \cap C' \neq \emptyset$, since $a \in S_1 \cap T$, then $|D' \cap S_1| \leq 1$. Noticing that Y_3 is a vertex-cut of $G - zv_1$, we have that $|Y_3| \geq 3$, and so $|B_1 \cap T| = 1, A_1 \cap T = \{x\}$. Obviously, $|Y_4| = 3$, and hence $A_1 \cap D' = \emptyset$, and so $A_1 = \{x\}$, which contradicts to that $|A_1| \geq 2$. So, we have that $S_1 \cap C' = \emptyset$, and so $|B_1 \cap C'| = 2$. Since $A_1 \cap T \neq \emptyset$, obviously, $\{z\} \cup (T - \{x\})$ would be a vertex-cut with cardinality less than 4, a contradiction.

From the above arguments, we can conclude that Subsubcase 2.1.2 does not occur.

Subsubcase 2.1.3. If $|B_1 \cap S| = |B \cap S_1| = 1$, then $|S \cap S_1| \leq 2$. We claim that $|S \cap S_1| < 2$. Otherwise, $|S \cap S'| = 2$. Then, $A \cap S_1 = \emptyset = S \cap A_1$. If $|A \cap A'| \geq 2$, then $\{x\} \cup (S \cap S_1)$ would be a vertex-cut of G with cardinality less than 4, a contradiction, and so $A \cap A_1 = \{x\}$. Note that $|X_2| = 3$. If $|A \cap B_1| \geq 2$, then by an argument similar to that used in Subcase 2.1, $A \cap B_1$ would be an E_0-edge-vertex-cut fragment contained in A, which contradicts to that A is an E_0-edge-vertex-cut end-fragment. Hence, $A \cap B_1 = \{z\}$, and so $|A| = 2$, which contradicts to that $|A| \geq 3$. Therefore, $|S \cap S_1| \leq 1$, and then $|X_3| \leq 3$, and so $B \cap B_1 = \emptyset$. Since

$A \cap B_1 = \{z\}$, we have that $|B_1| = 2$ and B_1 is a 1-edge-vertex-cut atom of G, say $B_1 = \{z, u\}$. Since C is a cycle and $(E(A) \cup [A, S]) \neq \emptyset$, we have that z is incident with at least two unremovable edges. From Corollary 4.2 we know that z is an inner vertex of some subgraph of G belong to $\Re$, which contradicts to that C does not pass through any subgraph of G belonging to $\Re$. The proof is now complete. □

Theorem 4.6. Let G be a 4-connected graph and C a cycle of G. If C passes through only one subgraph of G belonging to $\Re$, then there exists at least one removable edge of G in C.

Proof. By contradiction. Assume that $E(C) \subset E_N(G)$. Let C pass through the subgraph H of G that belongs to $\Re$, see the definitions of H in Definitions 3.1 through 3.6. If H is a maximal l-belt, from the assumption, it is easy to see that $\{x_2x_1, y_ly_{l+1}\} \cap E(C) \neq \emptyset$. If $x_2x_1 \in E(C)$, by letting $S = \{y_{l+2}, x_{l+2}, y_1\}, e = x_2x_1, B = \{x_2, \cdots, x_{l+1}, y_2, \cdots, y_{l+1}\}, A = G - e - S - B$, then $(e, S; A, B)$ is a separating group of G such that A does not contain any inner vertex of the maximal l-belt ($l \geq 1$); if $y_ly_{l+1} \in E(C)$, by letting $S = \{x_1, y_1, x_{l+2}\}, e = y_{l+1}y_{l+2}, B = \{x_2, \cdots, x_{l+1}, y_2, \cdots, y_{l+1}\}, A = G-e-S-B$, then $(e, S; A, B)$ is a separating group of G such that A does not contain any inner vertex of the maximal l-belt ($l \geq 1$). If H is a maximal l-co-belt, similarly, we have that $\{x_1x_2, y_1y_2\} \cap E(C) \neq \emptyset$, if $x_1x_2 \in E(C)$, by letting $S = \{y_{l+2}, x_{l+3}, y_1\}, e = x_2x_1, B = \{x_2, \cdots, x_{l+2}, y_2, \cdots, y_{l+1}\}, A = G - e - S - B$, then $(e, S; A, B)$ is a separating group of G such that A does not contain any inner vertex of the maximal l-co-belt ($l \geq 1$); if $y_1y_2 \in E(C)$, by letting $S = \{y_{l+2}, x_{l+3}, x_2\}, e = y_2y_1, B = \{x_3, \cdots, x_{l+2}, y_2, \cdots, y_{l+1}\}, A = G - e - S - B$, then $(e, S; A, B)$ is a separating group of G such that A does not contain any inner vertex of the maximal l-co-belt ($l \geq 1$). If H is a maximal l-bi-fan ($l \geq 1$), by letting $S = \{a, b, x_{l+3}\}, e = x_2x_1, B = \{x_2, \cdots, x_{l+2}\}, A = G - e - S - B$, then $(e, S; A, B)$ is a separating group of G such that A does not contain any inner vertex of the maximal l-bi-fan. If H is a helm, by letting $e = x_1v_1, S = \{v_2, v_3, v_4\}, B = \{a, x_1, x_2, x_3, x_4\}, A = G - e - S - B$, then $(e, S; A, B)$ is a separating group of G such that A does not contain any inner vertex of the helm. If H is a W-framework, then C must pass through x_1x_2, x_2x_3. In this case, by letting $e = x_2x_1, S = \{x_3, x_4, y_2\}, B = \{x_2, y_3\}, A = G - e - S - B$, then $(e, S; A, B)$ is a separating group of G such that A does not contain any inner vertex of the W-framework. If H is a W'-framework, by noticing that $\{x_1x_2, x_2y_2\} \cap E(C) \neq \emptyset$, then if $x_1x_2 \in E(C)$, by letting $S = \{y_2, x_3, y_4\}, B = \{x_2, y_3\}, A = G - x_1x_2 - S - B$, then $(x_1x_2, S; A, B)$ is a separating group of G such that A does not contain any inner vertex of the W'-framework; if $x_2y_2 \in E(C)$, by letting $S = \{x_1, y_3, v\}$ such that $v \in \Gamma_G(x_3)$, $B = \{x_2, x_3\}, A = G - x_2y_2 - S - B$, then the separating group $(x_2y_2, S; A, B)$ is a separating group of G such that A does not contain any inner vertex of the W'-framework.

Let $E_0 = E(C)$, then A is an E_0-edge-vertex-cut fragment of G such that it does not contain any inner vertex of H. Obviously, A contains an E_0-edge-vertex-cut end-fragment as its subgraph, say A'. It is easy to see that A' does not contain any inner vertex of H. Finally, by an argument analogous to that used in the proof of Theorem 4.5, we can show that A' contains an inner vertex of some subgraph of G belonging to $\Re$, which contradicts to that A' does not contain any inner vertex of any subgraph of G belonging to $\Re$. The proof is now complete. □

Finally, to end this paper we construct examples to show that the lower bounds for the numbers of removable edges of G that a cycle of G can contain in Theorems 4.5 and 4.6 are in some sense best possible, and we also construct an example to show that the conditions, i.e., the numbers of subgraphs of G belonging to $\Re$ that a cycle of G can pass through in Theorems 4.5 and 4.6 are in some sense best possible.

Let F be a maximal k-bi-fan such that $V(F) = \{a, b, z_1, z_2, \cdots, z_{k+3}\}$ and $E(F) = \{z_1z_2, z_2z_3, \cdots, z_{k+2}z_{k+3}, az_2, az_3, \cdots, az_{k+2}, bz_2, \cdots, bz_{k+2}\}$ where $k \geq 1$. Let L be a maximal l-belt such that $V(L) = \{x_1, x_2, \cdots, x_{l+2}, y_1, y_2, \cdots, y_{l+2}\}$ and $E(H) = E_1(H) \cup E_2(H)$ where $E_1(H) = \{x_1x_2, x_2x_3, \cdots, x_{l+1}x_{l+2}, y_1y_2, y_2y_3, \cdots, y_{l+1}y_{l+2}\}$ and $E_2(H) = \{y_1x_2, x_2y_2, y_2x_3, \cdots, y_lx_{l+1}, x_{l+1}y_{l+1}, y_{l+1}x_{l+2}\}$, in which $l \geq 1$.

Example 1. Identify the vertex a with x_1, vertex b with y_{l+2}, vertex z_{k+3} with x_{l+2}, vertex z_1 with y_1, respectively. Denote the resulting graph by G_1. Let $G = G_1 + ab + y_1x_{l+2}$. It is easy to see that G is a 4-connected graph. First, let $A = \{x_3, x_4, \cdots, x_{l+1}, y_2, y_3, \cdots, y_{l+1}\}, S = \{x_2, x_{l+2}, y_1\}, B = G - by_{l+1} - S - A$, then $(by_{l+1}, S; A, B)$ is a separating group of G, and so $by_{l+1} \in E_N(G)$. Since $y_1x_{l+2} \in E([S])$, from Theorem 2.4 we have that $y_1x_{l+2} \in E_R(G)$. Obviously, $(x_{l+2}z_{k+2}, S_1)$ is a separating pair such that $S_1 = \{a, b, z_2\}$, and (z_2y_1, S_2) is also a separating pair such that $S_2 = \{a, b, x_{l+2}\}$. It is easy to see that $z_iz_{i+1} \in E_N(G)$ where $i = 2, \cdots, k+1$. Pick up the cycle $C_1 = y_1x_{l+2}z_{k+2}z_{k+1}z_k \cdots z_2y_1$. Then, C_1 only passes through one subgraph of G belonging to $\Re$, and C_1 has only one removable edge y_1x_{l+2} of G. This shows that the result of Theorem 4.6 is in some sense best possible.

Example 2. First, delete the vertices z_1, z_{k+3} from F. Then, identify vertex z_2 with x_1, vertex z_{k+2} with y_{l+2}, respectively. Denote the resulting graph by G_2. Let $G = G_2 + ab + ay_1 + bx_{l+2} + y_1x_{l+2}$. It is easy to see that G is a 4-connected graph. Let $A = \{x_3, \cdots, x_{l+1}, y_2, \cdots, y_{l+1}\}, S = \{y_1, x_{l+2}, x_2\}, B = G - z_{k+2}y_{l+1} - S - A$, then $(z_{k+2}y_{l+1}, S; A, B)$ is a separating group of G, and so $z_{k+2}y_{l+1} \in E_N(G)$. Since $y_1x_{l+2} \in E([S])$, from Theorem 2.4 we have that $y_1x_{l+2} \in E_R(G)$. Obviously, (z_2x_2, S_1) is a separating group of G such that $S_1 = \{a, b, z_{k+2}\}$, and so $z_2x_2 \in E_N(G)$. By a similar argument, we can get that $ay_1, bx_{l+2} \in E_N(G)$. Since $ab \in E([S_1])$, we have $ab \in E_R(G)$. Pick up the cycle $C_2 = abx_{l+2}y_1a$. Then, C_2 does not pass through any subgraph of G belonging to $\Re$, and C_2 has exactly two removable edges ab, y_1x_{l+2} of G . This shows that the result of Theorem 4.5 is in some sense best possible.

The following example shows that if a cycle C of G passes through two subgraphs of G belonging to $\Re$, then it may not contain any removable edge of G.

Example 3. First, delete the vertices z_{k+3} from F. Then, identify the vertex a with x_1, vertex b with x_{l+2}, vertex z_{k+2} with y_{l+2}, vertex z_1 with y_1, respectively. Denote the resulting graph by G_3. Let $G = G_3 + ab + y_1x_{l+2}$. It is easy to see that G is a 4-connected graph. Pick up the cycle $C_3 = y_1y_2 \cdots y_{l+2}z_{l+2}z_{l+1} \cdots z_2y_1$. Then, C_3 passes through two subgraphs of G belonging to $\Re$. It is easy to see that $E(C_3) \subset E_N(G)$, and so C_3 does not contain any removable edge of G. This in some sense shows that the conditions of Theorems 4.5 and 4.6 are best possible.

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